Monografías Matemáticas *García de Galdeano* **37**, 139–147 (2012)

LYAPUNOV STABILITY FOR A RYDBERG ATOM IN A CIRCULARLY POLARIZED MICROWAVE FIELD AND A STATIC MAGNETIC FIELD

Manuel Iñarrea, Víctor Lanchares, Ana Isabel Pascual and José Pablo Salas

Abstract. The interaction of a Rydberg atom with a circularly polarized microwave field leads, with finely tuned parameters, to the creation of stable equilibrium positions similar to the well known Lagrangian equilibrium points in celestial mechanics (cf. [6]). Besides, the addition of a static magnetic field, perpendicular to the plane of polarization, can be used to manipulate the stability properties of the equilibria (cf. [9] and [10]).

The aim of this communication is the characterization of nonlinear stability properties for equilibrium points by making use of appropriate results, based on KAM theory. Special attention will be paid when resonance conditions take place between the fundamental frequencies of the system.

Keywords: Circularly microwave field, static magnetic field, equilibria, stability, resonances.

AMS classification: 70H08, 70H14, 37N05.

§1. Introduction

The dynamics of a hydrogen atom in the presence of a circularly polarized (CP) microwave field crossed with a magnetic field **B**, denoted hereafter by CP × **B**, is known to give rise to two different behaviors (cf. [9]). On the one hand, the electron can follow a perturbed Keplerian orbit, which can be studied under the point of view of classical mechanics, by means of perturbation methods (cf. [8]). On the other hand, the electron can be trapped in a region beyond the Stark saddle point, by properly tuning the external parameters of the problem.

The last case is specially interesting, because of the appearance of equilibrium points similar to the Lagrangian points in the restricted three body problem. These points are directly linked with the ionization threshold for the electron (cf. [6]). Besides this remarkable connection, the stability properties of these points are also of interest as they are the key to have a real trapping region. In this way, the main goal of this work is the characterization of nonlinear stability properties of these points as a function of the external parameters. We will not pay attention to the size of the region of stability, a question that deserves a further analysis and that is relevant to get an effective trapping region.

The problem we deal with has also a Celestial Mechanics counterpart, as it describes the dynamics of a dust particle subject to radiation pressure, the sun magnetic field and orbiting

an idealized planet that revolves around the sun in circular orbit. In this context, the existence of stable trapping regions associated with stable equilibrium points may account for dust clouds responsible for the phenomenon known as zodiacal light (cf. [12]).

§2. The problem

In atomic units, the Hamiltonian of the problem CP × **B**, in the dipole aproximation, is given by

$$\mathcal{H} = \frac{1}{2}(P_x^2+P_y^2+P_y^2) - \frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{\omega_c}{2}(xP_y - yP_x) + \frac{\omega_c^2(x^2+y^2)}{8} \pm f(x\cos\omega_f t + y\sin\omega_f t),$$

where the magnetic field is supposed to be parallel to the direction of the z-axis, ω_c is the cyclotron frequency, ω_f is the CP field frequency and f the electric field strength ($f > 0$). The plus or minus sign depends on the polarization direction of the microwave field.

The explicit time dependence in the Hamiltonian can be removed by going to a sinodic reference frame that rotates at the constant angular velocity ω_f, in such a way that the moving x-axis is aligned with the direction of the electric field. The new Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}(P_x^2 + P_y^2 + P_z^2) - \frac{1}{\sqrt{x^2+y^2+z^2}} - \left(\omega_f \pm \frac{\omega_c}{2}\right)(xP_y - yP_x) + \frac{\omega_c^2(x^2+y^2)}{8} \pm fx. \quad (1)$$

In this work, we study the planar model, that is the model restricted to the invariant set $z = P_z = 0$. In this way, by setting z and P_z to zero in (1), we obtain the Hamiltonian corresponding to the plane case:

$$\mathcal{H} = \frac{1}{2}(P_x^2 + P_y^2) - \frac{1}{\sqrt{x^2+y^2}} - \left(\omega_f \pm \frac{\omega_c}{2}\right)(xP_y - yP_x) + \frac{\omega_c^2(x^2+y^2)}{8} \pm fx. \quad (2)$$

