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METHOD FOR SOLVING A STOCHASTIC CONSERVATION LAW

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Abstract. This paper presents techniques introduced in a joint work with G. Vallet and P. Wittbold for solving the Cauchy problem for a multi-dimensional nonlinear conservation law with stochastic perturbation [2]. We propose to advance main difficulties in the use of deterministic tools for studying stochastic P.D.E., and alternative methods.

Keywords: Stochastic PDE, first-order hyperbolic equation, Cauchy problem, multiplicative stochastic perturbation, Young measures, Kruzhkov's entropy.

AMS classification: 60H15, 35R60, 35L60.

§1. Introduction

We are interested in the formal stochastic nonlinear conservation law of type:

$$\mathrm{du} - \mathrm{div}(\mathbf{f}(u))\mathrm{dt} = h(u)\mathrm{dw} \quad \text{in } \Omega \times \mathbb{R}^d \times]0, T[, \tag{1.1}$$

with an initial condition u_0 and $d \geq 1$.

In the sequel we assume that T is a positive number, $Q =]0, T[\times\mathbb{R}^d$ and that $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on the classical Wiener space $(\Omega, \mathcal{F}, P)$. These assumptions on W are made for convenience. Let us assume that

H_1: $\mathbf{f} = (f_1, .., f_d) : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz-continuous function and $\mathbf{f}(0) = \mathbf{0}$.

H_2: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$.

H_3: $u_0 \in L^2(\mathbb{R}^d)$.

We propose to present tools for showing existence and uniqueness result of the stochastic entropy solution to the above-mentioned problem. Our aim is to adapt the known methods for first-order nonlinear P.D.E. to noise perturbed ones.

Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. One needs to introduce the notion of entropy solution in order to discriminate the "physical solution".

Only few papers have been devoted to the study of multiplicative stochastic perturbation of nonlinear first-order hyperbolic problems in the $\mathbb{R}^d$ case. Let us mention, without exhaustiveness, the work of J. Feng and D. Nualart [7] where they introduced a notion of strong entropy solution in order to prove the uniqueness of the entropy solution for the Cauchy problem:

$$\mathrm{du} + \mathrm{div}(\mathbf{f}(u))\mathrm{dt} = \int_{z \in Z} \sigma(., u, z)\mathrm{dw}(t, z).$$

Using the vanishing viscosity and compensated compactness arguments, the authors established existence of strong entropy solutions only in the $1D$ case.
In the recent paper [3], G.-Q. Chen, Q. Ding and K. H. Karlsen propose to revisit the work of J. Feng and D. Nualart. They prove that the multidimensional stochastic problem is well-posed by using uniform spatial BV-bound. They show the existence of strong stochastic entropy solutions in $L^p \cap BV$ and develop a "continuous dependence" theory for stochastic entropy solutions in BV.
Finally, let us mention the paper by A. Debussche and J. Vovelle [5] concerning the d-dimensional problem with multiplicative noise

$$\mathrm{du} + f(u)_x \mathrm{dt} = h(u)\mathrm{dw},$$

which is considered on a torus. The authors use the kinetic formulation of the problem and prove existence and uniqueness of a kinetic solution.
The aim of C. Bauzet, G. Vallet and P. Wittbold in [2] is to complete those results by showing existence and uniqueness of solution in the $\mathbb{R}^d$ case under weaker assumptions on the data, with a "Hilbert space" approach. The authors propose a method of artificial viscosity to prove the existence of a solution. The compactness properties used are based on the theory of Young measures and on measure-valued solutions. Then, an appropriate adaptation of Kruzhkov's doubling variables technique, and of the way J. Feng and D. Nualart propose to treat the stochastic source term, is presented to prove that any stochastic entropy solution is equal to a solution given by the artificial viscosity method. Thus, the entropy inequalities seem to suffice for the uniqueness *via* Kato-type inequality. This yields the uniqueness of the measure-valued entropy solution, and, by standard arguments, this allows to deduce existence and uniqueness of the stochastic weak entropy solution.
We propose in this paper to present difficulties (brought by the stochastic perturbation) met by the authors in the use of classical tools from the deterministic setting, and techniques developed to treat the stochastic terms in [2].

