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DIRECTIONAL TRANSFORMS AND PSEUDO-COMMUTING PROPERTIES

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Abstract. Given a diagonal anisotropic expanding matrix $D \in \mathbb{Z}^{\mathrm{d}\times\mathrm{d}}$, we show that it is always possible to detect a unimodular matrix $A \in \mathbb{Z}^{\mathrm{d}\times\mathrm{d}}$ that satisfies the pseudo-commuting properties

$$DA = A^n D.$$

In particular when $\mathrm{d} = 2$ the more general relation

$$DA^m = A^n D$$

holds.

Keywords: Commuting property, integer matrix, shear matrix, directional transform.

§1. Introduction

Applications related to phenomena that can be described as functions $f \in L^2(\mathbb{R}^{\mathrm{d}})$ (e.g. signals, images etc) have received a strong impulse from the definition of wavelet transform and from the construction of efficient algorithms based on a multiresolution analysis (MRA) of $L^2(\mathbb{R}^{\mathrm{d}})$. When dealing with anisotropic phenomena, it is well known that wavelets do not provide optimally sparse representations. For these reasons, in the nineties, directional transforms were introduced and since then a huge amount of work has be done to solve efficiently anisotropic problems such as the detection of edges along anisotropic directions in two dimensional images. The literature on directional transforms includes many papers, especially for the two dimensional case (see e.g. [1] for directional wavelets, [3] for ridgelets, [5] for contourlets, [4] for curvelets, and the references therein).
In this panorama, a particular attention has to be given to the discrete shearlet transforms (see e.g. [7]) which have many interesting applications: both theoretical and practical. Thanks to their mathematical structure, they provide a MRA similar to those associated with classical wavelets. In the shearlets case, the classic (acceptable) dilation matrix M (tipically the dyadic matrix $2I$) is replaced by the product between an expanding anisotropic diagonal matrix

$$D_a = \begin{pmatrix} a^2 I_p & 0 \\ 0 & a I_{d-p} \end{pmatrix}, \quad a \in \mathbb{N},\ a \geq 2,\ p \leq \mathrm{d} \tag{1}$$

and a pseudo rotation matrix on $\mathbb{Z}^d$, the so called *shear* matrix

$$S_W = \begin{pmatrix} I_p & W \\ 0 & I_{d-p} \end{pmatrix}, \tag{2}$$

resulting in

$$M(W) = D_a S_W.$$

Shear matrices have interesting properties that make them attractive and efficient for the solution of applied problems. The main drawback is that the scaling matrix $M(W)$ has large determinant given by $\det(D_a) = a^{d+p}$ which takes minimum value for

$$D_2 = \begin{pmatrix} 4I_p & 0 \\ 0 & 2I_{d-p} \end{pmatrix}.$$

This leads to a quite substantial complexity in implementations since the complexity is directly related to the determinant of the scaling matrix.

This drawback was the starting point for studying the existence of different matrices with smaller determinant which, as shearlets, allow to write the scaling matrix as product between an anisotropic expanding diagonal matrix and a unimodular matrix. Our aim is to look for directional anisotropic expanding matrices $M = DA$, with properties equivalent to those of $M(W)$, but with smaller determinant, satisfying a pseudo-commuting property

$$DA = A^n D, \tag{3}$$

for some integer n.

§2. Notations and backgroud material

2.1. Preliminary definitions

A unimodular matrix $A \in \mathbb{Z}^{\mathrm{d}\times\mathrm{d}}$ is a matrix with $\det(A) = \pm 1$. A square matrix B is a periodic matrix if $B^{(\ell+1)} = B$ for ℓ a positive integer. If ℓ is the least such integer, then the matrix is said to have period ℓ or to be ℓ-periodic. The inverse of a unimodular matrix is a unimodular matrix. We say that a linear transformation Ξ on $\mathbb{R}^{\mathrm{d}}$ is an *a*cceptable dilation for $\mathbb{Z}^{\mathrm{d}}$ if it leaves $\mathbb{Z}^{\mathrm{d}}$ invariant, i.e., $\Xi\mathbb{Z}^{\mathrm{d}} \subset \mathbb{Z}^{\mathrm{d}}$, and all the eigenvalues of Ξ satisfy $|\lambda_i| > 1$. Equivalently, $\left\|\Xi^{-j}\right\| \to 0$ as $j \to \infty$ for some or any matrix norm. This implies in particular that, as j increases, $\Xi^{-j}\mathbb{Z}^{\mathrm{d}}$ tends to $\mathbb{R}^{\mathrm{d}}$.
An acceptable dilation Ξ is such that $|\det \Xi|$ is an integer ≥ 2. $\Xi \in \mathbb{Z}^{\mathrm{d}\times\mathrm{d}}$ is also called *expanding matrix* or *scaling matrix*. We say that an expanding matrix is anisotropic if at least two eigenvalues have different modulus.