As we are interested in the equilibria of the system given by the Hamiltonian (2) we have to solve the corresponding Hamilton equations equated to zero. These are

$$\begin{cases} \dot{x} = P_x + \omega y, \\ \dot{y} = P_y - \omega x, \\ \dot{P}_x = -\dfrac{x}{r^3} + \omega P_y \mp f - \dfrac{\omega_c^2}{4}x, \\ \dot{P}_y = -\dfrac{y}{r^3} - \omega P_x - \dfrac{\omega_c^2}{4}y, \end{cases}$$

where $\omega = \omega_f \pm \omega_c/2$ and $r = \sqrt{x^2+y^2}$.

From the system above it follows that an equilibrium point (x, y, P_x, P_y) must verify $P_x = y = 0$ and $P_y = \omega x$. Moreover, x must be a positive root of the equation

$$\omega_f(\omega_f \pm \omega_c)x^3 \mp fx^2 - 1 = 0,$$

or a negative root of the equation

$$\omega_f(\omega_f \pm \omega_c)x^3 \mp fx^2 + 1 = 0.$$

The discussion on the number of equilibria is summarized in the following proposition

Proposition 1. *For the positive sign in (2), there are two equilibrium points, one of them with $x < 0$ and the other one with $x > 0$. For the minus sign in (2), the number of equilibrium points depends on the sign of $\omega_f(\omega_f - \omega_c)$:*

- *If $\omega_f(\omega_f - \omega_c) > 0$, then there are 2 equilibria, one with $x > 0$ and another one with $x < 0$.*
- *If $\omega_f(\omega_f - \omega_c) = 0$, there is one equilibrium point with $x > 0$.*
- *If $\omega_f(\omega_f - \omega_c) < 0$, there are two equilibria if $f > F_c$, where $F_c = \sqrt[3]{\frac{27}{4}\omega_f^2(\omega_f - \omega_c)^2}$. In this case, the two of them verify $x > 0$. If $f \leq F_c$, no equilibria exist.*

To study the stability properties of the equilibrium points we introduce a function of the coordinates x and y, usually called the effective potential, in such a way that linear stable points correspond to relative maxima and minima of this function. In the positive case, the effective potential is given by

$$EP_p = fx - \frac{1}{2}\omega_f(\omega_c + \omega_f)(x^2 + y^2) - \frac{1}{\sqrt{x^2 + y^2}}.$$

As a result, the equilibrium with $x > 0$ is a maximum and the equilibrium with $x < 0$ is a saddle. In the negative case, the effective potential reads as

$$EP_n = -fx - \frac{1}{2}\omega_f(\omega_c - \omega_f)(x^2 + y^2) - \frac{1}{\sqrt{x^2 + y^2}},$$

and the character of the equilibria depends on the sign of $\omega_f(\omega_f - \omega_c)$. In this sense, if $\omega_f(\omega_f - \omega_c) > 0$, the equilibrium with $x > 0$ is a saddle and the equilibrium with $x < 0$ is a maximum. If $\omega_f(\omega_f - \omega_c) < 0$ and $f > F_c$, one of the positive equilibria is a saddle (we call x_s) and the other is a minimum (we call x_m).

The previous analysis shows that there are two different configurations for the effective potential, maximum-saddle or minimum-saddle (cf. [7]). It is known that saddle points correspond to unstable points and a minimum give rise to a nonlinear stable point (cf. [13]), as it follows from Dirichlet's theorem (cf. [14]). On the other hand, a maximum is a linear stable point but its character from the point of view of Lyapunov is not decided. In this way, there are well known counterexamples where a linear stable point of a Hamiltonian system is unstable in the Lyapunov sense (cf. [3]). To solve the question of nonlinear stability we will make use of KAM theory and the next sections are devoted to introduce the results we will use.

