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# NUMERICAL SIMULATION OF ANISOTHERMAL NEWTONIAN FLOWS

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**Abstract.** We are interested in the finite element approximation of the Navier-Stokes equations with variable density and with heat transfer. We discuss the choice of compatible discretizations and we investigate the stability of the Jacobian matrix in a simplified framework. We propose to introduce the mass flux and to use Raviart-Thomas elements for its discretization, nonconforming elements for the velocity and a DG method for the temperature. Finally, some numerical tests are presented.

*Keywords:* Compressible Navier-Stokes equations, anisothermal flow, finite elements, stabilization.

*AMS classification:* 65M60, 76M10, 80A20.

## §1. Introduction

We are interested here in the approximation of 2D anisothermal flows for Newtonian fluids. The governing equations are the momentum, mass and energy conservation laws together with the constitutive equation and a state equation, in a polygonal domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \rho\left(\dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) - \operatorname{div}\underline{\tau} + \nabla p = \mathbf{f}, \\ \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0, \\ \rho C_p\left(\dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right) - k\triangle T = Q, \\ \rho = \rho\,(p,T)\,, \\ \underline{\tau} = 2\eta\underline{D}(\mathbf{v}). \end{cases} \tag{1}$$

We close the system by imposing initial and boundary conditions. The unknowns are the velocity $\mathbf{v}$, the stress tensor $\underline{\tau}$, the pressure $p$, the temperature $T$ and the density $\rho$. The viscosity $\eta$, the thermal conductivity $k$ and the heat capacity $C_p$ are given constants.

This preliminary study is devoted to the development of a stable finite element approximation of problem (1) and to its implementation in the C++ library Concha. The further goal is the extension to more complex anisothermal flows, for instance to compressible gases or to viscoelastic non-Newtonian fluids. Therefore, we propose to keep the density as an unknown of the problem in order to allow the treatment of different state equations, such as $p = \rho RT$ for a gas or $\rho = \rho_0\,(1-\beta\,(T-T_0))$ for a polymeric liquid, with $R$ the gas constant, $\beta$ the dilatation coefficient and $\rho_0$ , $T_0$ some reference values.

As regards the constitutive law, it is obvious that in the Newtonian case the stress tensor can be eliminated from the equations, which is no longer possible when dealing with non-Newtonian fluids. For instance, for the polymeric liquids which we want to treat in the future the constitutive law can be usually written as follows:

$$\lambda\left(\frac{\partial}{\partial t}\underline{\tau} + \mathbf{v}\cdot\nabla\underline{\tau} - \underline{\tau}\nabla\mathbf{v}^T - \nabla\mathbf{v}\underline{\tau}\right) + \underline{\tau} + f(\underline{\tau}) = 2\eta\underline{D}(\mathbf{v})$$

and yields, at constant density and constant temperature, a three-fields formulation in $(\mathbf{v}, p, \underline{\tau})$. This aspect has been treated in the incompressible isothermal case in [3]. Here, we only focus on the velocity-pressure formulation for Newtonian fluids.

## §2. Choice of compatible discretizations

We present in the sequel some numerical difficulties related to the approximation of (1), as well as our choice of discretization.

### 2.1. Incompressible Navier-Stokes equations

We begin by considering the stationary Stokes equations:

$$\begin{cases} -\eta\triangle\mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \end{cases} \tag{2}$$

with homogeneous Dirichlet boundary conditions, for simplicity of presentation.

Its finite element discretization is very well studied in the literature and several methods exist, each one with its own advantages and disadvantages. Thus, one may employ finite element spaces for the velocity and the pressure which satisfy an inf-sup condition (see [2] for a review), or choose the two discrete spaces independently but then add a stabilization term in order to ensure the uniform coercivity of the matrix. Completely discontinuous discrete spaces can also be employed, leading to a discontinuous Galerkin (DG) method which is known to be flexible but quite expensive from a computational point of view.