First of all, we need to introduce some notations and make precise the functional setting.

- Denote by E the integral over Ω with respect to the probability measure P.
- For a given separable Banach space X we denote by $N^2_w(0,T,X)$ the space of the predictable X-valued processes (cf. [4]). This space is $L^2(]0,T[\times\Omega,X)$ endowed with the product measure $dt \otimes dP$ and the predictable σ-field $\mathcal{P}_T$(*i.e.* the σ-field generated by the sets $\{0\}\times\mathcal{F}_0$ and the rectangles $]s,t]\times A$ for any $A \in \mathcal{F}_s$).
- $\mathcal{E} = \{\eta \in C^{2,1}(\mathbb{R}),\ \eta \geq 0,\ \text{convex}, \eta(0) = 0,\ \text{supp}\ \eta'' \text{ compact}\}$, the set of smooth entropies.
- $\eta_\delta \in \mathcal{E}$ denotes a uniform approximation of the absolute value function

$$\eta'_\delta(r) = \begin{cases} 1 & , \quad r \geq \delta \\ \sin\left(\frac{\pi}{2\delta}r\right) & , \quad -\delta < r < \delta \\ -1 & , \quad r \leq -\delta. \end{cases}$$

- $\forall \eta \in \mathcal{E},\ F^\eta(a,b) = \displaystyle\int_a^b \eta'(\sigma - a)\mathbf{f}'(\sigma)\mathrm{d}\sigma.$

- $F(a,b) = Sgn_0(a-b)[\mathbf{f}(a)-\mathbf{f}(b)] = \lim_{\delta\to 0^+} F^{\eta_\delta}(a,b)$ denotes the entropy flux.
- $\forall u \in \mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d))$, $\forall k \in \mathbb{R}$, $\forall \eta \in \mathcal{E}$ and $\forall \varphi \in \mathcal{D}(\mathbb{R}^{d+1})$

$$\begin{aligned}\mu_{u,\eta,k}(\varphi) &= \int_{\mathbb{R}^d} \eta(u_0-k)\varphi(0)\mathrm{dx} + \int_Q \eta(u-k)\partial_t\varphi - F^\eta(u,k)\nabla\varphi\mathrm{dxdt} \\ &\quad + \int_Q \eta'(u-k)h(u)\varphi\mathrm{dxdw}(t) + \frac{1}{2}\int_Q h^2(u)\eta''(u-k)\varphi\mathrm{dxdt}.\end{aligned}$$

Definition 1. A function u of $\mathcal{N}_w^2(0,T,L^2(\mathbb{R}^d))$ is an entropy solution of the stochastic conservation law (1.1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$ if $u \in L^\infty(0,T,L^2(\Omega,L^2(\mathbb{R}^d)))$ and, for any $\varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$, any real k and any $\eta \in \mathcal{E}$

$$0 \le \mu_{u,\eta,k}(\varphi) \qquad P-\text{a.s.}$$

And, $ess \lim_{t\to 0^+} E\int_K |u(t,x)-u_0|\mathrm{dx} = 0$ for any compact set $K \subset \mathbb{R}^d$.

Remark 1. The condition on the initial data comes from the regularity of the solution $u \in L^\infty(0,T,L^2(\Omega,L^2(\mathbb{R}^d))$, we can follow the idea of F. Otto [8], as here the random variable doesn't bring new difficulty.

§2. The parabolic case

The following existence and uniqueness result is a classic one. One can refer to [4] and many others authors.

