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A FRACTAL PROCEDURE FOR THE COMPUTATION OF MIXED INTERPOLANTS

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Abstract. We develop a procedure from the fractal methodology for the computation of an interpolant born from the cooperation of two functions of different nature. In particular, we define an Iterated Function System whose attractor is a fractal interpolant to a set of data with mixing properties. If the maps of the System are chosen in a suitable way, the approximant constructed is differentiable.
Since the degree of smoothness can be a priori fixed, the methodology described may be used in order to reduce the regularity of the classical interpolants as polynomial, splines, etc.

Keywords: Fractal interpolation functions, iterated function systems.

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§1. Introduction

In this paper we propose a procedure for the definition of smooth fractal functions of interpolation, whose degree of regularity can be fixed a priori. The function is defined as the perturbation of a classical mapping with a criterion of proximity to another. In this way, the approximant constructed comes from the cooperation of two classical elements and the methodology of iterated funtion systems. After the construction, we give an upper bound of the uniform error committed on a compact interval.

In a second part we present an application of the procedure to the field of the numerical integration. In particular we propose a correction to the polynomial quadrature formulae for partitions with small number of points.

§2. Fractal Functions

In former papers ([4], [5]), we have studied an Iterated Function System $\{w_n(t,x)\}_{n=1}^N$ defined on the set $C = I \times \mathbb{R}$, where I is a compact interval, $I = [a,b] \subset \mathbb{R}$. The maps $w_n(t,x)$ are defined by

$$w_n(t,x) = (L_n(t), F_n(t,x)) \quad \forall\, n = 1,2,...,N,$$

where

$$\begin{cases} L_n(t) = a_n t + b_n, \\ F_n(t,x) = \alpha_n x + q_n(t). \end{cases} \tag{1}$$

The system is associated with a partition of the interval I

$$\Delta : a = t_0 < t_1 < \cdots < t_N = b.$$

The coefficients a_n and b_n are defined in terms of the nodes of the partition as

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0}, \qquad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0}, \tag{2}$$

and $F_n(t,x)$ satisfies some Lipschitz conditions ([1]). The multiplier α_n is a vertical scale factor of the transformation, such that $-1 < \alpha_n < 1$. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector.

Theorem 1. *[1, 2]: The iterated function system (IFS) defined above admits a unique attractor G. G is the graph of a continuous function $h : I \to \mathbb{R}$ interpolating the data ($h(t_n) = x_n$, for all $n = 0, 1, \ldots, N$).*

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t,x))\}_{n=1}^N$. It satisfies the functional equation:

$$h(t) = F_n(L_n^{-1}(t), h \circ L_n^{-1}(t)). \tag{3}$$

In this paper we study a particular case of a Fractal Interpolation Function (FIF). The maps q_n are defined as

$$q_n(t) = g \circ L_n(t) - \alpha_n b(t), \tag{4}$$

where g and b are continuous functions, $g, b : I \to \mathbb{R}$, such that $b(t_0) = g(t_0)$, $b(t_N) = g(t_N)$.

The attractor of the system is the graph of a continuous function $g^\alpha : I \to \mathbb{R}$ which interpolates to g at the nodes of the partition,

$$g^\alpha(t_n) = g(t_n) \qquad \forall\, n = 0, 1, \ldots, N. \tag{5}$$

The mapping g^α satisfies the functional equation (3)

$$g^\alpha(t) = g(t) + \alpha_n\,(g^\alpha - b) \circ L_n^{-1}(t) \qquad \forall\, t \in I_n. \tag{6}$$

Let $\mathcal{G}$ be the set of continuous functions

$$\mathcal{G} = \{f \in C[a,b] : f(t_0) = g(t_0), f(t_N) = g(t_N)\}.$$

$\mathcal{G}$ is a complete metric space with respect to the uniform norm. Define a mapping $T^\alpha : \mathcal{G} \to \mathcal{G}$ by

$$(T^\alpha f)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)). \tag{7}$$

for all $t \in [t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$.

T^α is a contraction mapping on the metric space $(\mathcal{G}, \|\cdot\|_\infty)$ and possesses a unique fixed point on $\mathcal{G}$, that is the FIF g^α.

The uniform distance between g^α and g is bounded in terms of the scale vector ([6]) and the map b,

$$\|g^\alpha - g\|_\infty \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|g - b\|_\infty \tag{8}$$

where $\|\cdot\|_\infty$ is the uniform norm defined as

$$\|f\|_\infty = \max\{|f(t)| : t \in I\} \tag{9}$$

and

$$|\alpha|_\infty = \max\{|\alpha_n| : \ n = 1, 2, ..., N\} \tag{10}$$

is the contractivity factor of the transformation T^α.

