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HIGH-PRECISION PERIODIC ORBIT CORRECTOR

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Abstract. An algorithm to compute periodic orbits of dynamical systems up to an arbitrary number of precision digits is presented. The algorithm is based on an optimized Newton-Raphson method combined with a new numerical ODE solver, TIDES that uses a Taylor series method. Finally, we present some numerical tests for the Lorenz model and the Hénon-Heiles Hamiltonian which show the quadratic convergence and the good behaviour of the proposed method.

Keywords: Periodic orbits, shooting method, Taylor series method, TIDES.
AMS classification: 37M20, 65P20.

§1. Introduction

Nowadays, more and more theoretical and applied problems need high-precision results. In Dynamical Systems we may find a large plethora of such problems, like studying the exponentially small splitting of separatrices, in the analysis of SNAs, in the study of complex singularities of systems like Lorenz model, and so on. Studying and locating the periodic orbits of dynamical systems give relevant information. So, the periodic orbits are an important topic in several physical applications and finding them accurately is of great importance in periodic orbit theory [2, 3, 4]. In this paper we propose a new algorithm to locate periodic orbits up to any arbitrary precision.

The only algorithm known on the literature capable of computing periodic orbits accurate and highly convergent is the method proposed by D. Viswanath [6] that is based on the Lindstedt-Poincaré technique. To introduce the problem and the new method here proposed, we describe briefly the Viswanath's technique. The problem is to find an isolated orbit of the dynamical system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^n$. Rescaling time using $\tau = \omega t$, we have the following one $\omega\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x})$. The starting guesses, ω_0 and $\mathbf{x}_0(\tau)$ must be sufficiently close to the periodic orbit. The aim is to improve approximations for ω_i and $\mathbf{x}_i(\tau)$ and each iteration is made up of a sequence of steps. Let $\omega_0\dot{\mathbf{y}}(\tau) = A(\tau)\mathbf{y} + \mathbf{r}(\tau) - \delta\omega\dot{\mathbf{x}}_0(\tau)$ be the *correction equation*, then compute the Fourier series for all n^2 entries of $A(\tau)$, n Fourier series for the residual $\mathbf{r}(\tau)$ and another n Fourier series for $\dot{\mathbf{x}}_0(\tau)$. The general solution of the above equation is written as, $\mathbf{y}(\tau) = Y(\tau)\mathbf{y}(0) + f_1(\tau) - \delta\omega f_2(\tau)$, where $Y(\tau)$ is the Fundamental solution of $\omega_0\dot{\mathbf{y}}(\tau) = A(\tau)\mathbf{y}$. We take into consideration that $Y(\tau)$, $f_1(\tau)$ and $f_2(\tau)$ are computed by using an accurate ODE solver in double precision. So, to obtain an arbitrary precision periodic orbit this algorithm uses several numerical techniques in a sophisticated way to use just double precision in the numerical integration of the ODE system.

As remarked, the method of D. Viswanath avoids the use of the integration of ODEs in multiple precision, but at the price of using a complicated algorithm. Therefore, we have tried

to develop a new algorithm for computing periodic orbits using a multiple precision ODE integrator. This method is described in the next section. Our algorithm is based on an optimized shooting method combined with TIDES (Taylor Integrator for Differential EquationS). This tool is an accurate numerical ODE integrator which allows us to integrate in multiple precision arithmetic. We remark that nowadays this method, the Taylor series method, is the only capable method to integrate and ODE system up to any desired precision level (any Runge-Kutta or similar numerical method for ODEs cannot be used for such a high-precision).

§2. The corrector algorithm

Let

$$\mathbf{x} = \mathbf{x}(t; \mathbf{y}), \quad t \in \mathbb{R}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \tag{1}$$

be the solution of the autonomous differential system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}); \quad \mathbf{x}(0) = \mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{2}$$

where $\mathbf{y}$ represents the initial conditions.