Among the inf-sup stable spaces, there are the conforming and the nonconforming approximations. We have chosen to use here low-order nonconforming finite elements either on triangles or on quadrilaterals, due to their well-known stability and their reduced stencil. Note that in the triangular case, the mass matrix is diagonal and we recover a divergence free discrete velocity. These spaces also present certain advantages concerning the adaptivity. We are using Crouzeix-Raviart [1] elements on triangles, respectively Rannacher-Turek [6] elements on quadrilaterals, whose degrees of freedom are the mean values across the edges. The finite dimensional spaces for the velocity are defined as follows:

$$\mathbf{V}_h^{CR} = \left\{\mathbf{v}\in\mathbf{L}^2(\Omega)\,;\ \forall\, T\in\mathcal{T}_h,\ \mathbf{v}|_T\in\mathbf{P}_1,\ \forall\, e\in\mathcal{S}_h,\ \int_e[\mathbf{v}]\,\mathrm{ds} = 0\right\},$$

$$\mathbf{V}_h^{RT} = \left\{\mathbf{v}\in\mathbf{L}^2(\Omega)\,;\ \forall\, T\in\mathcal{T}_h,\ \mathbf{v}|_T\in\mathbf{Q}_T,\ \forall\, e\in\mathcal{S}_h,\ \int_e[\mathbf{v}]\,\mathrm{ds} = 0\right\},$$

where $\mathbf{Q}_T = (Q_T)^2$ with $Q_T = \{v\,;\, v \circ \Psi_T \in \hat{Q}^{\text{rot}}\}$, $\hat{Q}^{\text{rot}} = \text{vect}\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}$ and $\Psi_T : \hat{T} \to T$ the bilinear one-to-one transformation of the square $\hat{T} = [-1, 1]^2$. We employ the usual notation $[\cdot]$ for the jump across en edge $e \in \mathcal{S}_h$ of the triangulation; the jump is equal to the trace if $e \subset \partial\Omega$. The pressure is looked for in

$$M_h = \left\{p \in L^2_0(\Omega);\ \forall\, T \in \mathcal{T}_h,\ p_{|T} \in P_0\right\}.$$

As regards now the instationary Navier-Stokes equations, it is well-known that the discretization of the additionnal nonlinear term $\mathbf{v} \cdot \nabla\mathbf{v}$ is more delicate since it necessitates stabilization. Several schemes such as SUPG, LPS or edge stabilization were proposed in the literature and are implemented in the library Concha. The approximation of the time derivative $\partial\mathbf{v}/\partial t$ is more standard, and several schemes (implicit and explicit Euler, Crank-Nicolson, BDF) are available in Concha. These aspects will be detailed in the next section, since their treatment is specific to the change of variables that we propose in the compressible case.

## 2.2. Compressible Navier-Stokes equations

The density $\rho$ is now an additionnal unknown, and we have to solve the following system :

$$\begin{cases} \rho\left(\dfrac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) - \eta\triangle\mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \dfrac{\partial\rho}{\partial t} + \text{div}(\rho\mathbf{v}) = 0 & \text{in } \Omega, \\ \rho = \rho(p) & \text{in } \Omega. \end{cases} \tag{3}$$

The numerical treatment of the convective term $\rho\mathbf{v}\cdot\nabla\mathbf{v}$ necessitates the design of adequate stabilization techniques, and is still an active and open research topic. To tackle it, we have chosen here to introduce the mass flux as an auxiliary variable $\mathbf{G} = \rho\mathbf{v}$ belonging to $\mathbf{H}(\text{div}, \Omega)$. For its discretization, we employ lowest-order Raviart-Thomas finite elements (see [7]), which are known to be $\mathbf{H}(\text{div}, \Omega)$-conforming. More precisely, $\mathbf{G}_h$ is looked for in the space $\mathbf{W}_h = \{\mathbf{w} \in \mathbf{H}(\text{div}, \Omega)\,;\, \forall\, T \in \mathcal{T}_h,\ \mathbf{w}|_T \in \mathbf{RT}_0\}$ where $\mathbf{RT}_0$ is defined as follows: $\mathbf{RT}_0 = \mathbf{P}_0 \oplus \mathbf{x}P_0$ on triangles, respectively $\mathbf{RT}_0 = P_1[x] \times P_1[y]$ on quadrilaterals. The degrees of freedom are the normal fluxes across the edges of the triangulation. It is useful to recall that the interpolation operator $\mathcal{E}_h$ of [7] satisfies, besides classical errors estimates, the following properties on every $T \in \mathcal{T}_h$ and $e \in \mathcal{S}_h$ respectively:

$$\text{div}\,(\mathcal{E}_h\mathbf{w}) = \pi_0\,(\text{div}\,\mathbf{w})\,, \qquad \mathcal{E}_h\mathbf{w}\cdot\mathbf{n} = \pi_0(\mathbf{w}\cdot\mathbf{n}),$$

where $\pi_0$ is the $L^2$-orthogonal projection on $P_0$.