Proposition 1. *For any positive ϵ, there exists a unique $u_\epsilon \in \mathcal{N}_w^2(0,T;H^1(\mathbb{R}^d))$ such that u_ϵ is a weak solution of the stochastic nonlinear parabolic problem*

$$du_\epsilon - [\epsilon\Delta u_\epsilon + div(\mathbf{f}(u_\epsilon))]\mathrm{dt} = h(u_\epsilon)\mathrm{dw} \quad in\ \Omega\times\mathbb{R}^d\times]0,T[, \tag{2.1}$$

with $u_\epsilon \in L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))$, $\partial_t[u_\epsilon - \int_0^\cdot h(u_\epsilon)\mathrm{dw}]$ and Δu_ϵ in $L^2(\Omega\times Q)$ and for the initial condition $u_0^\epsilon \in \mathcal{D}(\mathbb{R}^d)$.
Moreover, there exists a positive constant C such that,

$$\forall\epsilon > 0, \quad \|u_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))} + \epsilon\|u_\epsilon\|^2_{L^2(]0,T[\times\Omega;H_0^1(\mathbb{R}^d))} \le C.$$

Remark 2. We consider here $(u_0^\epsilon)_\epsilon$ a sequence approximating our initial condition u_0 in $L^2(\mathbb{R}^d)$. The regularities $\partial_t[u_\epsilon - \int_0^\cdot h(u_\epsilon)\mathrm{dw}]$ and Δu_ϵ in $L^2(\Omega\times Q)$ are not obvious, they come from the suitable choice of $u_0^\epsilon \in \mathcal{D}(\mathbb{R}^d)$. One refers to the work of G. Vallet [10].

Consider φ in $\mathcal{D}^+(\bar{Q})$, k a real number and $\eta \in \mathcal{E}$. Since $\eta(u_\epsilon - k)\varphi \in L^2(0,T,H^1(\mathbb{R}^d))$ a.s., it is possible to apply the Itô formula to the operator $\Psi(t,u_\epsilon) := \int_{\mathbb{R}^d}\eta(u_\epsilon - k)\varphi\mathrm{dx}$ and thus, P-a.s.:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \eta(u_\epsilon(T)-k)\varphi(T)\mathrm{dx} \\
= \quad & \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\mathrm{dx} + \int_Q \eta(u_\epsilon - k)\partial_t\varphi \mathrm{dxdt} \\
& -\epsilon \int_Q \eta''(u_\epsilon - k)\varphi \nabla u_\epsilon \nabla u_\epsilon \mathrm{dxdt} - \epsilon \int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi \mathrm{dxdt} \\
& - \int_Q \eta'(u_\epsilon - k)\mathbf{f}(u_\epsilon)\nabla\varphi \mathrm{dxdt} - \int_Q \eta''(u_\epsilon - k)\varphi \mathbf{f}(u_\epsilon)\nabla u_\epsilon \mathrm{dxdt} \\
& + \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi \mathrm{dxdw}(t) + \frac{1}{2}\int_Q h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi \mathrm{dxdt}.
\end{aligned}$$

Remark 3. Let us mention that in the deterministic setting, to propose a "viscous" entropy formulation, we test the parabolic regularization in $\tilde{u}_\epsilon$ with $\eta(\tilde{u}_\epsilon - k)\varphi$. In the stochastic case, testing the parabolic regularization with $\eta(u_\epsilon - k)\varphi$ is the same thing as applying Itô's derivation formula with $\Psi(t, u_\epsilon) := \int_{\mathbb{R}^d} \eta(u_\epsilon - k)\varphi dx$. Notice that the stochastic perturbation brings two new terms in this derivation formula: one containing an Itô integral, and another one containing the second-order derivative of η.

Since the support of η'' is compact, for any $i = 1, \ldots, d$, $\mathbb{R} \ni r \mapsto \eta''(r-k)f_i(r)$ is a bounded continuous function. Then, thanks to the chain-rule for Sobolev functions, we obtain the following viscous entropic formulation for any dP-measurable set A

$$\begin{aligned}
0 \quad \leq \quad & E\left[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_\epsilon - k)h(u_\epsilon)\varphi \mathrm{dxdw}(t)\right] \\
& -\epsilon E\left[1_A \int_Q \eta'(u_\epsilon - k)\nabla u_\epsilon \nabla\varphi \mathrm{dxdt}\right] \\
& +E\left[1_A \int_{\mathbb{R}^d} \eta(u_0^\epsilon - k)\varphi(0)\mathrm{dx}\right] \qquad (2.2) \\
& +E\left[1_A \int_Q \eta(u_\epsilon - k)\partial_t\varphi - F^\eta(u_\epsilon, k)\nabla\varphi + \frac{1}{2}h^2(u_\epsilon)\eta''(u_\epsilon - k)\varphi \mathrm{dxdt}\right] \\
& := E[1_A \mu^\epsilon_{u_\epsilon,\eta,k}(\varphi)].
\end{aligned}$$