§3. Lyapunov stability

One of the results from KAM theory is Arnold's theorem (cf. [1]) that guarantees nonlinear stability of a maximum for almost all set of admissible external parameters. Here we reproduce the version of this theorem given by Meyer and Schmidt in (cf. [11]).

Theorem 2. *Let be a two degrees of freedom Hamiltonian system expressed in variables* $(\Psi_1, \Psi_2, \psi_1, \psi_2)$ *as*

$$\mathcal{H} = \mathcal{H}_2(\Psi_1, \Psi_2) + \mathcal{H}_4(\Psi_1, \Psi_2) + \cdots + \mathcal{H}_{2N}(\Psi_1, \Psi_2) + \overline{\mathcal{H}}(\Psi_1, \Psi_2, \psi_1, \psi_2)$$

where it is verified that:

1. $\mathcal{H}$ *is a real analytic function in a neighborhood of the origin.*
2. *Each* $\mathcal{H}_{2k}$, $1 \leq k \leq N$ *is a homogeneous polynomial of degree* k *in* Ψ_1, Ψ_2 *with real coefficients independent of the angles. In particular,*

$$\mathcal{H}_2 = \omega_1 \Psi_1 - \omega_2 \Psi_2, \quad \omega_1, \omega_2 > 0, \tag{3}$$

$$\mathcal{H}_4 = \frac{1}{2}(A\Psi_1^2 - 2B\Psi_1\Psi_2 + C\Psi_2^2),$$

   where A, B *and* C *depend on the parameters of the Hamiltonian.*

3. $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\Psi_1, \Psi_2, \psi_1, \psi_2) = O((\Psi_1 + \Psi_2)^{2N+1})$.

With these conditions, the origin is a stable equilibrium if exists $2 \leq k \leq N$ *such as*

$$\mathcal{D}_{2k} = \mathcal{H}_{2k}(\omega_2, \omega_1) \neq 0,$$
$$\mathcal{D}_{2j} = \mathcal{H}_{2j}(\omega_2, \omega_1) = 0, \quad 2 \leq j < k.$$

The practical implementation of this theorem yields a great amount of computation work. First, the Hamiltonian must be expressed in action–angle variables, in such a way the quadratic part reduces to (3).

This is achieved by means of a linear transformation. Next, the most tricky part of the process, the Hamiltonian must be brought to the so-called Birkhoff normal form (cf. [2] and [15]) up to a certain order through the application of successive canonical transformations near to the identity. This process can be made in a recursive manner using the algorithm of the Lie–Deprit perturbation method (cf. [4]). This process simplifies if it is carried out in complex variables, returning to Poincare, or action-angle, variables at the end. Once the normalization has been completed Arnold's theorem can be applied.

We note that in the statements of theorem 2 there are some implicit assumptions. The first one is that the Hamiltonian is written in normal form, that is, the computation work is already supposed done. The second one is that the frequencies of the system ω_1 and ω_2 are not in resonance of order less or equal than $2N$, because the terms $\mathcal{H}_{2k}$, $1 \leq k \leq N$, only depend on the momenta Ψ_1 and Ψ_2. When the frequencies of the system verify a resonance condition, the normal form is no longer as those presented in the theorem and terms depending on the angles ψ_1 and ψ_2 appear.

To handle resonant cases we need a more general result. In this sense we will make use of a geometric criterion (cf. [5] and [13]), that extends Arnold's theorem to the resonant cases.

§4. The geometric criterium

Let us suppose that $\mathcal{H}_2$ can be written as in (3). Then, the Hamiltonian $\mathcal{H}$ can be brought to normal form, in such a way that $\mathcal{H}_2$ becomes a formal integral. Also let us assume that ω_1 and

ω_2 satisfy a n:m resonant condition of order greater or equal than two, that is, $n\omega_1 - m\omega_2 = 0$ with $n + m \geq 2$. Under these assumptions we introduce a set of action–angle variables named Lissajous variables, with a twofold goal. On the one hand, the formal integral depends on one of the actions and the normalization procedure can be viewed as the elimination of a fast variable by means of an averaging process.

