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VERY WEAK SOLUTIONS OF STOKES PROBLEM IN EXTERIOR DOMAIN

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Abstract. The existence and the uniqueness of very weak solutions of Stokes system are well known in the classical Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$ when Ω is bounded (see [3]). When Ω is an exterior domain, a similar approach would fail (in particular because Poincare's inequalities do not hold in such domains). Therefore, a specific functional framework based on density arguments is necessary to do this work.

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§1. Introduction

Let Ω' be a bounded connected open domain in $\mathbb{R}^3$ with boundary $\partial\Omega' = \Gamma$ of class $C^{1,1}$ representing an obstacle and let Ω its complement, i.e. $\Omega = \mathbb{R}^3 \setminus \overline{\Omega'}$. In this work, we are interested in the existence and the uniqueness of very weak solution concerning the Stokes problem in exterior domain:

$$-\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = h \ \text{ in } \ \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \ \text{ on } \ \Gamma, \qquad (\mathcal{S})$$

where $\boldsymbol{u}$ denote the velocity and q the pressure and both are unknown, $\boldsymbol{f}$ the external forces, h the compressibility condition and $\boldsymbol{g}$ the boundary condition for the velocity, the three functions being known. This problem is well done in 2005 by R. Farwig [4], with data $\boldsymbol{f} = \operatorname{div}\mathbb{F}_0$, h and $\boldsymbol{g}$ satisfying

$$\mathbb{F}_0 \in \boldsymbol{L}^r(\Omega),\ h \in L^r(\Omega),\ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma),\ 3 < p < \infty,\ \frac{1}{3} + \frac{1}{p} = \frac{1}{r}$$

yielding $\frac{3}{2} < r < 3$.

In this paper, we are interested in the following data:

$$\boldsymbol{f} = \operatorname{div}\mathbb{F}_0 + \nabla f_1, \quad h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma),$$

with

$$\mathbb{F}_0 \in \boldsymbol{L}^r(\Omega),\ f_1 \in W_0^{-1,p}(\Omega),\ \frac{3}{2} < p < \infty, \quad \text{and} \quad \frac{1}{3} + \frac{1}{p} = \frac{1}{r},$$

or

$$\mathbb{F}_0 \in \boldsymbol{W}^{0,r}_{-1}(\Omega),\ f_1 \in W^{-1,p}_{-1}(\Omega) \quad \text{and} \quad h \in W^{0,r}_{-1}(\Omega),$$

with

$$\frac{3}{2} < p < \infty,\ p \neq 3 \quad \text{and} \quad \frac{1}{3} + \frac{1}{p} = \frac{1}{r}.$$

§2. Basic concepts on Sobolev spaces

Let $x = (x_1, x_2, x_3)$ be a typical point in $\mathbb{R}^3$ and let $r = |x| = (x_1^2+x_2^2+x_3^2)^{1/2}$ denote its distance to the origin. We define the weight function $\rho(x) = 1 + r$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $1/p + 1/p' = 1$. Then, for any nonnegative integers m and real numbers $p > 1$ and α, setting

$$k = k(m,p,\alpha) = \begin{cases} -1, & \text{if } \frac{3}{p} + \alpha \notin \{1,\dots,m\}, \\ m - \frac{3}{p} - \alpha, & \text{if } \frac{3}{p} + \alpha \in \{1,\dots,m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha}(\Omega) = \{\, u \in \mathcal{D}'(\Omega);& \\ &\forall\, \lambda \in \mathbb{N}^3 :\ 0 \le |\lambda| \le k,\ \rho^{\alpha-m+|\lambda|}(\ln(1+\rho))^{-1} D^{\lambda} u \in L^p(\Omega); \\ &\forall\, \lambda \in \mathbb{N}^3 :\ k+1 \le |\lambda| \le m, \rho^{\alpha-m+|\lambda|} D^{\lambda} u \in L^p(\Omega)\,\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm:

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha}(\Omega)} = \Bigg(&\sum_{0\le|\lambda|\le k} \|\rho^{\alpha-m+|\lambda|}(\ln(1+\rho))^{-1} D^{\lambda} u\|^p_{L^p(\Omega)} \\ &+ \sum_{k+1\le|\lambda|\le m} \|\rho^{\alpha-m+|\lambda|} D^{\lambda} u\|^p_{L^p(\Omega)} \Bigg)^{1/p}. \end{aligned}$$

We note that the logarithmic weight only appears if $p = 3$ or $p = 3/2$ and all the local properties of $W^{1,p}_0(\Omega)$ (respectively, $W^{2,p}_0(\Omega)$) coincide with those of the corresponding classical Sobolev space $W^{1,p}(\Omega)$ (respectively, $W^{2,p}(\Omega)$). For $m = 1$ or $m = 2$ we set $\mathring{W}^{m,p}_{\alpha}(\Omega)$ as the adherence of $\mathcal{D}(\Omega)$ for the norm $\|\cdot\|_{W^{m,p}_{\alpha}(\Omega)}$. Then, the dual space of $\mathring{W}^{m,p}_{\alpha}(\Omega)$, denoting by $W^{-m,p'}_{-\alpha}(\Omega)$, is a space of distributions. When $\Omega = \mathbb{R}^3$, we have $\boldsymbol{W}^{1,p}_{\alpha}(\mathbb{R}^3) = \mathring{\boldsymbol{W}}^{1,p}_{\alpha}(\mathbb{R}^3)$.