We are next interested in the stability of the discrete steady problem. To highlight the structure of the corresponding operator, let us consider a simple state equation, let's say $\rho = C\,p$ with $C$ constant:

$$\begin{cases} -\eta\triangle\mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ -Cp\mathbf{v} + \mathbf{G} = \mathbf{0} & \text{in } \Omega, \\ \text{div}\,\mathbf{G} = 0 & \text{in } \Omega. \end{cases}$$

We apply Newton's method and at each iterate, we obtain the variational formulation:

$$\begin{cases} (\mathbf{v}_h,\ \mathbf{G}_h,\ p_h) \in \mathbf{V}_h \times \mathbf{W}_h \times M_h, \\ \forall \mathbf{v}'_h \in \mathbf{V}_h,\ \eta \displaystyle\int_{\Omega_h} \nabla \mathbf{v}_h \cdot \nabla \mathbf{v}'_h \,\mathrm{dx} - \int_\Omega p_h \operatorname{div} \mathbf{v}'_h \,\mathrm{dx} = \int_\Omega \mathbf{f} \cdot \mathbf{v}'_h \,\mathrm{dx}, \\ \forall \mathbf{G}'_h \in \mathbf{W}_h,\ -C \displaystyle\int_\Omega p^n_h \mathbf{v}_h \cdot \mathbf{G}'_h \,\mathrm{dx} - C \int_\Omega p_h \mathbf{v}^n_h \cdot \mathbf{G}'_h \,\mathrm{dx} + \int_\Omega \mathbf{G}_h \cdot \mathbf{G}'_h \,\mathrm{dx} = 0, \\ \forall p'_h \in M_h,\ \displaystyle\int_\Omega p'_h \operatorname{div} \mathbf{G}_h \,\mathrm{dx} = 0. \end{cases}$$

The corresponding Jacobian matrix can be written as follows:

$$\mathcal{J} = \begin{pmatrix} A & 0 & \vdots & B_1 \\ A_1 & I & \vdots & B_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & B_3 & \vdots & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B}_1 \\ \mathcal{B}_2 & 0 \end{pmatrix},$$

with $\mathcal{B}_1 \neq \mathcal{B}_2^T$ and $\mathcal{A}$ non-symmetric. In order to show that $\mathcal{J}$ is invertible, we shall apply a generalization of the Babuska-Brezzi theorem which was given by Nicolaides in [5]. We have then to check three discrete inf-sup conditions on $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{A}$ respectively, the latter one on $\operatorname{Ker}\mathcal{B}_2 \times \operatorname{Ker}\mathcal{B}_1$.

**Proposition 1.** *There exists $\beta_1 > 0$ independent of h such that*

$$\inf_{p \in M_h} \sup_{(\mathbf{v},\mathbf{G}) \in \mathbf{V}_h \times \mathbf{W}_h} \frac{-\int_\Omega p \operatorname{div} \mathbf{v} \,\mathrm{dx} - C \int_\Omega p \mathbf{v}^n_h \cdot \mathbf{G} \,\mathrm{dx}}{\|p\|_{0,\Omega} \left(|\mathbf{v}|_{1,h} + \|\mathbf{G}\|_{\mathbf{H}(\mathrm{div},\Omega)}\right)} \geq \beta_1.$$

*Proof.* The proof is identical to the one of the classical inf-sup condition for the two-fields formulation of the Stokes problem on $\mathbf{V}_h \times M_h$ (see for instance [2]), by taking $\mathbf{G} = \mathbf{0}$. □

**Proposition 2.** *There exists $\beta_2 > 0$ independent of h such that*

$$\inf_{p \in M_h} \sup_{\mathbf{G} \in \mathbf{W}_h} \frac{-\int_\Omega p \operatorname{div} \mathbf{G} \,\mathrm{dx}}{\|p\|_{0,\Omega} \|\mathbf{G}\|_{\mathbf{H}(\mathrm{div},\Omega)}} \geq \beta_2.$$