§3. Entropic formulation

We would like to pass to the limit in (2.2) with respect to ϵ. Because of the random variable, we are not able to use classical results of compactness. But the one given by the concept of Young measure is appropriate here, and the technique is based on the notion of narrow convergence of Young measures (or entropy processes), we refer to E.J. Balder [1] but also to R. Eymard, T. Gallouët and R. Herbin [6].

Since u_ϵ is a bounded sequence in $\mathcal{N}_w^2(0, T, L^2(\mathbb{R}^d))$ and thanks to the compact support of φ in $\mathbb{R}^d$, the associated Young measure sequence $\mathbf{u}_\epsilon$ converges (up to a subsequence still indexed in the same way) to an "entropy process" denoted by $\mathbf{u} \in L^\infty(0, T, L^2(\Omega\times\mathbb{R}^d\times]0, 1[))$. Precisely, given a Carathéodory function $\Psi(t,x,\omega,\lambda) : Q\times\Omega\times\mathbb{R}\to\mathbb{R}$ such that $\Psi(.,\mathbf{u}_\epsilon)$ is uniformly integrable, one has:

$$E\int_Q \psi(.,\mathbf{u}_\epsilon)dxdt \underset{\epsilon\to 0}{\longrightarrow} E\int_Q\int_0^1 \psi(.,\mathbf{u}(.,\alpha))d\alpha dxdt.$$

By assumptions on η, all the integrands in the third line of (2.2) are uniformly integrable and passing to the limit is possible in all the integrals. One is also able to pass to the limit in the first term of (2.2) using the weak continuity of the stochastic integral from $L^2(\Omega\times Q)$ to $L^2(\Omega\times\mathbb{R}^d)$, see [4]. Finally, the *a priori* estimate on ∇u_ϵ yields that the second term of (2.2) tends to 0 with ϵ.
Therefore at the limit one gets

$$\begin{aligned} 0 \quad\leq\quad & E\left[1_A\int_0^T\int_{\mathbb{R}^d}\int_0^1 \eta'(\mathbf{u}(.,\alpha)-k)h(\mathbf{u}(.,\alpha))\varphi \mathrm{d}\alpha\mathrm{dx}\mathrm{dw}(t)\right]\\ & +E\left[1_A\int_{\mathbb{R}^d}\eta(u_0-k)\varphi(0)\mathrm{dx}\right]\\ & +E\left[1_A\int_Q\int_0^1 \left[\eta(\mathbf{u}(.,\alpha)-k)\partial_t\varphi - F^\eta(\mathbf{u}(.,\alpha),k)\nabla\varphi\right]\mathrm{d}\alpha\mathrm{dxdt}\right]\\ & +\frac{1}{2}E\left[1_A\int_Q\int_0^1 h^2(\mathbf{u}(.,\alpha))\eta''(\mathbf{u}(.,\alpha)-k)\varphi\mathrm{d}\alpha\mathrm{dxdt}\right]. \end{aligned}$$

Remark 4. Since (u_ϵ) is bounded in the Hilbert space $\mathcal{N}^2_w(0,T,L^2(\mathbb{R}^d))$, by identification, one shows that $u_\epsilon \rightharpoonup \int_0^1 \mathbf{u}(.,\alpha)\mathrm{d}\alpha$ in the same space, and so $\int_0^1 \mathbf{u}(.,\alpha)\mathrm{d}\alpha$ is a predictable process. The interesting point is the measurability of $\mathbf{u}$ with respect to all its variables (t,x,ω,α). Revisiting the work of E. Yu. Panov [9] with the σ-field $\mathcal{P}_T\otimes L(\mathbb{R}^d)$, one shows that $\mathbf{u}$ is measurable for the σ-field $\mathcal{P}_T\otimes L(\mathbb{R}^d\times]0,1[)$.