Lissajous variables $(\Phi_1, \Phi_2, \phi_1, \phi_2)$ are specifically designed for each particular value of the resonance n:m and they are related with the Poincaré variables through the formulae

$$\begin{aligned} \Psi_1 &= \frac{\Phi_1 + \Phi_2}{2m}, \qquad \psi_1 = m(\phi_1 + \phi_2), \\ \Psi_2 &= \frac{\Phi_1 - \Phi_2}{2n}, \qquad \psi_2 = n(\phi_1 - \phi_2). \end{aligned}$$

Now, $\mathcal{H}_2$ turns to be simply $\nu\,\Phi_2$, being

$$\nu = \frac{\omega_1}{m} = \frac{\omega_2}{n}.$$

Besides, the Poisson's bracket $(\mathcal{H}_2, \mathcal{H}_j)$, needed to compute the normal form, is just

$$(\mathcal{H}_2, \mathcal{H}_j) = \nu \frac{\partial \mathcal{H}_j}{\partial \phi_2},$$

and the process of normalization is no more than an averaging over the ϕ_2 angle.

Moreover, the normal form is generated by the invariants (cf. [5]) M_1, M_2, C and S that, as functions of Lissajous variables, are given by

$$\begin{aligned} M_1 &= \frac{1}{2}\Phi_1, & C &= 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2}\cos 2nm\phi_1, \\ M_2 &= \frac{1}{2}\Phi_2, & S &= 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2}\sin 2nm\phi_1. \end{aligned} \tag{4}$$

In this way, the normal form up to order N is written as

$$\mathcal{H} = \mathcal{H}_2 + \sum_{j=3}^{N} \mathcal{H}_j,$$

where $\mathcal{H}_2 = 2\omega M_2$, and

$$\mathcal{H}_j = \sum_{2(\gamma_1+\gamma_2)+(n+m)(\gamma_3+\gamma_4)=j} a_{\gamma_1\gamma_2\gamma_3\gamma_4} M_1^{\gamma_1} M_2^{\gamma_2} C^{\gamma_3} S^{\gamma_4}, \qquad 3 \leq j \leq N,$$

with $a_{\gamma_1\gamma_2\gamma_3\gamma_4} \in \mathbb{R}$.

The invariants are not independent and they satisfy the equation

$$C^2 + S^2 = (M_1 + M_2)^n (M_1 - M_2)^m, \tag{5}$$

together with the restriction

$$M_1 \geq |M_2|\,. \tag{6}$$

Note that the reduced phase space is given by the equation (5) and the restriction (6). It is a set of surfaces of revolution, one for each constant value of M_2. Fixed a value for M_2, (5) is a surface of revolution with a vertex in the point $M_1 = |M_2|$, $C = S = 0$.

Once the reduced phase space is determined, it is possible to know the flow of the normalized system, when it is truncated to a prescribed order. Indeed, the flow results as the intersection of the normalized Hamiltonian function with the surface defined by (5). Based on this idea, in (cf. [5] and [13]), Elipe et al. established the following results, the first is the geometric criterion and the second one is a derived result from it.

Theorem 3. *Let us assume that the Hamiltonian is normalized up to a certain order $N \geq s$, being $\mathcal{H}_N$ the first term that does not vanish for $M_2 = 0$. Let us consider the two surfaces*

$$\mathcal{G}_1 = \{(C, S, M_1) \in \mathbb{R}^3; \quad \mathcal{H}_N(C, S, M_1, 0) = 0\},$$

and

$$\mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}^3; \quad C^2 + S^2 = M_1^s\}.$$

If the origin is an isolated point of intersection, then it is stable. In other case, if the two surfaces are not tangent, the origin is unstable.