If Ω is a Lipschitz exterior domain, then for $\alpha = 0$ we have

$$\mathring{\boldsymbol{W}}^{1,p}_0(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(\Omega),\ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \partial\Omega \right\},$$

and

$$\mathring{\boldsymbol{W}}^{2,p}_0(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{2,p}_0(\Omega),\ \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = \boldsymbol{0} \ \text{ on } \ \partial\Omega \right\},$$

where $\partial\boldsymbol{v}/\partial\boldsymbol{n}$ is the normal derivate of $\boldsymbol{v}$.

The spaces $\boldsymbol{W}^{1,p}_{\alpha}(\Omega)$ or $\boldsymbol{W}^{2,p}_{\alpha}(\Omega)$ sometimes contain some polynomial functions. We have for $m = 1$ or $m = 2$:

$$\mathcal{P}_j \subset \boldsymbol{W}^{m,p}_{\alpha}(\Omega) \quad \text{with} \quad \begin{cases} j = [m - (3/p + \alpha)], & \text{if } 3/p + \alpha \notin \mathbb{Z}^-, \\ j = -(3/p + \alpha), & \text{otherwise,} \end{cases}$$

where $[s]$ denotes the integer part of the real number s and $\mathcal{P}_j$ is the space of polynomials of degree less then j.

We recall the following Sobolev embeddings for $\alpha = 0$ or $\alpha = 1$

$$W^{1,p}_{\alpha}(\Omega) \hookrightarrow W^{0,p*}_{\alpha}(\Omega) \quad \text{where} \quad p* = \frac{3p}{3-p} \quad \text{and} \quad 1 < p < 3.$$

Consequently, by duality, we have

$$W^{0,q}_{-\alpha}(\Omega) \hookrightarrow W^{-1,p'}_{-\alpha}(\Omega) \quad \text{where} \quad q = \frac{3p'}{3+p'} \quad \text{and} \quad p' > 3/2.$$

On the other hand, if $3/p + \alpha \notin \{1, 2\}$, we have the following continuous embedding:

$$\boldsymbol{W}^{2,p}_{\alpha}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}_{\alpha-1}(\Omega) \hookrightarrow \boldsymbol{W}^{0,p}_{\alpha-2}(\Omega).$$

§3. Preliminary results

In the sequel, we need to introduce the following spaces:

$$\boldsymbol{\mathcal{D}}_{\sigma}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla\cdot\boldsymbol{\varphi} = 0\} \quad \text{and} \quad \boldsymbol{\mathcal{D}}_{\sigma}(\overline{\Omega}) = \left\{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega});\ \nabla\cdot\boldsymbol{\varphi} = 0\right\}.$$

Then, we show some density results that are essential for the proofs below. We begin by introducing the space

$$\boldsymbol{X}^{\ell}_{r,p}(\Omega) = \left\{\boldsymbol{\varphi} \in \mathring{\boldsymbol{W}}^{1,r}_{\ell}(\Omega);\ \nabla\cdot\boldsymbol{\varphi} \in \mathring{W}^{1,p}_{\ell}(\Omega)\right\}.$$

Thank's to Poincaré-type inequality (see [2]), we can equipped this space with the following norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^{\ell}_{r,p}(\Omega)} = \sum_{1\leq i,j\leq 3}\left\|\frac{\partial \boldsymbol{v}_i}{\partial x_j}\right\|_{\boldsymbol{W}^{0,r}_{\ell}(\Omega)} + \|\nabla\cdot\boldsymbol{v}\|_{W^{1,p}_{\ell}(\Omega)}.$$

Lemma 1. *Let Ω be a Lipschitz open set in $\mathbb{R}^3$ and suppose that $0 \leqslant 1/p - 1/r \leqslant 1/3$. We have the following properties:*

i) The space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{X}^1_{r,p}(\Omega)$ and, for all $\boldsymbol{q} \in \boldsymbol{W}^{-1,p}_{-1}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}^1_{r',p'}(\Omega)$, we have

$$\langle\nabla \boldsymbol{q}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{X}^1_{r',p'}(\Omega)]'\times\boldsymbol{X}^1_{r',p'}(\Omega)} = -\langle \boldsymbol{q}, \nabla\cdot\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,p}_{-1}(\Omega)\times\mathring{\boldsymbol{W}}^{1,p'}_{1}(\Omega)}.$$

ii) If in addition $p \neq 3$ and $r \neq 3$, then the space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{X}^0_{r,p}(\Omega)$ and, for all $\boldsymbol{q} \in \boldsymbol{W}^{-1,p}_{0}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}^0_{r',p'}(\Omega)$, we have

$$\langle\nabla \boldsymbol{q}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{X}^0_{r',p'}(\Omega)]'\times\boldsymbol{X}^0_{r',p'}(\Omega)} = -\langle \boldsymbol{q}, \nabla\cdot\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,p}_{0}(\Omega)\times\mathring{\boldsymbol{W}}^{1,p'}_{0}(\Omega)}.$$

Proof. The density of $\mathcal{D}(\Omega)$ in $\boldsymbol{X}^{\ell}_{r,p}(\Omega)$ relies on an adequate truncation procedure and regularization. The truncation function that we shall use has been defined by: $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \varphi(t) \leqslant 1$ for any $t \in \mathbb{R}^3$, and

$$\varphi(t) = \begin{cases} 1, & \text{if } 0 \leqslant |t| \leqslant 1,\\ 0, & \text{if } |t| \geqslant 2.\end{cases}$$