The solution of (2) is periodic if it verifies the periodicity condition

$$\mathbf{x}(T, \mathbf{y}) - \mathbf{y} = 0. \tag{3}$$

The Newton method is a common procedure to find the roots of this equation. Our algorithm is an iterative scheme that begins with a set $(\mathbf{y}_0, T_0)$ of approximate initial conditions. At each iteration we update the initial conditions $(\mathbf{y}_i, T_i)$ by adding them the corrections $(\Delta\mathbf{y}_i, \Delta T_i)$ that are obtained by expanding

$$\mathbf{x}(T_i + \Delta T_i; \mathbf{y}_i + \Delta\mathbf{y}_i) - (\mathbf{y}_i + \Delta\mathbf{y}_i) = 0,$$

in a Taylor series up to the first order

$$\mathbf{x}(T_i; \mathbf{y}_i) - \mathbf{y}_i + \left(\frac{\partial \mathbf{x}}{\partial \mathbf{y}} - I\right)\Delta\mathbf{y}_i + \left(\frac{\partial \mathbf{x}}{\partial t}\right)\Delta T_i = 0. \tag{4}$$

The $n \times n$ matrix $\partial\mathbf{x}/\partial\mathbf{y}$ is the fundamental matrix, i.e. the solution of the variational equations. This matrix evaluated at $(\mathbf{y}_i, T_i)$ is an approximation of the monodromy matrix M. I is the identity matrix of order n. The column vector $\partial\mathbf{x}/\partial\mathbf{t}$ represents the derivative of the solution with respect to the time, i.e., $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. This vector, evaluated in $(\mathbf{y}_i, T_i)$ corresponds to the expression $\mathbf{f}(\mathbf{y}_{T_i})$, where $\mathbf{y}_{T_i} = \mathbf{x}(T_i, \mathbf{y}_i)$. To do that, we use the accurate numerical ODE integrator TIDES [1] that computes simultaneously both, the solution and the partial derivatives of the solutions of (2). So, the previous equation is equivalent to the next one

$$(M - I)\Delta\mathbf{y}_i + \mathbf{f}(\mathbf{y}_{T_i})\Delta T_i = -(\mathbf{y}_{T_i} - \mathbf{y}_i). \tag{5}$$

To solve this linear system, it must take into account that varying $\Delta\mathbf{y}_i$ along the periodic orbit gives different representations of the same periodic orbit. Therefore, we impose the additional requirement that $\Delta\mathbf{y}_i$ must be orthogonal to the vector field at $\mathbf{y}_i$; i.e.,

$$\langle \mathbf{f}(\mathbf{y}_i), \Delta\mathbf{y}_i \rangle = 0. \tag{6}$$

2.1. Dissipative case

Equations (5) and (6) are written in a matrix form of dimension $(n+1)\times(n+1)$,

$$\begin{pmatrix} M-I & \mathbf{f}(\mathbf{y}_{T_i}) \\ (\mathbf{f}(\mathbf{y}_i))^T & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{y}_i \\ \Delta T_i \end{pmatrix} = \begin{pmatrix} \mathbf{y}_i - \mathbf{y}_{T_i} \\ 0 \end{pmatrix}. \tag{7}$$

In order to obtain the corrections $\Delta\mathbf{y}_i$ and ΔT_i, we use an iterative scheme which solves the linear system by using the Singular Value Decomposition Algorithm (SVD) [5] although it may use any known solver method for linear systems as the matrix is a non-singular square matrix.

2.2. Hamiltonian case

When the differential system (2) admits one or more integrals, a new constrain or vector of constrains, respectively, must be added to the periodicity condition (3). To maintain the new constrain, $\mathbf{G}(t;\mathbf{x}) = \mathbf{g}$, we impose the condition

$$\mathbf{G}(T_i+\Delta T_i;\mathbf{y}_i+\Delta\mathbf{y}_i)-\mathbf{g} \approx \mathbf{G}(T_i;\mathbf{y}_i)-\mathbf{g}+\left.\frac{\partial\mathbf{G}}{\partial\mathbf{x}}\right|_{(T_i;\mathbf{y}_i)}\Delta\mathbf{y}_i+\left.\frac{\partial\mathbf{G}}{\partial t}\right|_{(T_i;\mathbf{y}_i)}\Delta T_i = 0.$$

In a Hamiltonian problem we have the integral of energy $\mathcal{H}(\mathbf{x}) = H$. So, in this case, we add the following condition to the above linear system,

$$(\nabla_{\mathbf{x}}\mathcal{H})|_{(T_i;\mathbf{y}_i)}\,\Delta\mathbf{y}_i + (\mathcal{H}_t)|_{(T_i;\mathbf{y}_i)}\,\Delta T_i = H - H_{T_i}.$$