*Proof.* The proof is well-known, see [7]. For $p \in M_h$, one considers the auxiliary problem:

$$\begin{cases} -\triangle z = p & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

and takes $\mathbf{w} = \nabla z$ which belongs, thanks to the regularity of the Laplace operator, to $\mathbf{H}^a(\Omega)$ with $a > 1/2$. Let then the Raviart-Thomas interpolate $\mathbf{G} = \mathcal{E}_h \mathbf{w}$. According to the properties of $\mathcal{E}_h$, one has $\operatorname{div} \mathbf{G} = -p$ and $\|\mathbf{G}\|_{\mathbf{H}(\mathrm{div},\Omega)} \leq c\,\|p\|_{0,\Omega}$, which implies the uniform inf-sup condition on $\mathcal{B}_2$. □

**Proposition 3.** *There exists $\alpha > 0$ such that:*

$$\inf_{(\mathbf{v},\mathbf{G})\in\operatorname{Ker}\mathcal{B}_2} \sup_{(\mathbf{v}',\mathbf{G}')\in\operatorname{Ker}\mathcal{B}_1} \frac{\mathcal{A}((\mathbf{v},\mathbf{G}),(\mathbf{v}',\mathbf{G}'))}{\|(\mathbf{v},\mathbf{G})\|_{\mathbf{H}^1_0(\Omega)\times\mathbf{H}(\mathrm{div},\Omega)} \|(\mathbf{v}',\mathbf{G}')\|_{\mathbf{H}^1_0(\Omega)\times\mathbf{H}(\mathrm{div},\Omega)}} \geq \alpha,$$

$$\inf_{(\mathbf{v}',\mathbf{G}')\in\operatorname{Ker}\mathcal{B}_1} \sup_{(\mathbf{v},\mathbf{G})\in\operatorname{Ker}\mathcal{B}_2} \frac{\mathcal{A}((\mathbf{v},\mathbf{G}),(\mathbf{v}',\mathbf{G}'))}{\|(\mathbf{v},\mathbf{G})\|_{\mathbf{H}^1_0(\Omega)\times\mathbf{H}(\mathrm{div},\Omega)} \|(\mathbf{v}',\mathbf{G}')\|_{\mathbf{H}^1_0(\Omega)\times\mathbf{H}(\mathrm{div},\Omega)}} > 0.$$

*Proof.* These two inf-sup conditions translate the fact that the matrix $\mathcal{A}$ is invertible on $\operatorname{Ker}\mathcal{B}_2 \times \operatorname{Ker}\mathcal{B}_1$. Since $\mathcal{A} = \left(\begin{smallmatrix} A & 0 \\ A_1 & I \end{smallmatrix}\right)$ is block triangular, it is therefore sufficient to show the invertibility of $A$. Thanks to the discrete Poincaré inequality on the nonconforming spaces, $A$ is uniformly invertible on the whole space $\mathbf{V}_h$. Thus, the statement is established. □

Let us now discuss the complete system (3). One may choose between two options: write the particular derivative of the first equation in conservative form $\partial\mathbf{G}/\partial t + \operatorname{div}(\mathbf{G}\otimes\mathbf{v})$, or keep $\rho\partial\mathbf{v}/\partial t + (\mathbf{G}\cdot\nabla)\mathbf{v}$. We have chosen here the latter variant. For the discretization of the convective term, we propose the stabilization:

$$\int_{\mathcal{T}_h} (\mathbf{G}_h\cdot\nabla)\mathbf{v}_h\cdot\mathbf{v}'_h\,\mathrm{dx} \approx -\int_{\mathcal{T}_h}\Big((\operatorname{div}\mathbf{G}_h)\mathbf{v}_h\cdot\mathbf{v}'_h + (\mathbf{G}_h\cdot\nabla)\mathbf{v}'_h\cdot\mathbf{v}_h\Big)\mathrm{dx} + \int_{\mathcal{S}_h}\mathbf{F}_e(\mathbf{G}_h,\mathbf{v}_h)\cdot[\mathbf{v}'_h]\;\mathrm{ds},$$