Sufficient conditions for the smoothness of order p of g^α are (see the reference [7]):

$$g, b \in C^p(I) \quad \text{and} \quad \begin{cases} g^{(r)}(t_0) = b^{(r)}(t_0), \\ g^{(r)}(t_N) = b^{(r)}(t_N), \end{cases} \quad r = 0, 1, \ldots, p, \tag{11}$$

$$|\alpha|_\infty < \frac{1}{N^p},$$

$$\alpha_n = \text{cte} \qquad \forall\, n = 1, 2, \ldots, N.$$

In order to satisfy the condition (11), we can choose as b a Hermite polynomial osculating g at the extremes of the interval I.

§3. Correction of a classical interpolant with fractal methodology

In this section we present an interpolant born from the cooperation of two approximants of different nature, first developed in previous works [8, 3]. The fractal function is defined first as perturbation of one classical. The additional condition of proximity to another interpolant provides a problem of convex optimization whose solution is a fractal element with mixing properties.

Theorem 2 (Collage Theorem [2])**.** *Let (X, d) be a complete metric space and let T be a contraction map on X with contractivity factor $c \in [0, 1)$. Then, for any $f \in X$*

$$d(f, \tilde{f}) \le \frac{1}{1-c}\, d(f, Tf),$$

where $\tilde{f}$ is the fixed point of T.

We consider two classical interpolants (S and P) of a set of data. We construct the fractal function P^α associated to P, defined in the previous section ($g = P$). Now we apply the collage theorem for $X = \mathcal{G}$, $f = S$, $\tilde{f} = P^\alpha$ and $T = T^\alpha$.

The distance here is the uniform metric and $T = T^\alpha$ is the contraction (7), so that $\|T^\alpha S - S\|_\infty < \varepsilon$ implies $\|S - P^\alpha\|_\infty < \frac{\varepsilon}{1-|\alpha|_\infty}$ and P^α will be a fractal interpolant close to S.

We look for a smooth function, for instance $P^\alpha \in C^1(I)$, and then we may set the problem of finding α^* solving the optimization

$$\min_\alpha \|T^\alpha S - S\|_\infty = \min_\alpha c(\alpha)$$

where $|\alpha|_\infty \le \delta < 1/N$, according to the condition 2 for the smoothness of P^α. The map b must have a contact of first order with P at the extremes of the interval.

The classical interpolants S (polynomial, spline) are piecewise smooth and consequently by the definition of T^α, $T^\alpha S - S$ also is. $c(\alpha)$ is non-differentiable in general, but its convexity can be proved and thus, the problem

$$(CP) \begin{cases} \min_\alpha c(\alpha), \\ |\alpha|_\infty \le \delta < 1/N, \end{cases}$$

is a constrained convex optimization problem. The existence of solution is clear if c is a continuous function as $\mathcal{B}_\delta = \{\alpha \in \mathbb{R}^N : |\alpha|_\infty \leq \delta < 1/N\}$ is a compact set of $\mathbb{R}^N$. In a previous paper [8] we proved that c is continuous, and (CP) convex, so that (CP) is a problem of constrained convex optimization with some solution.

If α^* is the optimum scale ($\alpha^* = \alpha_n$, $\forall n = 1, 2, \ldots, N$), the expression $c(\alpha^*)/(1 - |\alpha^*|_\infty)$ provides an upper bound of the uniform distance $\|P^{\alpha^*} - S\|_\infty$ according to the Collage Theorem.

Figures 1 and 2 display a polynomial interpolant P and a cubic spline S (respectively) to the set of data $D = \{(0, 1), (1/4, 5), (1/2, 2), (3/4, 4), (1, 3)\}$. Figure 3 shows the corresponding fractal P^{α^*} defined by the method described. The order of regularity is $p = 1$. The loss of smoothness can be observed.

The following result provides an upper bound of the uniform error of the fractal interpolant P^{α^*} with respect to the original function X.

Theorem 3. *If $X(t)$ is the original continuous function providing the interpolation data and α^* is the optimum scale, the following error estimate is obtained:*

$$\|X - P^{\alpha^*}\|_\infty \leq E_P + \frac{l^4}{(N-1)4!2^4}\|P^{(4)}\|_\infty, \tag{12}$$

where E_P is an upper bound of the interpolation error corresponding to P, l is the length of the interval I, $N + 1$ is the number of points of the partition and b (4) is a Hermite polynomial with a contact of first order with P at the extremes of the interval.