Taking into account that the Hamiltonian does not depend on the time, the second term of the addition is cancelled. So, the constrain condition has the form

$$(\nabla_{\mathbf{x}}\mathcal{H})|_{(T_i;\mathbf{y}_i)}\,\Delta\mathbf{y}_i = H - H_{T_i}. \tag{8}$$

Hamiltonian condition (8) is computed using TIDES and Mathematica's operator gradient, $\nabla_{\mathbf{x}}\mathcal{H}$. The matrix of the new linear system has dimension $(n+2)\times(n+1)$. So, we wish to find the least-norm solution to an overdetermined set of linear equations and for this, we use the SVD Algorithm for constructing the singular value decomposition of the matrix. Here, we have for the Hamiltonian case the matrix form,

$$\begin{pmatrix} M-I & \mathbf{f}(\mathbf{y}_{T_i}) \\ (\mathbf{f}(\mathbf{y}_i))^T & 0 \\ (\nabla_{\mathbf{x}}\mathcal{H})|_{(T_i;\mathbf{y}_i)} & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{y}_i \\ \Delta T_i \end{pmatrix} = \begin{pmatrix} \mathbf{y}_i - \mathbf{y}_{T_i} \\ 0 \\ H - H_{T_i} \end{pmatrix}. \tag{9}$$

§3. ODE's, partial derivatives and multiple precision with TIDES

To compute the correction, as well as to solve the linear system (7) and (9), we have to compute the matrix of the systems. For that, we need to integrate the ODE (2) and to compute the partial derivatives of its solution (1) with respect to the initial condition $\mathbf{y}$. To do that we

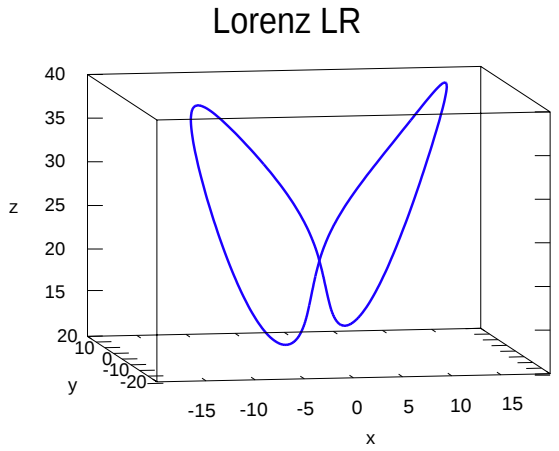

Figure 1: The periodic orbit LR of the Lorenz model.

use the software TIDES [1], that consists of a C library and a MATHEMATICA precompiler that writes a C program which permits to compute simultaneously both, the solution and the partial derivatives of the solution of (2), in double or multiple precision, by using the Taylor Series Method (TSM).

Usually, the matrix of partial derivatives $\mathbf{\Phi} = \partial\mathbf{x}/\partial\mathbf{y}$ of the solution with respect to the initial condition is computed by using the variational equations $\dot{\mathbf{\Phi}} = (\partial\mathbf{f}/\partial\mathbf{x}) \cdot \mathbf{\Phi}$, that are different for each problem and sometimes very difficult to formulate. In TIDES, instead of formulate the variational equations, we use the Taylor series expression

$$\mathbf{x}(t) = \sum_i \mathbf{x}^{[i]}\, h^i, \quad h = t - t_0, \quad \mathbf{x}^{[i]} = \frac{1}{i!}\frac{d\mathbf{x}^{(i)}(t_0)}{dt^i},$$

to create iterative formulas to compute simultaneously both, the solution and the partial derivatives. This simplifies the process and permits to extend it to any differential equation and work with any precision without difficulties. Obviously, to use the Taylor series method the second member of the differential equations has to be a smooth enough function.

§4. Tests

This method has proved its applicability with two paradigmatic examples, Lorenz model and Hénon-Heiles Hamiltonian. The classical Lorenz model is given by the ordinary differential equation

$$\dot{x} = \sigma(y - x), \quad \dot{y} = -xz + rx - y, \quad \dot{z} = xy - bz. \tag{10}$$

In this work, we will take the classical Saltzman values of the parameters $b = 8/3$, $\sigma = 10$ and $r = 28$ and the initial conditions $(x, y, z) = (-13.764, -19.579, 27)$ and a period $T = 1.5586$ (with just five correct digits). So, we have computed the LR periodic orbit up to one hundred