2.2. Shear dilation matrices

Shearlet scaling matrices are given by the interaction between a diagonal expanding matrix with integer entries and a pseudo rotation matrix. The diagonal expanding matrix gives a particular anisotropic dilation called, in the shear literature, *parabolic scaling*, typically taking the form

$$D_2 = \begin{pmatrix} 4I_p & 0 \\ 0 & 2I_{d-p} \end{pmatrix}, \quad p \leq \mathrm{d}. \tag{4}$$

The pseudo rotation matrix on $\mathbb{R}^{\mathrm{d}}$, the so called *shear* matrix, is defined in block form as

$$S_W = \begin{pmatrix} I_p & W \\ 0 & I_{\mathrm{d}-p} \end{pmatrix}, \quad W \in \mathbb{R}^{p\times(\mathrm{d}-p)},\ p \leq \mathrm{d}. \tag{5}$$

Shear matrices on $\mathbb{Z}^d$ are unimodular matrices satisfying the following property (see [8])

$$S_{-W} = S_W^{-1} \tag{6}$$

More in general we have

$$S_{jW} = S_W^j \quad \text{and} \quad S_W S_{W'} = S_{W+W'}. \tag{7}$$

The scaling matrix has then the form

$$M(W) = D_2 S_W.$$

The unimodular matrix S_W and the parabolic scaling matrix D_2 obey to the following rule that we call *pseudo-commuting* property

$$D_2 S_W = S_W^2 D_2. \tag{8}$$

In fact

$$\begin{aligned} D_2 S_W &= \begin{pmatrix} 4I_p & 0 \\ 0 & 2I_{d-p} \end{pmatrix} \begin{pmatrix} I_p & W \\ 0 & I_{d-p} \end{pmatrix} = \begin{pmatrix} 4I_p & 4W \\ 0 & 2I_{d-p} \end{pmatrix} \\ &= \begin{pmatrix} I_p & 2W \\ 0 & I_{d-p} \end{pmatrix} \begin{pmatrix} 4I_p & 0 \\ 0 & 2I_{d-p} \end{pmatrix} \\ &= S_{2W} D_2 = S_W^2 D_2. \end{aligned}$$

§3. Pseudo-commuting property

The pseudo-commuting property (8) is crucial in subdivision schemes, as it allows to give explicit formulas for the iterated matrices and hence to know the sampling grid for the signal to be analysed. In fact, if we consider a stationary subdivision scheme with scaling matrix $M(W)$, relations and (7) and (8) give after j iterations

$$\begin{aligned} M(W)_j &= \prod_{i=1}^{j} (D_2 S_W) \\ &= D_2 S_W \cdot D_2 S_W \dots D_2 S_W \cdot D_2 S_W \cdot D_2 S_W \\ &= D_2 S_W \cdot D_2 S_W \dots D_2 S_W \cdot D_2 S_W S_{2W} D_2 \\ &= D_2 S_W \cdot D_2 S_W \dots D_2 S_W \cdot D_2 S_{3W} D_2 \\ &= D_2 S_W \cdot D_2 S_W \dots D_2 S_W \cdot S_{6W} D_2^2 \\ &\vdots \\ &= S_{\ell W} D_2^j = S_W^\ell D_2^j, \end{aligned} \tag{9}$$

with $\ell = \sum_{i=1}^{j} 2^i$. And thus $M(W)_j^{-1} = D_2^{-j} S_W^{-\ell}$ where $S_W^{-\ell}$ is a unimodular matrix. Then $M(W)_j^{-1}\mathbb{Z}^{\mathrm{d}} = D_2^{-j} S_W^{-\ell}\mathbb{Z}^{\mathrm{d}} = D_2^{-j}\mathbb{Z}^{\mathrm{d}}$. Hence we need to know the signal on the simpler grid $D_2^{-j}\mathbb{Z}^{\mathrm{d}}$.