Now a separability argument for the norm of $H^1(Q)$ yields the existence of a Young measure solution in the sense of the following definition.

Definition 2. $\mathbf{u}\in\mathcal{N}^2_w\left(0,T,L^2(\mathbb{R}^d\times]0,1[)\right)\cap L^\infty\left(0,T,L^2(\Omega\times\mathbb{R}^d\times]0,1[)\right)$ is a (Young) measure-valued entropy solution of (1.1) with the initial data $u_0\in L^2(\mathbb{R}^d)$ if for any $\eta\in\mathcal{E}$ and any $(k,\varphi)\in\mathbb{R}\times\mathcal{D}^+([0,T]\times\mathbb{R}^d)$,

$$0\leq\int_0^1 \mu_{\mathbf{u},\eta,k}(\varphi)d\alpha,\quad P-\text{a.s.}$$

And, $ess\lim_{t\to 0^+} E\int_{K\times]0,1[}|\mathbf{u}(t)-u_0|\mathrm{dx}\mathrm{d}\alpha = 0$, for any compact set $K\subset\mathbb{R}^d$.

§4. Local Kato inequality

The aim of this section is to discuss about the way of obtaining the following interior Kato inequality, which permits to prove that the measure-valued solution is an entropy solution in the sense of Definition 1.

Proposition 2. *Let* $\mathbf{u_1}$, $\mathbf{u_2}$ *be Young measure-valued entropy solutions to (1.1) with initial data* $u_{1,0}, u_{2,0} \in L^2(\mathbb{R}^d)$*, respectively. Then, for any nonnegative function* φ *in* $\mathcal{D}(\overline{Q})$*, it holds*

$$\begin{aligned} 0 \quad \leq \quad & \int_{\mathbb{R}^d} |u_{1,0} - u_{2,0}|\varphi(0)\mathrm{dx} + E\int_{Q\times]0,1[^2} \Big|\mathbf{u_1}(t,x,\alpha) - \mathbf{u_2}(t,x,\beta)\Big|\partial_t\varphi \mathrm{dxdtd}\alpha\mathrm{d}\beta \\ & -E\int_{Q\times]0,1[^2} F\Big(\mathbf{u_1}(t,x,\alpha), \mathbf{u_2}(t,x,\beta)\Big).\nabla\varphi\mathrm{dxdtd}\alpha\mathrm{d}\beta. \end{aligned} \tag{4.1}$$

Proof. We propose here to present stages of the proof introduced in [2], emphasizing on differences with the deterministic setting, and stochastic calculus tools chosen. The main idea is to use Kruzhkov's doubling variables method. Let us apply the usual technique and advice when we meet difficulties. For this, we consider two measure-valued solutions $\mathbf{u_1}$, $\mathbf{u_2}$ and those inequalities P-a.s.:

$$0 \leqslant \int_0^1 \mu_{\mathbf{u_1}(t,x,\alpha),\eta_\delta,k_2}(\psi)d\alpha \qquad ; \qquad 0 \leqslant \int_0^1 \mu_{\mathbf{u_2}(s,y,\beta),\eta_\delta,k_1}(\psi)\mathrm{d}\beta, \tag{4.2}$$

where $k_1, k_2 \in \mathbb{R}$ and $\psi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$.
Notice that, comparing with the deterministic case, the stochastic perturbation of our conservation law brings new terms in the entropy inequalities, ones containing an Itô integral:

$$\begin{aligned} &\int_0^1 \int_Q \eta'_\delta(\mathbf{u_1}(t,x,\alpha) - k_2)h(\mathbf{u_1})\psi\mathrm{dxdw}(t)\mathrm{d}\alpha \\ &\int_0^1 \int_Q \eta'_\delta(\mathbf{u_2}(s,y,\beta) - k_1)h(\mathbf{u_2})\psi\mathrm{dydw}(s)\mathrm{d}\beta, \end{aligned} \tag{4.3}$$