Now, let $\boldsymbol{v} \in \boldsymbol{X}^{\ell}_{r,p}(\Omega)$ and $\widetilde{\boldsymbol{v}}$ be the extension by $\mathbf{0}$ of $\boldsymbol{v}$ to $\mathbb{R}^3$, then we have $\widetilde{\boldsymbol{v}} \in \boldsymbol{X}^{\ell}_{r,p}(\mathbb{R}^3)$. We begin to apply the cut off functions φ_k, defined on $\mathbb{R}^3$ for any $k \in \mathbb{N}^*$, by $\varphi_k(x) = \varphi(x/k)$. Set $\boldsymbol{v}_k = \varphi_k \widetilde{\boldsymbol{v}}$. It is easy to prove that $\boldsymbol{v}_k \to \widetilde{\boldsymbol{v}}$ in $\boldsymbol{X}^{\ell}_{r,p}(\mathbb{R}^3)$ when $k \to \infty$. Now, we start the regularization of our sequence $\boldsymbol{v}_k$. In a first step we consider that Ω' is strictly star-shaped with respect to one of its points which is taken to the origin. Under this assumption, we set $\boldsymbol{v}_{k,\theta}(x) = \boldsymbol{v}_k(\theta x)$ for any real number $\theta > 1$ and $x \in \mathbb{R}^3$. Then $\boldsymbol{v}_{k,\theta} \in \boldsymbol{X}^{\ell}_{r,p}(\mathbb{R}^3)$ and supp $\boldsymbol{v}_{k,\theta}$ is compact in Ω. Moreover

$$\lim_{\theta\to 1} \boldsymbol{v}_{k,\theta} = \boldsymbol{v}_k \ \text{ in } \ \boldsymbol{X}^{\ell}_{r,p}(\mathbb{R}^3).$$

Consequently, for any real number $\epsilon > 0$ small enough, the restriction of $\rho_\epsilon * \boldsymbol{v}_{k,\theta}$ to Ω belongs to $\mathcal{D}(\Omega)$ and

$$\lim_{\epsilon\to 0}\lim_{\theta\to 1}\lim_{k\to\infty} \rho_\epsilon * \boldsymbol{v}_{k,\theta} = \widetilde{\boldsymbol{v}} \ \text{ in } \ \boldsymbol{X}^{1}_{r,p}(\mathbb{R}^3),$$

where ρ_ϵ is a mollifier. Consequently, $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{X}^{\ell}_{r,p}(\Omega)$. In the case where Ω' is only a Lipschitz open set in $\mathbb{R}^3$, we have to recover Ω' by a finite number of star open sets and partitions of unity. Clearly, it suffices to apply the above argument to each of these sets to derive the desired result on the entire domain. □

Remark 1. Observe that for $\boldsymbol{f} \in (\boldsymbol{X}^{\ell}_{r,p}(\Omega))'$ with $\ell = 1$ or $\ell = 0$, there exist $\mathbb{F}_0 = (f_{ij})_{1\le i,j\le 3} \in \boldsymbol{W}^{0,r'}_{-\ell}(\Omega)$ and $f_1 \in W^{-1,p'}_{-\ell}(\Omega)$ such that:

$$\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1. \tag{1}$$

Moreover,

$$\|\boldsymbol{f}\|_{[\boldsymbol{X}^{\ell}_{r,p}(\Omega)]'} = \max\Big\{ \big\|f_{ij}\big\|_{\boldsymbol{W}^{0,r'}_{-\ell}(\Omega)}, 1 \le i,j \le 3, \|f_1\|_{W^{-1,p'}_{-\ell}(\Omega)} \Big\}.$$

Conversely, if $\boldsymbol{f}$ satisfies (1), then $\boldsymbol{f} \in (\boldsymbol{X}^{\ell}_{r,p}(\Omega))'$.

Giving a meaning to the trace of a very weak solution of the Stokes problem is not trivial: remember that we are not in the classical variational framework. In this way, we need to introduce some spaces. First, we consider the space:

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \Big\{ \boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega),\ \boldsymbol{\psi}|_\Gamma = 0, (\nabla\cdot\boldsymbol{\psi})|_\Gamma = 0 \Big\}.$$

The following lemma gives another characterization to the space $\boldsymbol{Y}_{p',\ell}(\Omega)$ very useful in the Green's formula defined in Corolllary 4.

Lemma 2. *We have the identity*

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \left\{ \boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega),\ \boldsymbol{\psi}|_\Gamma = 0, \frac{\partial\boldsymbol{\psi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n}|_\Gamma = 0 \right\} \tag{2}$$

and the range space of the normal derivative $\gamma_1 : \boldsymbol{Y}_{p',\ell}(\Omega) \longrightarrow \boldsymbol{W}^{1/p,p'}(\Gamma)$ *is*