From the previous arguments, the following question arises: given a diagonal anisotopic expanding matrix $D \in \mathbb{Z}^{\mathrm{d}\times\mathrm{d}}$, is it possible to find some $n \in \mathbb{N}$, and a unimodular matrix A, such that

$$DA = A^n D? \tag{10}$$

Or more in general is it possible to find $m, n \in \mathbb{N}$, such that

$$DA^m = A^n D? \tag{11}$$

§4. Matrices satisfying a pseudo commuting property

In this section we show that, given a diagonal expanding and anisotropic matrix D, it is always possible to determine a unimodular matrix A that pseudo-commutes with D.

First we consider in §4.1 and in §4.2 the cases $\mathrm{d} = 2$ and $\mathrm{d} = 3$ which have an immediate relevance in applications, and in §4.3 the case $\mathrm{d} > 3$.

4.1. case $\mathrm{d} = 2$

In what follows we need some results on the powers of unimodular matrices that are recalled in Appendix 5.1.

Theorem 1. *Let*

$$D = \begin{pmatrix} kr & 0 \\ 0 & k \end{pmatrix} \in \mathbb{Z}^{2\times 2}, \quad r \in \mathbb{Q}_+ \setminus \{1\}, \tag{12}$$

be an anisotropic diagonal expanding matrix. Let $A \in \mathbb{Z}^{2\times 2}$ be a non-diagonal unimodular matrix, and $DA = A^n D$ for some n. Then A has one of the following forms

- $A = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ *and necessarily $r = n$;*
- $A = \pm\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ *and necessarily $r = \frac{1}{n}$.*

If trace$(A) < 0$, n must be odd.

Proof. The assumption $DA = A^n D$ implies that $D^{-1}AD = D^n$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$DAD^{-1} = \begin{pmatrix} kr & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{kr} & 0 \\ 0 & \frac{1}{k} \end{pmatrix} = \begin{pmatrix} a & rb \\ \frac{c}{r} & d \end{pmatrix}.$$

Let $t := \mathrm{trace}(A)/2$. Being A unimodular (see Lemma 5), we get

$$A^n = U_{n-1}(t)A - U_{n-2}(t)I \quad \text{if} \quad \det(A) = 1, \tag{13}$$

$$A^n = P_{n-1}(t)A + P_{n-2}(t)I, \quad \text{if} \quad \det(A) = -1, \tag{14}$$

where U_n are the Chebyshev polynomials of second kind, and P_n are the Pell polynomials.
Let us consider the case $\det(A) = 1$. Relation (13) gives

$$A^n = \begin{pmatrix} aU_{n-1}(t) - U_{n-2}(t) & bU_{n-1}(t) \\ cU_{n-1}(t) & dU_{n-1}(t) - U_{n-2}(t) \end{pmatrix}. \tag{15}$$

The corresponding elements of A^n and of DAD^{-1} have to be equal. For the non diagonal entries, we must impose

$$bU_{n-1}(t) = br, \quad cU_{n-1}(t) = c/r. \tag{16}$$

The two relation are both satisfied in the trivial case $b = c = 0$. In the non trivial case, the assumption on D, forces $b = 0$ or $c = 0$, and being A unimodular, we have $a = d = 1$ and $t = 1$, or $a = d = -1$ and $t = -1$. When $t = 1$, $U_{n-1}(1) = n$ (see Lemma 6), then $r = n$, otherwise if $t = -1$ we obtain $U_{n-1}(-1) = (-1)^{n-1}n = r$.
Let us consider the diagonal elements, we must impose

$$aU_{n-1}(t) - U_{n-2}(t) = a, \quad dU_{n-1}(t) - U_{n-2}(t) = d. \tag{17}$$

Summing the two equations we get $2tU_{n-1}(t) - 2U_{n-2}(t) = 2t$ that, using the recurrence relation (38), becomes