where $\mathbf{F}_e(\mathbf{G}_h,\mathbf{v}_h) = (\mathbf{G}_h\cdot\mathbf{n}_e)^+\,\mathbf{v}_h^{\mathrm{in}} + (\mathbf{G}_h\cdot\mathbf{n}_e)^-\,\mathbf{v}_h^{\mathrm{ex}}$ represents the numerical flux and $\mathbf{n}_e$ is a unit normal to the edge $e$ . For a given piecewise continuous function $\varphi$, we have denoted $\varphi^{\mathrm{ex}}(\mathbf{x}) = \lim_{\varepsilon\to 0}\varphi(\mathbf{x}-\varepsilon\mathbf{n}_e)$, $\varphi^{\mathrm{in}}(\mathbf{x}) = \lim_{\varepsilon\to 0}\varphi(\mathbf{x}+\varepsilon\mathbf{n}_e)$ and $[\varphi] = \varphi^{\mathrm{ex}}-\varphi^{\mathrm{in}}$. Then we end up with another matrix $\mathcal{A}^* = \left(\begin{smallmatrix} A^* & A_2 \\ A_1 & I \end{smallmatrix}\right)$ instead of $\mathcal{A}$, for which the inf-sup conditions of Proposition 3 should be established. Note that for $\operatorname{div}\mathbf{G}_h = 0$, the diffusion-convection operator $A^*$ is uniformly coercive on $\mathbf{V}_h$ since one can show that

$$A^*(\mathbf{v}_h,\mathbf{v}_h) = \eta\,|\mathbf{v}_h|^2_{1,h} + \frac{1}{2}\int_{\mathcal{S}_h}|\mathbf{G}_h\cdot\mathbf{n}_e|\,[\mathbf{v}_h]\cdot[\mathbf{v}_h]\;\mathrm{ds}.$$

For the discretization of the time derivative $\rho\partial\mathbf{v}/\partial t$, we have employed the BDF (*Backward Differential Formula*) scheme of order 2, for its robustness and stability. The variable at $t_{n+1}$ is expressed in terms of the solutions at the two previous time steps as follows:

$$\rho_{n+1}\frac{\partial\mathbf{v}_{n+1}}{\partial t} \approx \rho_{n+1}\left(\frac{1}{\triangle t}\left(\frac{3}{2}\mathbf{v}_{n+1} - 2\mathbf{v}_n + \frac{1}{2}\mathbf{v}_{n-1}\right) + O\left(\triangle t^2\right)\right).$$

The coercivity of the diagonal blocks corresponding to the velocity and the pressure is thus enhanced, but the block $\mathcal{B}_1$ is also modified and a new inf-sup condition should be satisfied.

## 2.3. Anisothermal flow

Taking into account the thermodynamics is essential in order to obtain realistic simulations.

The energy equation is convection-dominated due to the large value of the heat capacity coefficient $C_p$. We have chosen to employ a DG method for its discretization, which is known to be well-adapted to such problems (see for instance Lesaint and Raviart [4]). In order to reduce the computational cost and also because $k \ll 1$ while $\rho C_p \approx 10^6$, we use here piecewise constant elements for $T$. Thus, the discrete diffusion operator on $T$ is reduced to the stabilization term on the edges while the convective term $\mathbf{G}\cdot\nabla T$ is approximated similarly to [4]. The density is approximated by the same finite elements as the temperature.

For the analysis of the corresponding discrete problem, one could apply twice the general results of [5]. To illustrate this, let us consider for the sake of simplicity the steady case and let us neglect the convection in the momentum law. Then the governing equations are:

$$\begin{cases} -\eta\triangle\mathbf{v} + \nabla p = \mathbf{f}, \\ \mathbf{G} - \rho\mathbf{v} = \mathbf{0}, \\ -k\triangle T + C_p\mathbf{G}\cdot\nabla T = 0, \\ \rho + \rho_0\beta T = \rho_0\left(1 + \beta T_0\right), \\ \operatorname{div}\mathbf{G} = 0, \end{cases}$$

and the Jacobian matrix of the discrete problem in the unknowns $(\mathbf{v}, \mathbf{G}, T, \rho, p)$ can be written as follows

$$\mathcal{J}' = \begin{pmatrix} \mathcal{A}' & \mathcal{B}_1' \\ \mathcal{B}_2' & 0 \end{pmatrix}, \quad \text{with:} \quad \mathcal{A}' = \begin{pmatrix} A & 0 & 0 & 0 \\ A_1 & I & 0 & B_1 \\ 0 & B_2 & D & 0 \\ 0 & 0 & B_3 & I \end{pmatrix}, \quad \mathcal{B}_1' = \begin{pmatrix} B_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{B}_2' = \begin{pmatrix} 0 \\ B_5 \\ 0 \\ 0 \end{pmatrix}^T.$$