and others containing the second derivative of η_δ:

$$\begin{aligned} &\frac{1}{2}\int_0^1 \int_Q h^2(\mathbf{u_1}(t,x,\alpha))\eta''_\delta(\mathbf{u_1} - k_2)\psi\mathrm{dtdxd}\alpha \\ &\frac{1}{2}\int_0^1 \int_Q h^2(\mathbf{u_2}(s,y,\beta))\eta''_\delta(\mathbf{u_2} - k_1)\psi\mathrm{dsdyd}\beta. \end{aligned} \tag{4.4}$$

Usually, we take in (4.2) $k_1 = \mathbf{u_1}(t,x,\alpha)$, $k_2 = \mathbf{u_2}(s,y,\beta)$, $\psi(t,x,s,y) = \varphi(s,y)\rho_m(x-y)\rho_n(t-s)$ with $\varphi \in \mathcal{D}^+([0,T]\times\mathbb{R}^d)$, $supp\varphi(t,.) \subset K$ a compact set of $\mathbb{R}^d$, ρ_n and ρ_m the usual mollifier sequences in $\mathbb{R}$ and $\mathbb{R}^d$ respectively, with $supp\rho_n \subset [-\frac{2}{n}, 0]$. Then, we integrate with respect to (s,y,β) for the first inequality, with respect to (t,x,α) for the second one, we add those two new inequalities and pass to the limit on δ, n and m.
In our case, there is a problem with this technique when we treat the stochastic integrals (4.3). Indeed, because of the definition of the Itô integral, we require an $\mathcal{F}_t$-measurability for replacing k_2 and an $\mathcal{F}_s$-measurability for replacing k_1, which are not satisfied by $k_1 = \mathbf{u_1}(t,x,\alpha)$ and $k_2 = \mathbf{u_2}(s,y,\beta)$ because we ignore if $s > t$ or $s < t$ (recall that $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 \leqslant s \leqslant t$).
For this reason, we consider the entropy formulations (4.2) with the same real k, and multiply by a kernel of convolution $\rho_l(\mathbf{u_1}(t,x,\alpha) - k)$ the inequality coming from $\mu_{\mathbf{u_2},\eta_\delta,k}$ and integrate with respect to (t,x,α); multiply by $\rho_l(\mathbf{u_2}(s,y,\beta) - k)$ the inequality coming from $\mu_{\mathbf{u_1},\eta_\delta,k}$ and

integrate with respect to (s, y, β), we add those two inequalities, then integrate over k in $\mathbb{R}$ all the formulation and take the expectation, we get:

$$0 \leqslant E\int_Q \int_{]0,1[^2} \int_{\mathbb{R}} \mu_{\mathbf{u_1},\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(\mathbf{u_2}(s,y,\beta)-k)\mathrm{dkd}\alpha\mathrm{d}\beta\mathrm{dsdy}$$

$$+E\int_Q \int_{]0,1[^2} \int_{\mathbb{R}} \mu_{\mathbf{u_2},\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(\mathbf{u_1}(t,x,\alpha)-k)\mathrm{dkd}\beta\mathrm{d}\alpha\mathrm{dtdx}.$$

With a judicious order for the passage to the limit, we are able to avoid our measurability problem in the stochastic integral. Indeed, we first pass to the limit on n, then we get the same time everywhere (t or s), and the problem of measurability with respect to the σ-field $\mathcal{P}_T$ is forgotten. Then passing to the limit on l, we get back that $\mathbf{u_1}(t,x,\alpha)$ and $\mathbf{u_2}(s,y,\beta)$ replace k in our formulation, as we wished at the beginning.
Now it remains to pass to the limit on δ and m in the entropy inequality. The second delicate point appears with terms containing the second derivative of η_δ (4.4) when we want to pass to the limit on δ. Indeed, because of the presence of η_δ'', we are not able to identify the limit of those terms, all we can say is that the limit exists (Tanaka formula), and the problem is that we need to know the limit to obtain the local Kato inequality. For this reason, we decide to consider a viscous regular solution u_ϵ instead of $\mathbf{u_2}$ and keep a measure-valued solution $\mathbf{u_1} := \hat{\mathbf{u}}$. Indeed, the suitable regularity of such a solution allows us to apply the Itô formula. Following the concept of J. Feng and D. Nualart for treating the stochastic term, the idea remains on combining terms containing η_δ'' with others coming from stochastic calculus. Let us mention that the passage to the limit on n and l on terms containing η_δ'' gives:

$$\begin{aligned}
& \frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(\hat{\mathbf{u}})\eta_\delta''(\hat{\mathbf{u}}(s,x,\alpha)-u_\epsilon(s,y))\rho_m(x-y)\varphi\mathrm{d}\alpha\mathrm{dsdxdy} \\
& +\frac{1}{2}E\int_{\mathbb{R}^d}\int_Q\int_0^1 h^2(u_\epsilon)\eta_\delta''(u_\epsilon(s,y)-\hat{\mathbf{u}}(s,x,\alpha))\rho_m(x-y)\varphi\mathrm{d}\alpha\mathrm{dsdxdy} \\
:= \quad & A+B.
\end{aligned}$$

The technical point is to combine appropriately those annoying terms with stochastic integrals coming from (4.3) using the following judicious remark: the martingale property of the Itô integral allows us to write the stochastic integrals in this way:

$$\begin{aligned}
& E\int_Q\int_{\mathbb{R}}\int_{s-2/n}^{s}\int_{\mathbb{R}^d}\int_0^1 \eta_\delta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\mathrm{d}\alpha\varphi\rho_m(x-y)\rho_n(t-s)\mathrm{dxdw}(t) \\
& \qquad \times\rho_l(u_\epsilon(s,y)-k)\mathrm{dkdyds} \\
= \quad & E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta_\delta'(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\mathrm{d}\alpha\rho_n(t-s)\mathrm{dw}(t)\varphi\rho_m(x-y)\mathrm{dx} \\
& \qquad \times[\rho_l(u_\epsilon(s,y)-k)-\rho_l(u_\epsilon(s-2/n,y)-k)]\mathrm{dkdyds} \\
:= \quad & C_{n,l}.
\end{aligned}$$

Here the choice of u_ϵ instead of $\mathbf{u_2}$ is crucial. Indeed, the regularity of u_ϵ allows us to apply

Itô's formula with $\mathrm{d}u_\epsilon = [\epsilon\Delta u_\epsilon + div\mathbf{f}(u_\epsilon)]\mathrm{dt} + h(u_\epsilon)\mathrm{dw} = A_\epsilon \mathrm{dt} + h(u_\epsilon)\mathrm{dw}$ and to get:

$$\begin{aligned} & \rho_l(u_\epsilon(s,y)-k) - \rho_l(u_\epsilon(s-2/n,y)-k) \\ = \ & \int_{s-\frac{2}{n}}^{s} \rho_l'(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\mathrm{d}\sigma + \int_{s-\frac{2}{n}}^{s} \rho_l'(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\mathrm{dw}(\sigma) \\ & + \frac{1}{2}\int_{s-\frac{2}{n}}^{s} \rho_l''(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\mathrm{d}\sigma, \end{aligned}$$

which wasn't possible with a measure-valued solution.
Thus, by integration by parts with respect to the variable k, it comes:

$$\begin{aligned} C_{n,l} \ = \ & -E\int_Q\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{s-2/n}^{s}\int_0^1 \eta_\delta''(\hat{\mathbf{u}}-k)h(\hat{\mathbf{u}})\mathrm{d}\alpha\rho_n(t-s)\mathrm{dw}(t)\varphi\rho_m(x-y)\mathrm{dx} \\ & \times\Big[\int_{s-\frac{2}{n}}^{s} \rho_l(u_\epsilon(\sigma,y)-k)A_\epsilon(\sigma,y)\mathrm{d}\sigma + \int_{s-\frac{2}{n}}^{s} \rho_l(u_\epsilon(\sigma,y)-k)h(u_\epsilon(\sigma,y))\mathrm{dw}(\sigma) \\ & + \frac{1}{2}\int_{s-\frac{2}{n}}^{s} \rho_l'(u_\epsilon(\sigma,y)-k)h^2(u_\epsilon(\sigma,y))\mathrm{d}\sigma\Big]\mathrm{dkdyds} \\ \to_{n,l} \ & -E\int_{\mathbb{R}^d}\int_Q\Big[\int_0^1 \eta_\delta''(\hat{\mathbf{u}}(s,x,\alpha)-u_\epsilon(s,y))h(\hat{\mathbf{u}}(s,x))h(u_\epsilon(s,y))\mathrm{d}\alpha\Big]\varphi\rho_m(x-y)\mathrm{dsdydx} \\ := \ & C. \end{aligned}$$

Thus,

$$\begin{aligned} A+B+C \ = \ & \frac{1}{2}E\int_Q\int_{\mathbb{R}^d}\int_0^1 [h(\hat{\mathbf{u}})-h(u_\epsilon)]^2\eta_\delta''(u_\epsilon(s,y)-\hat{\mathbf{u}}(s,x,\alpha))\rho_m(x-y)\varphi\mathrm{d}\alpha\mathrm{dydxds} \\ \to_\delta \ & 0. \end{aligned}$$

In summary, this is the plan of the proof. By doing stochastic computations on the Itô integral and passing to the limit (with classical techniques) with respect to n, l, δ, ϵ, m in this order on

$$\begin{aligned} 0 \leqslant \ & E\int_Q\int_0^1\int_{\mathbb{R}} \mu_{\hat{\mathbf{u}},\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(u_\epsilon(s,y)-k)\mathrm{dkd}\alpha\mathrm{dsdy} \\ & + E\int_Q\int_0^1\int_{\mathbb{R}} \mu^\epsilon_{u_\epsilon,\eta_\delta,k}(\varphi(s,y)\rho_m(x-y)\rho_n(t-s))\rho_l(\hat{\mathbf{u}}(t,x,\alpha)-k)\mathrm{dkd}\alpha\mathrm{dtdx}, \end{aligned}$$

we finally obtain the local Kato inequality. □

Proposition 3. *The measure-valued solution is unique. Moreover, it is the unique entropy solution.*

Proof. As in the deterministic case, set $\omega = \|f'\|_\infty$, $\hat{u}_0 = u_0$, $\gamma(t) = \frac{(T-t)^+}{T}$, and denote by ψ any nonincreasing regular function with $1_{]-\infty,K]} \leq \psi \leq 1_{]-\infty,K+1]}$, where $K > 0$. Then, considering $K = R + \omega T$ for any $R > 0$ and $\varphi(t,x) = \psi(|x| + \omega t)\gamma(t)$ in (4.1) implies that, $\mathbf{u}(t,x,\beta) = \hat{\mathbf{u}}(t,x,\alpha)$ for almost any $x \in B(0,R)$, $t \in]0,T[$, $\omega \in \Omega$, $\alpha,\beta \in]0,1[$. Thus, on the one hand $\mathbf{u} = \hat{\mathbf{u}}$; on the other hand $\mathbf{u}(t,x,\alpha) = u(t,x)$ is independent of α, hence an entropy solution in the sense of Definition 1.

□

Proposition 4. *Entropy solutions satisfy a "contraction principle": if u_1, u_2 are entropy solutions of (1.1) corresponding to initial data $u_{1,0}, u_{2,0} \in L^2(\mathbb{R}^d)$, respectively, then, for any positive K and time t, $E \int_{B(0,K-\omega t)} |u_1 - u_2| \mathrm{dx} \leq \int_{B(0,K)} |u_{1,0} - u_{2,0}| \mathrm{dx}$.*

Proof. This is a consequence of the previous proof when passing to the limit when ψ converges to $1_{]-\infty,K]}$. □

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