$$U_n(t) - U_{n-2}(t) = 2t. \tag{18}$$

If $t = 1$, (18) is is always verified. If $t = -1$, condition (18) becomes $(-1)^n(n+1)-(-1)^{n-2}(n-1) = -2$, and it is satisfied when n is odd. In this case, $U_{n-1}(-1) = (-1)^{n-1}n$ gives again $n = r$, and all admitted matrices are thus $A = \begin{pmatrix} \pm 1 & w \\ 0 & \pm 1 \end{pmatrix}$, for any $\omega \in \mathbb{Z}$. We now consider $b = 0$ and $c \neq 0$. As before, $t = 1$ or $t = -1$. In this case, (16) forces $U_{n-1}(t) = n = 1/r$ and all admitted matrices are thus $A = \begin{pmatrix} \pm 1 & 0 \\ w & \pm 1 \end{pmatrix}$, for any $\omega \in \mathbb{Z}$.
All such matrices are, up to a sign, shear matrices, verifying $DA = A^nD$ (only for odd values of n in case the $t = -1$).
The case $\det(A) = -1$ uses the same arguments but with the Pell's polynomials P_i (see relation (14)) and it is easy to show that it provides only trivial cases. □

We observe that the exponent n depends on the anisotropicity ratio of the elements of D, and that among the matrices D of the form (12), we find the parabolic dilation matrix D_2 ($r = n = 2$ and $k = 2$) which is also the matrix of this class with the smallest determinant.
As a final note, if A verifies $DA = A^nD$, then A^T undergoes the dual relation

$$A^TD = D(A^T)^n.$$

Thus a dual version of Theorem 1 can be given by using the dual relation

$$DA^n = AD.$$

We turn now to the more general relation $DA^m = A^nD$, $m > 1$.

Theorem 2. *Let*

$$D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \mathbb{Z}^{2\times 2} \tag{19}$$

be an anisotropic expanding matrix and $A \in \mathbb{Z}^{2\times 2}$ *be a non-diagonal unimodular matrix. The identity*

$$DA^m = A^n D, \tag{20}$$

is satisfied if

(1) $A^q = I$, *and* m, n *such that* $m = \ell_1 q$, $n = \ell_2 q$, *for some* $\ell_1, \ell_2 \in \mathbb{Z}$;

(2) $D = \begin{pmatrix} r\beta & 0 \\ 0 & \beta \end{pmatrix}$, m, n *such that* $r = \dfrac{n}{m}$, *and* $A = \pm \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$;

(3) $D = \begin{pmatrix} r\beta & 0 \\ 0 & \beta \end{pmatrix}$, m, n *such that* $r = \dfrac{m}{n}$, *and* $A = \pm \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}^T$.

In the two last cases, $\mathrm{trace}(A) < 0$ *implies* $n - m \in 2\mathbb{Z}$.

Proof. Case (1) is trivially satisfied and implies that $m = n + \ell q$ for some $\ell \in \mathbb{Z}$. The assumption $DA^m = A^n D$ implies that $D^{-1}A^m D = A^n$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad b \neq 0 \,\text{and/or}\, c \neq 0.$$

From Lemma 5, we know that either

$$A^n = \begin{pmatrix} aU_{n-1}(t) - U_{n-2}(t) & bU_{n-1}(t) \\ cU_{n-1}(t) & dU_{n-1}(t) - U_{n-2}(t) \end{pmatrix}$$

or

$$A^n = \begin{pmatrix} aP_{n-1}(t) + P_{n-2}(t) & bP_{n-1}(t) \\ cP_{n-1}(t) & dP_{n-1}(t) + P_{n-2}(t) \end{pmatrix},$$

depending on the sign of $\det(A)$, where $t = \mathrm{trace}(A)/2$, U_n are the Chebyshev polynomials of second type and P_n are the Pell polynomials.
If $\det(A) = 1$,

$$DA^m D^{-1} = \begin{pmatrix} aU_{m-1}(t) - U_{m-2}(t) & b\frac{\alpha}{\beta}U_{m-1}(t) \\ c\frac{\beta}{\alpha}U_{m-1}(t) & dU_{m-1}(t) - U_{m-2}(t) \end{pmatrix}.$$

The diagonal and the off–diagonal elements of A have to satisfy

$$aU_{m-1}(t) - U_{m-2}(t) = aU_{n-1}(t) - U_{n-2}(t), \quad dU_{m-1}(t) - U_{m-2}(t) = dU_{n-1}(t) - U_{n-2}(t), \tag{21}$$

$$bU_{n-1}(t) = b\frac{\alpha}{\beta}U_{m-1}(t), \quad cU_{n-1}(t) = c\frac{\beta}{\alpha}U_{m-1}(t). \tag{22}$$

First, let us assume that $U_{m-1}(t) = 0$, which implies that either $t = 0$, i.e., $d = -a$, and $m \in 2\mathbb{N}$ or $t = \pm\frac{1}{2}$ and $m \in 3\mathbb{N}$.