The inf-sup conditions on $\mathcal{B}_1'$ and $\mathcal{B}_2'$ are the same as in the previous section, so one only has to check the inf-sup condition for $\mathcal{A}'$ on $\operatorname{Ker}\mathcal{B}_2' \times \operatorname{Ker}\mathcal{B}_1'$ in order to conclude that $\mathcal{J}'$ is invertible. For this purpose, one can decompose $\mathcal{A}'$ in mixed form as follows

$$\mathcal{A}' = \begin{pmatrix} \mathcal{A}'' & \mathcal{B}_1'' \\ \mathcal{B}_2'' & C \end{pmatrix},$$

where

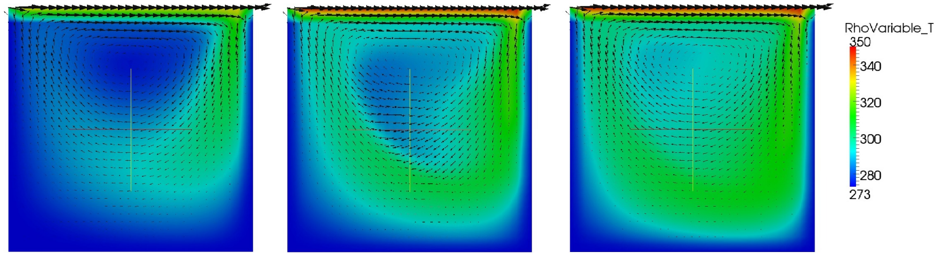
$$\mathcal{A}'' = \begin{pmatrix} A & 0 & 0 \\ A_1 & I & 0 \\ 0 & B_2 & D \end{pmatrix}, \quad \mathcal{B}_1'' = \begin{pmatrix} 0 \\ B_1 \\ 0 \end{pmatrix}, \quad \mathcal{B}_2'' = \begin{pmatrix} 0 \\ 0 \\ B_3 \end{pmatrix}^T, \quad C = I.$$

Since $C$ is clearly positive definite, we can establish inf-sup conditions for $\mathcal{A}''$, $\mathcal{B}_1''$ and $\mathcal{B}_2''$. Note that the latter is obvious, since $B_3$ corresponds to $\int_\Omega(\rho_0\beta)T\rho\,\mathrm{dx}$. Moreover, $A$ being the nonconforming diffusion operator on $\mathbf{v}$ and $D$ the DG diffusion-convection operator on $T$, they are uniformly coercive, so that the block triangular matrix $\mathcal{A}''$ is clearly invertible.

In the unsteady case, the time-discretization enforces the coercivity of the diagonal blocks $A$ and $D$, but also modifies the bilinear forms $\mathcal{B}_2'$ and $\mathcal{B}_1''$. Finally, when taking into account the convective term $\mathbf{G}\cdot\nabla\mathbf{v}$, the first line of $\mathcal{A}''$ is modified and the matrix is no longer block triangular; nevertheless, the new diagonal block $A$ is still uniformly coercive.