$$\boldsymbol{Z}_{p'}(\Gamma) = \Big\{ \boldsymbol{z} \in \boldsymbol{W}^{1/p,p'}(\Gamma);\ \ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \Big\}.$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega)$ such that $\boldsymbol{u} = \mathbf{0}$ on Γ. Then $\operatorname{div} \boldsymbol{u} = (\partial \boldsymbol{u}/\partial \boldsymbol{n}) \cdot \boldsymbol{n}$ on Γ and the identity (2) holds. Moreover, it is clear that $\operatorname{Im}(\gamma_1) \subset \boldsymbol{Z}_{p'}(\Gamma)$. Conversely, let $\boldsymbol{\mu} \in \boldsymbol{Z}_{p'}(\Gamma)$. As Ω' is bounded, we can fix once for all a ball B_{R_o}, centered at the origin and with radius R_0, such that $\overline{\Omega'} \subset B_{R_o}$. Thus we have the existance of $\boldsymbol{u} \in \boldsymbol{W}^{2,p'}(\Omega_{R_0})$ such that $\boldsymbol{u} = \mathbf{0}$, $\partial \boldsymbol{u}/\partial \boldsymbol{n} = \boldsymbol{\mu}$ on $\Gamma \cup \partial B_{R_o}$ $(\Omega_{R_0} = \Omega \cap B_{R_0})$. The function $\boldsymbol{u}$ can be extended by zero outside B_{R_o} and owing to its boundary conditions on ∂B_{R_o}, the extended function, still denoted by $\boldsymbol{u}$, belongs to $\boldsymbol{W}^{2,p'}_{\ell}(\Omega)$, for any ℓ since its support is bounded. Since $\boldsymbol{\mu} \cdot \boldsymbol{n} = 0$ on Γ, we have $\boldsymbol{u} \in \boldsymbol{Y}_{p',\ell}(\Omega)$ and $\boldsymbol{\mu} \in \operatorname{Im}(\gamma_1)$. □

Secondly, we shall use the space:

$$\boldsymbol{T}^{\ell}_{r,p}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{-\ell}(\Omega);\ \Delta \boldsymbol{v} \in [\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]' \right\},$$

equipped with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{T}^{\ell}_{r,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{-\ell}(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]'} .$$

We also introduce the following space:

$$\boldsymbol{H}^{r}_{p,\ell}(\operatorname{div}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega);\ \nabla \cdot \boldsymbol{v} \in \boldsymbol{W}^{0,r}_{\ell-1}(\Omega) \right\}.$$

This space is equipped with the graph norm. The following lemma proves that the tangential trace of functions $\boldsymbol{v} \in \boldsymbol{T}^{\ell}_{r,p}(\Omega)$ belong to the dual space of $\boldsymbol{Z}_{p'}(\Gamma)$, wich is

$$(\boldsymbol{Z}_{p'}(\Gamma))' = \left\{ \mu \in \boldsymbol{W}^{-1/p,p}(\Gamma);\ \mu \cdot \boldsymbol{n} = 0 \right\}.$$

Observe that we can decompose $\boldsymbol{v}$ into its tangential and normal parts, that is: $\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$. The proof of the following lemma is similar to the case of bounded domain (see [3]).

Lemma 3. *Suppose that $3/2 < p < \infty$ and $1/p + 1/3 = 1/r$. Then the space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{T}^{0}_{r,p}(\Omega)$. If in addition $p \neq 3$, we have $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{T}^{1}_{r,p}(\Omega)$.*

Corollary 4. *Let $3/2 < p < \infty$ and $1/p + 1/3 = 1/r$. Then the mapping $\gamma_\tau : \boldsymbol{v} \longrightarrow \boldsymbol{v}_\tau|_\Gamma$ on the space $\mathcal{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\boldsymbol{T}^{0}_{r,p}(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma)$, and we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{T}^{0}_{r,p}(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{Y}_{p',0}(\Omega)$,*

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\boldsymbol{X}^{0}_{r',p'}(\Omega)]' \times \boldsymbol{X}^{0}_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi}\, dx - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

If in addition $p \neq 3$, we have for any $\boldsymbol{v} \in \boldsymbol{T}^{1}_{r,p}(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{Y}_{p',1}(\Omega)$,

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\boldsymbol{X}^{1}_{r',p'}(\Omega)]' \times \boldsymbol{X}^{1}_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi}\, dx - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

The following lemma gives a precise sense to the normal trace of functions $\boldsymbol{v} \in \boldsymbol{H}^{r}_{p,\ell}(\operatorname{div}, \Omega)$ and the proof is very classical.

Lemma 5. *Let Ω be a Lipschitz open set in $\mathbb{R}^3$. Suppose that $0 \leqslant 1/r - 1/p \leqslant 1/3$ and $\ell = 0$ or $\ell = 1$. Then* *i) The space* $\mathcal{D}(\overline{\Omega})$ *is dense in* $\boldsymbol{H}^r_{p,\ell}(\mathrm{div},\Omega)$.

ii) The mapping $\gamma_n : \boldsymbol{v} \longrightarrow \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma$ *on the space* $\mathcal{D}(\overline{\Omega})$ *can be extended by continuity to a linear and continuous mapping, still denoted by* γ_n*, from* $\boldsymbol{H}^r_{p,\ell}(\mathrm{div},\Omega)$ *into* $\boldsymbol{W}^{-1/p,p}(\Gamma)$*. If in addition* $1/r - 1/p = 1/3$ *and* $3/2 < p < \infty$*, we have the following Green formula: for any* $\boldsymbol{v} \in \boldsymbol{H}^r_{p,\ell}(\mathrm{div},\Omega)$ *and* $\varphi \in W^{1,p'}_{1-\ell}(\Omega)$*,*

$$\int_\Omega \boldsymbol{v}\cdot\nabla\varphi\,dx + \int_\Omega \varphi\,\nabla\cdot\boldsymbol{v}\,dx = \langle \boldsymbol{v}\cdot\boldsymbol{n},\varphi\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}\,.$$

§4. Very weak solutions in $\boldsymbol{L}^p(\Omega)\times W^{-1,p}_0(\Omega)$

In this section, we prove the existence and the uniqueness of very weak solutions to the Stokes problem via an argument of duality. We begin by specifying the meaning of very weak variational formulation.

Let

$$\boldsymbol{f} \in [\boldsymbol{X}^0_{r',p'}(\Omega)]',\ h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \tag{3}$$

with

$$\frac{3}{2} < p < \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{3} = \frac{1}{r} \tag{A$_1$}$$

yielding $1 < r < 3$.