In the first case, the eigenvalues of A are $\pm i$, hence $A^2 = -I$ and $A^4 = I$, i.e. A is 4−periodic. In addition, $U_{n-1}(0) = 0$, and the equality of the diagonal elements yield $U_{n-2}(0) = U_{m-2}(0)(\neq 0)$. By Lemma 6, $n = m + 4\ell$ for some $\ell \in \mathbb{Z}$.
If $m \in 3\mathbb{N}$ and $t = 1/2$, the characteristic polynomial $\chi(A)$ of A, satisfies

$$0 = \chi_A(A) = A^2 - 2tA + \det(A)\, I = A^2 - A + I, \tag{23}$$

hence $A^2 = A - I$, $A^3 = A^2 - A = -I$ and therefore $A^6 = I$, that is A is 6−periodic. Now we have, $U_{n-1}(\frac{1}{2}) = 0$, and $U_{n-2}(\frac{1}{2}) = U_{m-2}(\frac{1}{2})(\neq 0)$. By Lemma 6 we get $n = m + 6\ell$ for some $\ell \in \mathbb{Z}$. In the same way, $m \in 3\mathbb{N}$ and $t = -1/2$ yield $A^3 = I$ and $n = m + 3\ell$ for some $\ell \in \mathbb{Z}$.
Then if A is such that $U_{m-1}(t) = 0$, A is a periodic matrix of order q ($q = 3, 4, 6$) and m, n such that $n = m + q\ell$ for some $\ell \in \mathbb{Z}$, (case (1)).
Next, let us assume that $U_{m-1}(t) \neq 0$. If $bc \neq 0$, (22) yields

$$\frac{U_{n-1}(t)}{U_{m-1}(t)} = \frac{\alpha}{\beta} \quad \text{and} \quad \frac{U_{n-1}(t)}{U_{m-1}(t)} = \frac{\beta}{\alpha}$$

contradicting the assumption D is anisotropic. Hence either $b = 0$ or $c = 0$ and then $t \in \{-1, 1\}$. Summing the identities (21) for the diagonal elements into

$$2tU_{n-1}(t) - 2U_{n-2}(t) = 2tU_{m-1}(t) - 2U_{m-2}(t)$$

and using (38), we get

$$U_n(t) - U_{n-2}(t) = U_m(t) - U_{m-2}(t), \tag{24}$$

which is always satisfied if $t = 1$. In the case $t = -1$, Lemma 6 implies that $n = m + 2\ell$ for some $\ell \in \mathbb{Z}$. If $b \neq 0$ and $c = 0$, $A = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ for some $w \in \mathbb{Z}$, and (22) gives

$$\frac{U_{n-1}(t)}{U_{m-1}(t)} = \frac{n}{m}.$$

then α, β need to be such that the ratio $r := \frac{\alpha}{\beta} = \frac{n}{m}$ (case (2)) that means

$$D = \begin{pmatrix} r\beta & 0 \\ 0 & \beta \end{pmatrix}.$$

While if $b = 0$ and $c \neq 0$, $A = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}^T$, and the previous arguments hold with $r := \frac{\alpha}{\beta} = \frac{m}{n}$, (case (3)).
If $\det(A) = -1$, we perform identical computations with the Pell polynomials, where the situation is simpler since Pell polynomials have real zeros only for odd orders and $t = 0$. Going on as in the case $\det(A) = 1$, the identities of the diagonal elements give

$$P_n(t) + P_{n-2}(t) = P_m(t) + P_{m-2}(t). \tag{25}$$

If we assume that $P_{m-1}(t) = 0$, m must be even, leading to $A^2 = I$ (see (23)) and (25) forces n even. If $P_{m-1}(t) \neq 0$, again $bc \neq 0$ contradicts the assumption on D. As before $b = 0$ or

$c = 0$, also in this case $t = 0$ and then m odd. The identity (25) (see Lemma 6) is satisfied only when also n is odd. But now the ratio $P_{n-1}(0)/P_{m-1}(0)$ is equal to one and then $\alpha = \beta$, which contradicts the assumption on D.
We conclude that in the case $\det(A) = -1$, the identity (20) holds when A is periodic of order two and m, n even. □

4.1.1. Example

For $D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ we have $r = \frac{3}{2}$, $m = 2$, $n = 3$ and

$$DA^2 = A^3 D$$

where $A = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$.