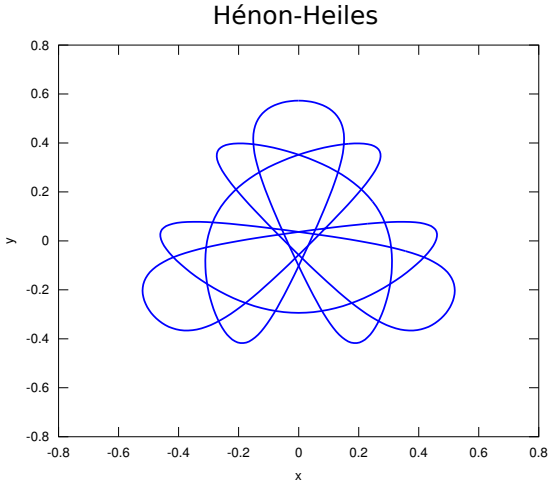

Figure 2: Stable orbit of the Hénon-Heiles problem.

digits of precision and we have obtained the next corrected initial conditions:

$$\begin{aligned} x &= -13.763809685186058958073230618459671664631238848297 \\ &\quad 7262250012134287600807969160127487947892682627184 6, \\ y &= -19.578732026230613926743618660803430026955625649678 3 \\ &\quad 6659773539464894683802943693730174080864746261638, \\ z &= 27.000675803239826810615080341095213706029740774444411 \\ &\quad 8670671293676283528368654572216408019214409963 86, \\ T &= 1.558652210716174727567870209212696070528480548997243 3 \\ &\quad 9358895215783190198756258880854355851082660142 36. \end{aligned}$$

It is well known that the chaotic attractor of the Lorenz model presents the shape of the wings of a butterfly. There are infinite unstable periodic orbits foilated to this attractor. Emphasize that the periodic orbit is labelled LR (see Figure 1) to indicate the sequence in which it moves, so it does one loop on the left and another one on the right.

On the other hand, the Hénon-Heiles problem is given by the Hamiltonian:

$$\mathcal{H}((x,y),(X,Y)) = \frac{1}{2}(X^2+Y^2) + \frac{1}{2}(x^2+y^2) + x^2 y - \frac{1}{3}y^3. \tag{11}$$

where (x,y) and (X,Y) represent the position and velocity vectors, respectively. In Figure 2, we show a stable periodic orbit with initial conditions $(x, y, X, Y) = (0, 0.5729, 0.2171, 0)$ and period $T = 32.378$ (with just five correct digits). Therefore, we have computed this stable periodic orbit of the Hénon-Heiles problem up to one hundred digits of precision and we have

Lorenz		Hénon-Heiles	
No. of iterations	$-\log_{10}\lvert Error\rvert$	No. of iterations	$-\log_{10}\lvert Error\rvert$
1	2.7520	1	3.6921
2	5.1912	2	6.8717
3	11.8648	3	12.6562
4	23.4265	4	25.3239
5	48.3370	5	49.9962
6	96.2892	6	98.9237

Table 1: Error estimates in each iteration of the algorithm.

obtained the next corrected initial conditions:

$$
\begin{aligned}
x &= -0.000150852895948940244967994122847932917483952699107544563475305500642961278934915784918583104063109323 7,\\
y &= 0.572953013702243503485047780986604731598634850331710532006078094336809906598678277432868844316463124 3,\\
X &= 0.217061265411614242231713359875990249988665243558856472953144979955559363023951319004682831231660212 2,\\
Y &= 0.000170045774300978394917283712731703737938515537398104937187223860955588249283317582514765005808513602 0,\\
T &= 32.377740342141170771017492618542347145372047305088163047770250177582275991709264013775490885582548 81.
\end{aligned}
$$

The computational complexity of the numerical solution of an ODE system using a TSM as TIDES with $D = -\log_{10}(TOL)$ number of digits is $O(D^4)$, using variable-precision arithmetic up to one hundred digits of precision, variable-order and variable-stepsize. Moreover, it is well known that the Newton method has quadratic convergence, so the previous algorithm which has been presented in the second section, is quadratically convergent too. We achieve the preset tolerance in six iterations with about one hundred digits of fixed precision arithmetic for both, the Lorenz model and the Hénon-Heiles Hamiltonian. As we can see in the Table 1, the number of digits of precision in the initial conditions of the periodic orbits is doubled at each iteration.

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