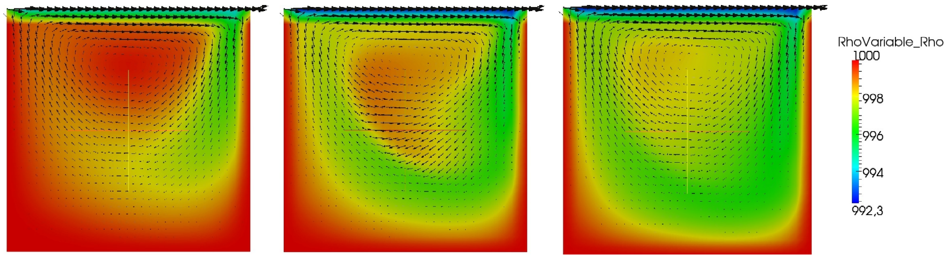
## §3. Numerical experiments

We present some of our first numerical results, carried out on two academic tests. Two different fluids have been considered, a polymer with a high viscosity and a liquid with physical




(a) $t = 60\,\mathrm{s}$ (b) $t = 90\,\mathrm{s}$ (c) $t = 120\,\mathrm{s}$

Figure 1: Polymer flow: temperature at different time steps




(a) $t = 60\,\mathrm{s}$ (b) $t = 90\,\mathrm{s}$ (c) $t = 120\,\mathrm{s}$

Figure 2: Polymer flow: density at different time steps

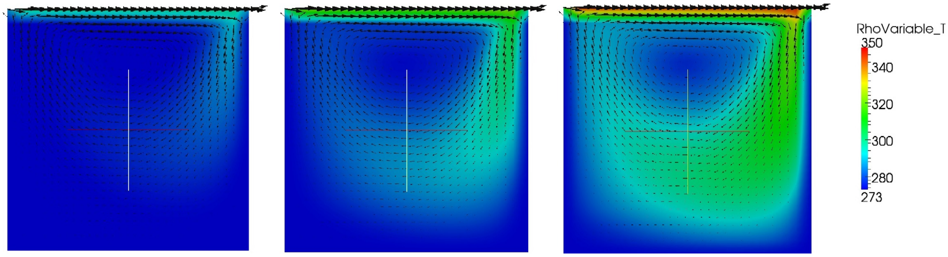
properties similar to those of water. We have taken into account the gravity force and we have considered an affine dependence of the density on the temperature, which corresponds to the case of polymers and which yields the state equation: $\rho = \rho_0\,(1 - \beta\,(T - T_0))$. The next tests are carried out on quadrilateral meshes. The parameters which are common to the numerical experiments are given in the table below:

| Parameter | Value |
|---|---|
| $\rho_0$: initial density | $1000\,\mathrm{kg/m^3}$ |
| $T_0$: initial temperature | $273\,\mathrm{K}$ |
| $\beta$: dilatation coefficient | $10^{-4}\,\mathrm{m^3/kg\cdot K}$ |

## 3.1. Driven cavity: polymer flow

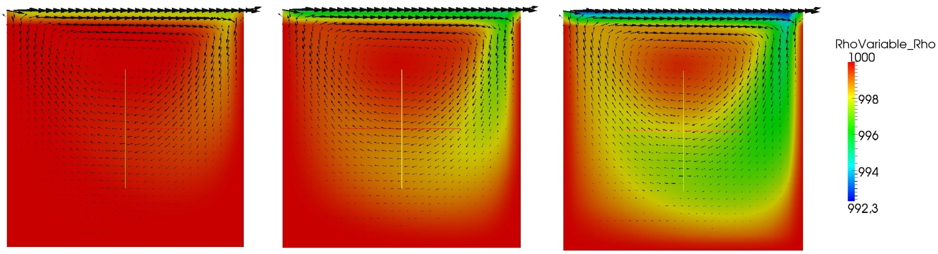
We consider first the driven cavity test in a square $\Omega$. We impose a velocity $\mathbf{v} = (0.03, 0)\,\mathrm{m/s}$ on the top boundary and $\mathbf{0}$ elsewhere, while the temperature equals $350\,\mathrm{K}$ on the top and $273\,\mathrm{K}$ elsewhere. The parameters specific to a polymeric liquid are: the heat capacity $C_p = 2000\,\mathrm{J/kg\cdot K}$, the thermal conductivity $k = 0.05\,\mathrm{W/m\cdot K}$ and the viscosity $\eta = 1000\,\mathrm{Pa\cdot s}$.

One can see in Figures 1 and 2 the evolution of the temperature and of the density. The results of the simulation are physically acceptable. The vortex drags the warm fluid towards the bottom. Due to the gravity force, this one raises slowly to the top and thus it warms the fluid situated between the upper edge and the warm convected fluid.