Remark 1. The unimodular matrices $A \in \mathbb{Z}^{\mathrm{d}\times\mathrm{d}}$ that can pseudo-commute with D are only (up to a sign) the shear matrices when $m = 1$ and shear (up to a sign) or periodic matrices when $m > 1$.

4.2. The case $\mathrm{d} = 3$

We want to see what happens when $\mathrm{d} = 3$. In this case we focus on $m = 1$ and on matrices D of the form

$$D = \begin{pmatrix} krs & 0 & 0 \\ 0 & kr & 0 \\ 0 & 0 & k \end{pmatrix}, \quad r, s \in \mathbb{Q}_+, \ (r, p) \neq (1, 1). \tag{26}$$

We start by considering unimodular matrices A of the form

$$\left(\begin{array}{c|c} B & \mathbf{v} \\ \hline \mathbf{0} & \lambda \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} \lambda & \mathbf{v}^T \\ \hline \mathbf{0} & B \end{array}\right) \tag{27}$$

where B is a 2×2 matrix, $\mathbf{v}$ is a column vector of dimension 2.
We observe that $\det(A) = \lambda \det(B)$, A unimodular implies B unimodular and $|\lambda| = 1$. In what follows we consider, for the sake of brevity,

$$\left(\begin{array}{c|c} B & \mathbf{v} \\ \hline \mathbf{0} & \lambda \end{array}\right), \tag{28}$$

and it easy to show that the results of Proposition 3 hold also for

$$\left(\begin{array}{c|c} \lambda & \mathbf{v}^T \\ \hline \mathbf{0} & B \end{array}\right),$$

via an exchange of the roles of s and r. The block form of A allows to write explicitly the nth power of A.

$$A^n = \left(\begin{array}{c|c} B^n & \left(\sum_{i=0}^{n-1} (\lambda)^{n-1-i} B^i\right) \mathbf{v} \\ \hline 0 & (\lambda)^n \end{array}\right) \tag{29}$$

Let us write

$$D = \left(\begin{array}{c|c} \overline{D} & 0 \\ \hline 0 & k \end{array}\right), \tag{30}$$

where $\overline{D} = \begin{pmatrix} krs & 0 \\ 0 & kr \end{pmatrix}$. The identity $DA = A^n D$ is equivalent to $DAD^{-1} = A^n$, and

$$DAD^{-1} = \left(\begin{array}{c|c} \overline{D}B\overline{D}^{-1} & \overline{D}\mathbf{v}/k \\ \hline 0 & \lambda \end{array}\right). \tag{31}$$

If $s = 1$, D takes the form

$$D = \left(\begin{array}{c|c} krI_2 & 0 \\ \hline 0 & k \end{array}\right) \tag{32}$$

and

$$DAD^{-1} = \left(\begin{array}{c|c} B & r\mathbf{v} \\ \hline 0 & \lambda \end{array}\right). \tag{33}$$

In the following proposition, we characterize a matrix A of the form (28) satisfying $DA = A^n D$.

Proposition 3. *Let D of the form (26), and A as (28). The pseudo-commuting property $DA = A^n D$ holds when $A = \left(\begin{array}{c|c} B & \mathbf{v} \\ \hline \mathbf{0} & \lambda \end{array}\right)$ has one of the following forms*

(i) $\mathbf{v} = \mathbf{0}$, $\lambda = \pm 1$, *B a shear matrix (up to a sign), $s = n$ or B $n-$periodic and $s = 1$ (if $\lambda = -1$, n is odd);*

(2i) *for some* $\mathbf{v} \neq \mathbf{0}$

(2i.1) $\lambda = 1$, $B = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, *$s = n$, and $r = 1$ or $r = n$;*

(2i.1.1) $\lambda = 1$, $B = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}^T$, *$s = \frac{1}{n}$, and $r = 1$ or $r = \frac{1}{n}$;*