Definition 1 (Very weak solution for the Stokes problem)**.** Suppose that (A$_1$) is satisfied and let $\boldsymbol{f}$, h and $\boldsymbol{g}$ verifying (3). We say that $(\boldsymbol{u},q) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}_0(\Omega)$ is a very weak solution of $(\mathcal{S})$ if the following equalities hold: For any $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega)$ and $\pi \in W^{1,p'}_0(\Omega)$,

$$-\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,dx - \langle q,\nabla\cdot\boldsymbol{\varphi}\rangle_{W^{-1,p}_0(\Omega)\times\mathring{W}^{1,p'}_0(\Omega)} = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma, \tag{4}$$

$$\int_\Omega \boldsymbol{u}\cdot\nabla\pi\,dx = -\int_\Omega h\pi\,dx + \langle \boldsymbol{g}\cdot\boldsymbol{n},\pi\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}\,, \tag{5}$$

where the dualities on Ω and Γ are defined by

$$\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\boldsymbol{X}^0_{r',p'}(\Omega)]'\times\boldsymbol{X}^0_{r',p'}(\Omega)}\,, \quad \langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}\,.$$

Note that if $3/2 < p < \infty$ and $1/p + 1/3 = 1/r$, we have:

$$W^{1,p'}_0(\Omega) \hookrightarrow L^{r'}(\Omega) \quad \text{and} \quad \boldsymbol{Y}_{p',0}(\Omega) \hookrightarrow \boldsymbol{X}^0_{r',p'}(\Omega),$$

which means that all the brackets and integrals have a sense.

Proposition 6. *Under the assumptions of Definition 1, the following two statements are equivalent:*

i) $(\boldsymbol{u},q) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}_0(\Omega)$ *is a very weak solution of* $(\mathcal{S})$*,*

ii) $(\boldsymbol{u},q)$ *satisfies the system* $(\mathcal{S})$ *in the sense of distributions.*

Proof. **i) ⇒ ii)** Let $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ a very weak solution of $(\mathcal{S})$, then if we take $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ and $\pi \in \mathcal{D}(\Omega)$ we can deduce by (4) and (5) that

$$-\Delta \boldsymbol{u} + \nabla \boldsymbol{q} = \boldsymbol{f} \text{ in } \Omega \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h \text{ in } \Omega,$$

and that $\boldsymbol{u} \in \boldsymbol{T}^0_{r,p}(\Omega)$. Now let $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega) \subset \boldsymbol{X}^0_{r',p'}(\Omega)$, then we have

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_\Omega = \langle -\nabla \boldsymbol{q} + \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega .$$

As (A_1) is satisfied, it follows from Corollary 4 that

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_\Omega = -\int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{\varphi}\, dx + \left\langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma$$

and since $1/r - 1/p = 1/3$, it follows from Lemma 1 ii) that

$$\langle \nabla \boldsymbol{q}, \boldsymbol{\varphi} \rangle_\Omega = -\langle \boldsymbol{q}, \nabla \cdot \boldsymbol{\varphi} \rangle_{W_0^{-1,p}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)} .$$

Thus we have

$$-\int_\Omega \boldsymbol{u} \Delta \boldsymbol{\varphi} dx + \left\langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma = \langle \boldsymbol{q}, \nabla \cdot \boldsymbol{\varphi} \rangle_{W_0^{-1,p}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)} + \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega ,$$

and we can deduce that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega)$

$$\left\langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma = \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma .$$

Now let $\boldsymbol{\mu} \in \boldsymbol{W}^{1/p,p'}(\Gamma)$, then we have $\langle \boldsymbol{u}_\tau - \boldsymbol{g}_\tau, \boldsymbol{\mu} \rangle_\Gamma = \langle \boldsymbol{u}_\tau - \boldsymbol{g}_\tau, \boldsymbol{\mu}_\tau \rangle_\Gamma$. It is clear that $\boldsymbol{\mu}_\tau \in \boldsymbol{Z}_{p'}(\Gamma)$, thus it follows from Lemma 2 that there exists $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega)$ such that $\partial \boldsymbol{\varphi}/\partial \boldsymbol{n} = \boldsymbol{\mu}_\tau$ on Γ. Then from this we can deduce that $\boldsymbol{u}_\tau = \boldsymbol{g}_\tau$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. From the equation $\nabla \cdot \boldsymbol{u} = h$, we deduce that $\boldsymbol{u} \in \boldsymbol{H}^r_{p,1}(\mathrm{div}, \Omega)$, then it follows from Lemma 5 ii), that for any $\boldsymbol{\pi} \in \boldsymbol{W}_0^{1,p'}(\Omega)$,

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{\pi} \rangle_\Gamma = \langle \boldsymbol{g} \cdot \boldsymbol{n}, \boldsymbol{\pi} \rangle_\Gamma .$$

Consequently $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n}$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and finally $\boldsymbol{u} = \boldsymbol{g}$ on Γ.

ii)⇒i) We suppose that $(\boldsymbol{u}, q)$ satisfies the system $(\mathcal{S})$ in the sense of distributions. Then for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega) \hookrightarrow \boldsymbol{X}^0_{r',p'}(\Omega)$ we have

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_\Omega = \langle \boldsymbol{f} - \nabla q, \boldsymbol{\varphi} \rangle_\Omega ,$$

Using Corollary 4 and Lemma 1 ii) we prove (4).