(a) $t = 25$ s (b) $t = 50$ s (c) $t = 75$ s

Figure 3: Water flow: temperature at different time-steps




(a) $t = 25$ s (b) $t = 50$ s (c) $t = 75$ s

Figure 4: Water flow: density at different time steps

## 3.2. Driven cavity: water flow

The specific parameters are now: $C_p = 4186\,\mathrm{J/kg}\cdot\mathrm{K}$, $k = 0.6\,\mathrm{W/m}\cdot\mathrm{K}$ and $\eta = 0.001\,\mathrm{Pa}\cdot\mathrm{s}$. We show in Figures 3 and 4 the temperature and the density (as well as the velocity field) at different time steps. Since the water has a turbulent flow, the stabilization employed in this case is not so efficient; we couldn't simulate a time interval as long as previously.

## 3.3. Confined flow

The domain is now a rectangle of sides 12 cm and 4 cm. We consider the polymeric liquid previously described and we impose $\mathbf{v} = \mathbf{0}$ on the whole boundary, a constant temperature 273 K on the top and a temperature depending on time and on the abscissa $x$ on the bottom: $T(x,t) = 273 + 100t + 250x$ if $T < 350$ and $T(x,t) = 350$ otherwise. On the vertical boundaries, a homogeneous Neumann condition is set for the temperature.

We show in Figures 5 and 6 the first component of the velocity and the density, as well as the streamlines, at the end of the simulation. Due to the gravity force and to the non-symmetric boundary condition on the bottom, the heat goes up slowly and generates a velocity field.

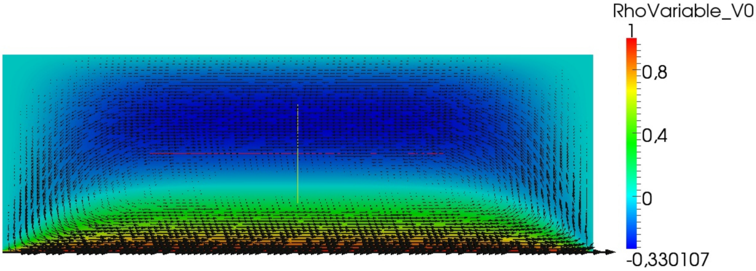



Figure 5: Confined flow: first component of the velocity at the end of simulation

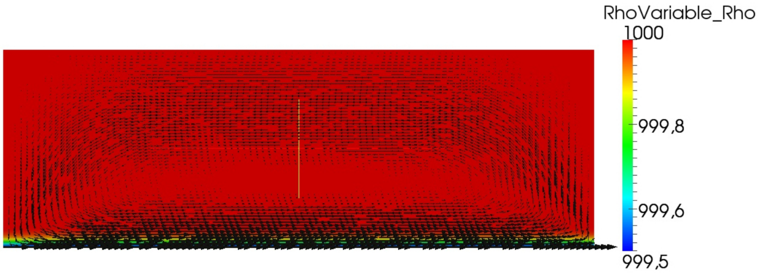



Figure 6: Confined flow: density at the end of simulation

## §4. Conclusion and further developments

The anisothermal Navier-Stokes model set up in this work is a first step towards the numerical simulation of more complex flows with heat tranfer, by using the library Concha. We have proposed a finite element method based on the introduction of an additionnal unknown, the mass flux, and investigated the stability of the Jacobian matrix in a simplified framework. In perspective, this study should be extended to a more general case. It will also be interesting to compare this approach with the classical one, written only in the primitive variables.

Although the considered model presents some simplifications (simplified state equation, absence of viscous dissipation in the energy equation), it contains the main difficulties related to this type of problem: compressibility, turbulent flow, dominant convection, significant number of unknowns etc. From a numerical point of view, its treatment necessitated the enrichment of the library Concha in order to take into account a variable density, as well as the implementation of a specific stabilization for certain nonlinear convective terms.

The first numerical results are encouraging, and show that the code gives physically acceptable results. More numerical experiments and comparisons with other softwares such as PolyFlow® or OpenFoam should be carried out in order to further validate the code. As future improvements, we think of using adaptive time steps, iterative solvers and also a local elimination procedure for the mass flux, which amounts to a different stabilization of $\rho\mathbf{v}\cdot\nabla\mathbf{v}$.

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