(2i.2) $\lambda = -1$, $B = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, *$s = n$(odd), and $r = 1$ or $r = \frac{1}{n}$;*

(2i.2.1) $\lambda = -1$, $B = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}^T$, *$s = \frac{1}{n}$ (n odd), and $r = 1$ or $r = n$;*

(2i.3) $\lambda = 1$, *B a $n-$periodic matrix such that* $B\mathbf{v} = \mathbf{v}$, $s = 1$, *and $r = n$.*

Proof. To get the proof, we need to equate the blocks in (29) and(31). In all cases $\overline{D}B\overline{D}^{-1} = B^n$, is verified (see Theorem 1) when $s = n$ and $B = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, $s = \frac{1}{n}$ and $B = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}^T$ or when $\overline{D} = krI_2$, which means $s = 1$ and $B = B^n$, i.e B a $n-$ periodic unimodular matrix. If $\mathbf{v} = \mathbf{0}$, the conditions (*i*) are trivially satisfied. If $\mathbf{v} \neq \mathbf{0}$, let us consider the case $\lambda = 1$, $s = n$, $B = \pm\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. For instance $B = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, we exploit the property (7) and the equality

$\sum_{i=0}^{n-1} B^i \mathbf{v} = \overline{D}\mathbf{v}/k$ becomes

$$C\mathbf{v} := \begin{pmatrix} rn & 0 \\ 0 & r \end{pmatrix}^{-1} \begin{pmatrix} n & \frac{n(n-1)}{2}w \\ 0 & n \end{pmatrix} \mathbf{v} = \mathbf{v}, \tag{34}$$

that is satisfied when $\mathbf{v}$ eigenvector corresponding to the eigenvalue 1 of $C = \begin{pmatrix} \frac{1}{r} & \frac{(n-1)}{2r}w \\ 0 & \frac{1}{r} \end{pmatrix}$ that occurs when $r = 1$ or $r = n$. The case $s = 1/n$ is left to the reader.

Now we consider $\lambda = -1$, n has to be odd and, from (7), $\sum_{i=0}^{n-1}(-1)^i B^i = \begin{pmatrix} 1 & (n-1)w \\ 0 & 1 \end{pmatrix}$. The matrix C is now $\begin{pmatrix} \frac{1}{rn} & (n-1)w \\ 0 & \frac{1}{r} \end{pmatrix}$, and the equality $\sum_{i=0}^{n-1} B^i \mathbf{v} = \overline{D}\mathbf{v}/k$ is satisfied by choosing $r = 1$ or $r = 1/n$ and $\mathbf{v}$ eigenvector of C corresponding to the eigenvalues 1 of C. The case $s = 1/n$ is left to the reader. Let us conclude with the last case $s = 1$, B $n-$periodic such that $B\mathbf{v} = \mathbf{v}$. Then

$$B^i\,\mathbf{c} = B^i(B\mathbf{v}) = B^{i+1}\mathbf{v} = B^{i+1}(B\,\mathbf{v}) = B^{i+2}\mathbf{v} = \cdots = B^{n-1}\mathbf{v} = \mathbf{v}.$$

If $\lambda = 1$, $\sum_{i=0}^{n-1} \mathbf{v} = n\mathbf{v}$ and $\overline{D}\mathbf{v}/k = r\mathbf{v}$, then $r = n$. The case $\lambda = -1$ leads to $r = 1$ which contradicts the assumption D anisotropic. □

4.2.1. Example

As an example, consider $k = 2$, $r = 3$, $s = 1$, $n = 3$, and

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 \\ -8 & -5 \end{pmatrix},$$

with $\det(B) = -1$, $B^2 = I$, $\mathbf{v} = (6 - 8)^T$, then

$$A = \begin{pmatrix} 5 & 3 & 6 \\ -8 & -5 & -8 \\ 0 & 0 & 1 \end{pmatrix}$$

is such that $DA = A^3 D$.

D of the form (26) and A unimodular with at least one zero non diagonal element pseudo-commute (see [6]) when A takes the form (i) or ($2i$). Moreover, if A is a generic matrix and $s = 1$ (D of the form (32)), again A and D pseudo-commute if A is of the form (i) or ($2i$). We conjecture that in general for any D as in (26), the pseudo-commuting property is satisfied only when A assumes one of the forms allowed by Proposition 3.