Now from the equation $\nabla \cdot \boldsymbol{u} = h$, we can deduce that for any $\pi \in W_0^{1,p'}(\Omega)$

$$\int_\Omega \pi \nabla \cdot \boldsymbol{u}\, dx = \int_\Omega h\pi\, dx,$$

this integral has a sense because we have $W_0^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$. Using Lemma 5 ii) we deduce (5).

□

Before stating the theorem of the existense and the uniqueness of the very weak solution for Stokes problem, we need to introduce the following null spaces for $\alpha \in \{-1,0,1\}$ and $k \in \{0,1,2\}$:

$$\mathcal{N}_{\alpha}^{k,p}(\Omega) = \Big\{ (\boldsymbol{u},\pi) \in \boldsymbol{W}_{\alpha}^{k,p}(\Omega) \times W_{\alpha}^{k-1,p}(\Omega);\ T(\boldsymbol{u},\pi) = (\boldsymbol{0},\boldsymbol{0}) \text{ in } \Omega \quad \text{and} \quad \boldsymbol{u}|_{\Gamma} = 0 \Big\},$$

with

$$T(\boldsymbol{u},\pi) = (-\Delta\, \boldsymbol{u} + \nabla\, \pi, \operatorname{div} u).$$

If $p \notin \{3/2, 3\}$, we can prove that $\mathcal{N}_{1}^{2,p}(\Omega) = \mathcal{N}_{0}^{1,p}(\Omega) = \mathcal{N}_{-1}^{0,p}(\Omega)$. Note that if $\boldsymbol{u} \in \boldsymbol{W}_{-1}^{0,p}(\Omega)$ and $-\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{0}$ in Ω with $\pi \in W_{-1}^{-1,p}(\Omega)$, then the tangential component $\boldsymbol{u}_{\tau}$ of $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and if $\operatorname{div} \boldsymbol{u} = 0$ in Ω, then $\boldsymbol{u} \cdot \boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. That means that $\boldsymbol{u} = \boldsymbol{0}$ on Γ makes sense.

Theorem 7. *Let Ω be an exterior domain with $C^{1,1}$ boundary and let p and r satisfy* (A$_1$) *and let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfying* (3). *Then the Stokes problem $(\mathcal{S})$ has exactly one solution $\boldsymbol{u} \in \boldsymbol{L}^{p}(\Omega)$ and $\boldsymbol{q} \in \boldsymbol{W}_{0}^{-1,p}(\Omega)$ if and only if for any $(\boldsymbol{v},\eta) \in \mathcal{N}_{0}^{2,p'}(\Omega)$:*

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle - \langle h, \eta\rangle + \langle \boldsymbol{g}, (\eta \boldsymbol{I} - \nabla \boldsymbol{v}) \cdot \boldsymbol{n}\rangle_{\Gamma} = 0.$$

Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^{p}(\Omega)} + \|q\|_{W_{0}^{-1,p}(\Omega)} \leq C(\|\boldsymbol{f}\|_{[\boldsymbol{X}_{r',p'}^{0}(\Omega)]'} + \|h\|_{L^{r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\,.$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u},\boldsymbol{q}) \in \boldsymbol{L}^{p}(\Omega) \times \boldsymbol{W}_{0}^{-1,p}(\Omega)$ such that for any $\boldsymbol{w} \in \boldsymbol{Y}_{p',0}(\Omega)$ and $\pi \in W_{0}^{1,p'}(\Omega)$ it holds:

$$\int_{\Omega} \boldsymbol{u}\cdot(-\Delta \boldsymbol{w} + \nabla\pi)\, dx - \langle q, \nabla\cdot \boldsymbol{w}\rangle_{W_{0}^{-1,p}(\Omega)\times \mathring{W}_{0}^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega} - \Big\langle \boldsymbol{g}_{\tau}, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\Big\rangle_{\Gamma} + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_{\Gamma} - \int_{\Omega} h\,\pi\, dx.$$

Let T be a linear form defined by:

$$T : \boldsymbol{L}^{p'}(\Omega) \times \mathring{W}_{0}^{1,p'}(\Omega) \longrightarrow \mathbb{R}$$

$$(\boldsymbol{F},\varphi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega} - \Big\langle \boldsymbol{g}_{\tau}, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\Big\rangle_{\Gamma} + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_{\Gamma} - \int_{\Omega} h\,\pi\, dx,$$

with $(\boldsymbol{w},\pi) \in \boldsymbol{W}_{0}^{2,p'}(\Omega) \times \boldsymbol{W}_{0}^{1,p'}(\Omega)$ is a solution of the following Stokes problem:

$$-\Delta \boldsymbol{w} + \nabla\,\pi = \boldsymbol{F} \quad \text{and} \quad \nabla\cdot \boldsymbol{w} = \varphi \ \text{ in } \Omega, \quad \boldsymbol{w} = 0 \ \text{ on } \Gamma,$$

and satisfying the following estimate (see [1, Theorem 3.1]):

$$\inf_{(\boldsymbol{v},\eta)\in \mathcal{N}_{0}^{2,p'}(\Omega)} \Big(\|\boldsymbol{w}+\boldsymbol{v}\|_{\boldsymbol{W}_{0}^{2,p'}(\Omega)} + \|\pi+\eta\|_{W_{0}^{1,p'}(\Omega)}\Big) \leqslant C\Big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\varphi\|_{W_{0}^{1,p'}(\Omega)}\Big). \tag{6}$$