Theorem 4. *Let us assume that $\mathcal{H}_s$ (s is the order of resonance) is the first term in the normal form that does not vanish for $M_2 = 0$. If s is odd ($s \geq 3$), then $\mathcal{H}_s(C, S, M_1, 0) = \gamma C + \eta S$ with $\gamma^2 + \eta^2 \neq 0$ and the origin is an unstable equilibrium. If s is even ($s \geq 4$), then $\mathcal{H}_s(C, S, M_1, 0) = a_s M_1^{s/2} + \gamma C + \eta S$ with $a_s^2 + \gamma^2 + \eta^2 \neq 0$ and the stability of the origin depends on the relative values of a_s^2 and $\gamma^2 + \eta^2$: if $a_s^2 > \gamma^2 + \eta^2$, the origin is a stable equilibrium, whereas if $a_s^2 < \gamma^2 + \eta^2$, the origin is unstable.*

§5. Resonant cases

Using the previous results, we study the Lyapunov stability of the maximum when it is verified a resonant condition. We start with the resonance of order three to be followed by the fourth order resonance.

5.1. 1:2 resonance

For a 1:2 resonance ($\omega_1 = 2\omega_2$), the term of order 3 in the normal form can be expressed in complex variables as

$$\mathcal{H}_3 = a_{1002} u V^2 + a_{0120} U v^2,$$

where $a_{1002} = i a_{0120}$ and $a_{0120} \in \mathbb{C}$. Therefore

$$\mathcal{H}_3 = a_{0120}(U v^2 - i u V^2).$$

Expressed in Lissajous invariants, $\mathcal{H}_3$ can be written as

$$\mathcal{H}_3 = a_s S,$$

with $a_s = -a_{0120}\sqrt{2}$.

Then, the two surfaces, from the geometric criterion, are given by

$$\mathcal{G}_1 = \{(C, S, M_1) \in \mathbb{R}^3 \,|\, a_s S = 0\},$$
$$\mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}^3 \,|\, C^2 + S^2 = M_1^3, \quad M_1 \geq 0\}.$$

It is clear that if $a_s \neq 0$, then $\mathcal{G}_1 \cap \mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}^3 \,|\, S = 0, C = \pm M_1^{3/2}\}$ and therefore, the equilibrium is unstable. By the contrary, if $a_s = 0$, then $\mathcal{H}_3(M_2 = 0) = 0$ and also $\mathcal{H}_3 \equiv 0$, and, in consequence, we need more terms of the normal form to decide about the stability. We have to compute the next term in the normal form, that is, $\mathcal{H}_4$. In this way the next nonzero term in the normal form $\mathcal{H}_4$ can be written, in complex variables, as

$$\mathcal{H}_4 = a_{1200} u U^2 + a_{1111} u U v V + a_{0012} v V^2.$$

Once expressed in terms of the real invariants, we have that

$$\mathcal{H}_4(M_2 = 0) = \frac{29.877}{\omega^4 x_0^8} M_1^2,$$

where x_0 is the x coordinate of the equilibrium point and $\omega = \omega_c + \frac{\omega_f}{2}$. Therefore the origin is the only point in the intersection $\mathcal{G}_1 \cap \mathcal{G}_2$ and, as a consequence of the geometric criterion, the equilibrium is stable.

5.2. 1:3 resonance

In presence of a 1:3 resonance, the term of fourth order in the normal form $\mathcal{H}_4$ evaluated at $M_2 = 0$ is given, in complex variables, by

$$\mathcal{H}_4 = a_{2200} u^2 U^2 + a_{1111} u U v V + a_{0022} v^2 V^2 + a_{1003} u V^3 + a_{0130} U v^3.$$

Expressed in Lissajous invariants, $\mathcal{H}_4$ can be written as

$$\mathcal{H}_4(M_2 = 0) = a_m M_1^2 + a_c C + a_s S,$$

being a_m, a_c, a_s dependent on the parameters of the problem and the coordinates of the equilibrium point.