4.3. The case $\mathrm{d} > 3$

Proposition 4. *Let* $\mathrm{d} > 3$ *and consider*

$$D := \left(\begin{array}{c|c} krI_p & 0 \\ \hline 0 & kI_{\mathrm{d}-p} \end{array}\right), \tag{35}$$

and let

$$A = \left(\begin{array}{c|c} B & W \\ \hline 0 & I_{\mathrm{d}-p} \end{array}\right), \tag{36}$$

where $B \in \mathbb{Z}^{p\times p}$ is a unimodular n–periodic matrix, $W \in \mathbb{Z}^{p\times(\mathrm{d}-p)}$, and $B^{n-1}\,W = W$. Then the relation $DA = A^n D$ holds if $r = n$.

Proof. We have that

$$DAD^{-1} = \left(\begin{array}{c|c} B & rW \\ \hline 0 & I_{\mathrm{d}-p} \end{array}\right), \quad A^n = \left(\begin{array}{c|c} B^n & (\sum_{j=0}^{n-1} B^j)W \\ \hline 0 & I_{\mathrm{d}-p} \end{array}\right) \tag{37}$$

From the assumptions on B

$$B^i\,W = B^i(B\,W) = B^{i+1}W = B^{i+1}(B\,W) = B^{i+2}W = \cdots = B^{n-1}W = W.$$

Then

$$(\sum_{i=0}^{n-1} B^i)W = nW,$$

and

$$A^n = \left(\begin{array}{c|c} B^n & nW \\ \hline 0 & I_{\mathrm{d}-p} \end{array}\right).$$

Taking $r = n$, the relation $AD = DA^n$ is satisfied. □

Therefore we have a simple mechanism to obtain non-shear matrices A satisfying $DA = A^n D$ for some diagonal matrix (35): take r so that $r = n$, B such that $B^n = B$, with $\lambda = 1$ as eigenvalue with $\mathrm{d} - p$ linear independent eigenvectors and juxtapose those as columns of W. In this way $B\,W = W$ and we can build A via (36).
Note that choosing $B = I_p$ we get a shear matrix.

§5. Appendix

5.1. Powers of unimodular matrices

The main tools for powers of unimodular matrices are Chebyshev polynomials of second kind $U_i(t)$, and Pell polynomials $P_i(t)$. The first ones are obtained by the recurrence relation

$$U_i(t) = 2tU_{i-1}(t) - U_{i-2}(t), \qquad i \geq 2, \tag{38}$$

initialized with $U_0(t) = 1$, $U_1(t) = 2t$, the second ones by

$$P_i(t) = 2tP_{i-1}(t) + P_{i-2}(t), \qquad i \geq 2, \tag{39}$$

with $P_i = U_i$, $i = 0, 1$.
The relation between matrices and polynomials is established by the following lemma, whose first statement is cited in [2] and which is easily verified by induction.

Lemma 5. *If* $A \in \mathbb{Z}^{2\times 2}$ *satisfies* $\det(A) = 1$ *then* $A^n = U_{n-1}(t)A - U_{n-2}(t)I$*, if* $\det(A) = -1$ *then* $A^n = P_{n-1}(t)A + P_{n-2}(t)I$*, where* $t := \operatorname{trace}(A)/2$.

Some more properties of the polynomials, which are also easily verified by induction (see [6]), are listed in the following lemma.

Lemma 6.

1. $U_n(\pm 1) = (\pm 1)^n(n+1)$.
2. $U_n(0) = \begin{cases} 0 & \text{if } n = 2m+1 \\ 1 & \text{if } n = 4m \\ -1 & \text{if } n = 4m+2 \end{cases}$
3. $U_n(1/2) = \begin{cases} 1 & \text{if } n = 6m \text{ or } n = 6m+1 \\ 0 & \text{if } n = 6m+2 \text{ or } n = 6m+5 \\ -1 & \text{if } n = 6m+3 \text{ or } n = 6m+4 \end{cases}$
4. $U_n(-1/2) = \begin{cases} 1 & \text{if } n = 3m \\ -1 & \text{if } n = 3m+1 \\ 0 & \text{if } n = 3m+2 \end{cases}$
5. $P_n(0) = \begin{cases} 1 & \text{if } n = 2m \\ 0 & \text{if } n = 2m+1 \end{cases}$

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