Then we have for any pair $(\boldsymbol{F},\varphi) \in \boldsymbol{L}^{p'}(\Omega) \times \mathring{W}_{0}^{1,p'}(\Omega)$ and for any $(\boldsymbol{v},\eta) \in \mathcal{N}_{0}^{2,p'}(\Omega)$

$$\begin{aligned}
&\Big|\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega} - \Big\langle \boldsymbol{g}_{\tau}, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\Big\rangle_{\Gamma} + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_{\Gamma} - \int_{\Omega} h\,\pi\, dx\Big| \\
&\quad = \Big|\langle \boldsymbol{f}, \boldsymbol{w}+\boldsymbol{v}\rangle_{\Omega} - \Big\langle \boldsymbol{g}_{\tau}, \frac{\partial (\boldsymbol{w}+\boldsymbol{v})}{\partial \boldsymbol{n}}\Big\rangle_{\Gamma} + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi+\eta\rangle_{\Gamma} - \int_{\Omega} h\,(\pi+\eta)\, dx\Big| \\
&\quad \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{X}_{r',p'}^{0}(\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Omega)} + \|h\|_{L^{r}(\Omega)}\Big)\Big(\|\boldsymbol{w}+\boldsymbol{v}\|_{\boldsymbol{W}_{0}^{2,p'}(\Omega)} + \|\pi+\eta\|_{W_{0}^{1,p'}(\Omega)}\Big).
\end{aligned}$$

Using (6), we prove that

$$\Big|\langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega - \Big\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\Big\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma - \int_\Omega h\,\pi\,dx\Big|$$
$$\leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{X}^0_{r',p'}(\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Omega)} + \|h\|_{L^r(\Omega)}\Big)\Big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}_0(\Omega)}\Big),$$

from this we can deduce that the linear form T is continuous on $\boldsymbol{L}^{p'}(\Omega)\times W^{1,p'}_0(\Omega)$ and according to the Riesz' Theorem we deduce that there exists a unique $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}_0(\Omega)$ solution of $(\mathcal{S})$ satisfying the appropriate estimate. □

§5. Very weak solutions in $\boldsymbol{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega)$

Here, we are interested in the case of the following assumptions:

$$\boldsymbol{f}\in[\boldsymbol{X}^1_{r',p'}(\Omega)]',\ h\in W^{0,r}_{-1}(\Omega)\quad\text{and}\quad \boldsymbol{g}\in\boldsymbol{W}^{-1/p,p}(\Gamma), \tag{7}$$

with

$$\frac{3}{2}<p<\infty,\quad p\neq 3\quad\text{and}\quad \frac{1}{p}+\frac{1}{3}=\frac{1}{r}, \tag{A$_2$}$$

yielding $1<r<3$.

Definition 2 (Very weak solution for the Stokes problem)**.** Suppose that (A_2) is satisfied and let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfying (7). We say that $(\boldsymbol{u}, q)\in\boldsymbol{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega)$ is a very weak solution of $(\mathcal{S})$ if the following equalities hold: For any $\boldsymbol{\varphi}\in\boldsymbol{Y}_{p',1}(\Omega)$ and $\pi\in W^{1,p'}_1(\Omega)$,

$$-\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,dx - \langle q, \nabla\cdot\boldsymbol{\varphi}\rangle_{W^{-1,p}_{-1}(\Omega)\times \mathring{W}^{1,p'}_1(\Omega)} = \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega - \Big\langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\Big\rangle_\Gamma,$$

$$\int_\Omega \boldsymbol{u}\cdot\nabla\pi\,dx = -\int_\Omega h\pi dx + \langle\boldsymbol{g}\cdot\boldsymbol{n},\pi\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)},$$

where the dualities on Ω and Γ are defined by:

$$\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\boldsymbol{X}^1_{r',p'}(\Omega)]'\times\boldsymbol{X}^1_{r',p'}(\Omega)},\quad \langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

Note that if $3/2<p<\infty$ and $1/p+1/3=1/r$, we have:

$$W^{1,p'}_1(\Omega)\hookrightarrow W^{0,r'}_1(\Omega),\quad\text{and}\quad \boldsymbol{Y}_{p',1}(\Omega)\hookrightarrow\boldsymbol{X}^1_{r',p'}(\Omega),$$

which means that all the brackets and integrals have a sense. As previously we prove under the assumption (A_2), that if $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfying (7), then $(\boldsymbol{u}, q)\in\boldsymbol{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega)$ is a very weak solution of $(\mathcal{S})$ if and only if $(\boldsymbol{u}, q)$ satisfy the system $(\mathcal{S})$ in the sense of distributions.

Theorem 8. *Let Ω be an exterior domain with $C^{1,1}$ boundary and let p and r satisfy* (A_2) *and let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfying* (7)*. Then the Stokes problem $(\mathcal{S})$ has a solution $\boldsymbol{u}\in\boldsymbol{W}^{0,p}_{-1}(\Omega)$ and $\boldsymbol{q}\in\boldsymbol{W}^{-1,p}_{-1}(\Omega)$ if and only if, for any $(\boldsymbol{v},\eta)\in\mathcal{N}^{2,p'}_1(\Omega)$:,*

$$\langle\boldsymbol{f},\boldsymbol{v}\rangle - \langle h,\eta\rangle + \langle\boldsymbol{g}, (\eta\boldsymbol{I}-\nabla\boldsymbol{v})\cdot\boldsymbol{n}\rangle_\Gamma = 0.$$