In our problem it is always verified that $a_m^2 > a_c^2 + a_s^2$. Therefore, as a consequence of Theorem 4, the equilibrium is always stable.

Acknowledgements

We thank financial support from Ministerio de Ciencia e Innovación (Project #MTM2008–03818/MTM) from Spain.

References

[1] Arnold, V. I. The stability of the equilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case. *Soviet Math. Dokl. 2* (1961), 247–249.

[2] Birkhoff, G. D. *Dynamical Systems*, vol. 9. American Mathematical Society, Providence, Rhode Island, 1991.

[3] Cherry, T. M. On periodic solutions of Hamiltonian systems of differential equations. *Phil. Trans. Roy. Soc. 227* (1928), 137–221.

[4] Deprit, A. Canonical transformations depending on a small parameter. *Celestial Mechanics and Dynamical Astronomy* (1969), 12–30. doi:10.1007/BF01230629.

[5] Elipe, A., Lanchares, V., and Pascual, A. I. On the stability of equilibria in two degrees of freedom Hamiltonian systems under resonances. *J. Nonlinear Sci. 15* (2005), 305–319. doi:10.1007/s00332-004-0674-1.

[6] Farrelly, D., and Uzer, T. Ionization mechanism of Rydberg atoms in a circularly polarized microwave field. *Phys. Rev. Lett. 74* (1995), 1720–1723. doi:10.1103/PhysRevLett.74.1720.

[7] Iñarrea, M., Lanchares, V., Pascual, A. I., and Salas, J. P. Electronic traps in a circularly polarized microwave field and a static magnetic field: Stability analysis. *Monografías De La Real Academia De Ciencias De Zaragoza 14* (1999), 114–118.

[8] Lanchares, V., Iñarrea, M., and Salas, J. P. Bifurcations in the hydrogen atom in the presence of a circularly polarized microwave field and a static magnetic field. *Phys. Rev. A. 56*, 3 (1997), 1839–1843. doi:10.1103/PhysRevA.56.1839.

[9] Lee, E., Brunello, F., and Farrelly, D. Single atom quasi–penning trap. *Phys. Rev. Lett. 75* (1995), 3461–3643. doi:10.1103/PhysRevLett.75.3641.

[10] Lee, E., Brunello, F., and Farrelly, D. Coherent states in a Rydberg atom: Classical mechanics. *Phys. Rev. A. 55* (1997), 2203–2221. doi:10.1103/PhysRevA.55.2203.

[11] Meyer, K. R., and Schmidt, D. S. The Stability of the Lagrange Triangular Point and a Theorem of Arnold. *Journal of Differential Equations 62* (1986), 222–236. doi:http://math.uc.edu/~meyer/jde86.pdf.

[12] Moulton, F. R. *An introduction to Celestial Mechanics*. Dover, New York, 1970. 2nd edition.

[13] Pascual, A. I. *Doctoral Thesis: Sobre la estabilidad de sistemas hamiltonianos de dos grados de libertad bajo resonancias.* Servicio de Publicaciones de la Universidad de La Rioja, Logroño, La Rioja, Spain, 2005. Available from: http://www.unirioja.es/servicios/sp/tesis/119.shtml.

[14] Siegel, C. L., and Moser, L. K. *Lectures on Celestial Mechanics*, vol. 187. Springer–Verlag, New York, 1971.

[15] Verhulst, F. *Nonlinear Differential Equations and Dynamical Systems*. Springer–Verlag, New York, 1991.

Manuel Iñarrea, Víctor Lanchares, Ana Isabel Pascual and José Pablo Salas.
Grupo de Dinámica No Lineal.
Universidad de La Rioja.
manuel.inarrea@unirioja.es, vlancha@unirioja.es, aipasc@unirioja.es,
josepablo.salas@unirioja.es