In $\boldsymbol{W}^{0,p}_{-1}(\Omega)\times\boldsymbol{W}^{-1,p}_{-1}(\Omega)$*, each solution is unique up to an element of* $\mathcal{N}^{0,p}_{-1}(\Omega)$ *and there exists a constant* $C>0$ *depending only on* p *and* Ω *such that*

$$\inf_{(\boldsymbol{v},\eta)\in\mathcal{N}^{1,p}_{0}(\Omega)}\Big(\|\boldsymbol{u}+\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{-1}(\Omega)}+\|q+\eta\|_{W^{-1,p}_{-1}(\Omega)}\Big)\leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{X}^{1}_{r',p'}(\Omega)]'}+\|h\|_{W^{0,r}_{-1}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big).$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u},\boldsymbol{q})\in\boldsymbol{W}^{0,p}_{-1}(\Omega)\times\boldsymbol{W}^{-1,p}_{-1}(\Omega)$ such that for any $\boldsymbol{w}\in\boldsymbol{Y}_{p',0}(\Omega)$ and $\pi\in W^{1,p'}_{1}(\Omega)$ it holds:

$$\int_\Omega\boldsymbol{u}\cdot(-\Delta\boldsymbol{w}+\nabla\pi)\,dx-\langle q,\nabla\cdot\boldsymbol{w}\rangle_{W^{-1,p}_{-1}(\Omega)\times\mathring{W}^{1,p'}_{1}(\Omega)}$$
$$=\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\Big\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\Big\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\pi\rangle_\Gamma-\int_\Omega h\,\pi\,dx.$$

Let T be a linear form defined from $\Big(\boldsymbol{W}^{0,p'}_{1}(\Omega)\times\mathring{W}^{1,p'}_{1}(\Omega))\perp\mathcal{N}^{1,p}_{0}(\Omega)\Big)$ onto $\mathbb{R}$ by:

$$T(\boldsymbol{F},\varphi)=\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\Big\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\Big\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\pi\rangle_\Gamma-\int_\Omega h\,\pi\,dx,$$

with $(\boldsymbol{w},\pi)\in\boldsymbol{W}^{2,p'}_{1}(\Omega)\times\boldsymbol{W}^{1,p'}_{1}(\Omega)$ is a solution of the following Stokes problem:

$$-\Delta\boldsymbol{w}+\nabla\pi=\boldsymbol{F}\quad\text{and}\quad\nabla\cdot\boldsymbol{w}=\varphi\ \text{ in }\ \Omega,\quad\boldsymbol{w}=0\ \text{ on }\ \Gamma,$$

and satisfying the following estimate (see [1, Theorem 3.1]):

$$\inf_{(\boldsymbol{v},\eta)\in\mathcal{N}^{2,p'}_{1}(\Omega)}\Big(\|\boldsymbol{w}+\boldsymbol{v}\|_{\boldsymbol{W}^{2,p'}_{1}(\Omega)}+\|\pi+\eta\|_{W^{1,p'}_{1}(\Omega)}\Big)\leqslant C\Big(\|\boldsymbol{F}\|_{\boldsymbol{W}^{0,p'}_{1}(\Omega)}+\|\varphi\|_{W^{1,p'}_{1}(\Omega)}\Big).\tag{8}$$

Then for any pair $(\boldsymbol{F},\varphi)\in(\boldsymbol{W}^{0,p'}_{1}(\Omega)\times\mathring{W}^{1,p'}_{1}(\Omega))\perp\mathcal{N}^{1,p}_{0}(\Omega)$ and for any $(\boldsymbol{v},\eta)\in\mathcal{N}^{2,p'}_{1}(\Omega)$

$$\Big|\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\Big\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\Big\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\pi\rangle_\Gamma-\int_\Omega h\,\pi\,dx\Big|$$
$$=\Big|\langle\boldsymbol{f},\boldsymbol{w}+\boldsymbol{v}\rangle_\Omega-\Big\langle\boldsymbol{g}_\tau,\frac{\partial(\boldsymbol{w}+\boldsymbol{v})}{\partial\boldsymbol{n}}\Big\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\pi+\eta\rangle_\Gamma-\int_\Omega h\,(\pi+\eta)\,dx\Big|$$
$$\leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{X}^{1}_{r',p'}(\Omega]'}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Omega)}+\|h\|_{W^{0,r}_{-1}(\Omega)}\Big)\Big(\|\boldsymbol{w}+\boldsymbol{v}\|_{\boldsymbol{W}^{2,p'}_{1}(\Omega)}+\|\pi+\eta\|_{W^{1,p'}_{1}(\Omega)}\Big).$$

Using (8), we prove that

$$\Big|\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\Big\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\Big\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\pi\rangle_\Gamma-\int_\Omega h\,\pi\,dx\Big|$$
$$\leq C\left(\|\boldsymbol{f}\|_{[\boldsymbol{X}_{p'}(\Omega]'}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Omega)}+\|h\|_{W^{0,r}_{-1}(\Omega)}\right)\left(\|\boldsymbol{F}\|_{\boldsymbol{W}^{0,p'}_{1}(\Omega)}+\|\varphi\|_{W^{1,p'}_{1}(\Omega)}\right),$$

From this we derive that the linear form T is continuous on $\Big(\boldsymbol{W}^{0,p'}_{1}(\Omega)\times\mathring{W}^{1,p'}_{1}(\Omega)\perp\mathcal{N}^{1,p}_{0}(\Omega)\Big)$ and according to the Riesz' Theorem, we deduce that there exists $(\boldsymbol{u},q)\in(\boldsymbol{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega))$ solution of $(\mathcal{S})$ unique up to an element of $\mathcal{N}^{1,p}_{0}(\Omega)$ and satisfying the appropriate estimate. □

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