Proof. It is clear that

$$\|X - P^{\alpha^*}\|_\infty \leq \|X - P\|_\infty + \|P - P^{\alpha^*}\|_\infty.$$

In the reference [7] (expression (2.53) for $k = 0$ and $p = 1$) it is proved that

$$\|P - P^{\alpha^*}\|_\infty \leq \frac{|\alpha^*|}{1 - |\alpha^*|} \frac{l^4}{4!\, 2^4} \|P^{(4)}\|_\infty.$$

The inequality $|\alpha^*| < 1/N$ provides the bound proposed. □

§4. Fractal quadrature

The procedure described is applied now for the computation of a numerical integration. Let us denote M_0 the integral of the interpolant P^{α^*} on the interval I.

M_0 can be computed using the fixed point equation (6)

$$M_0 = \int_I P^{\alpha^*}(t)\, dt = \sum_{n=1}^{N} \int_{I_n} (\alpha_n P^{\alpha^*} \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t))\, dt,$$

that is to say

$$M_0 = \left(\sum_{n=1}^{N} \alpha_n \int_{I_n} P^{\alpha^*} \circ L_n^{-1}(t)\, dt \right) + Q_0,$$

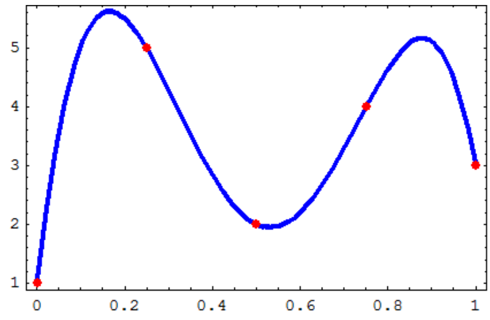

Figure 1: A polynomial interpolant P

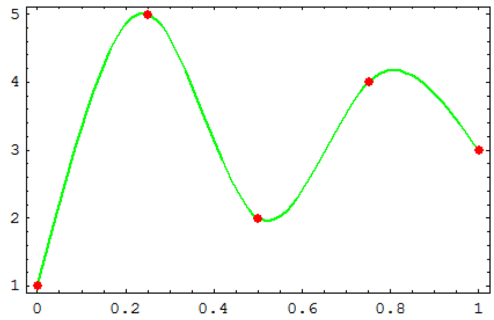

Figure 2: A cubic spline S

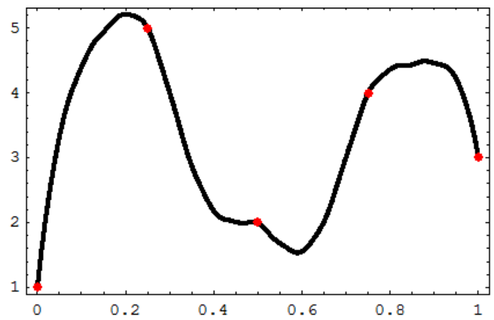

Figure 3: Fractal interpolant P^{α^*} computed from the maps of Figures 1 and 2

where

$$Q_0 = \int_I Q(t)\, dt \tag{13}$$

and

$$Q(t) = q_n \circ L_n^{-1}(t) \quad \text{if} \quad t \in I_n. \tag{14}$$

With the change $L_n^{-1}(t) = \widetilde{t}$, bearing in mind (1),

$$M_0 = \sum_{n=1}^{N} \alpha_n a_n M_0 + Q_0$$

and

$$M_0 = \frac{Q_0}{1 - \sum_{n=1}^{N} \alpha_n a_n}.$$

In this case,

$$q_n(t) = P \circ L_n(t) - \alpha_n b(t)$$

and thus

$$Q_0 = \int_I P(t)\, dt - \sum_{n=1}^{N} \alpha_n \int_{I_n} b \circ L_n^{-1}(t)\, dt.$$

With the same change $L_n^{-1}(t) = \widetilde{t}$,

$$Q_0 = C_0 - B_0\, (\sum_{n=1}^{N} \alpha_n a_n),$$

where C_0 is the polynomial quadrature

$$C_0 = \int_I P(t)\, dt$$

and

$$B_0 = \int_I b(t)\, dt.$$

Since $a_n = 1/N$ and $\alpha_n = \alpha^*$,

$$Q_0 = (C_0 - \alpha^* B_0)$$

and

$$M_0 = \frac{(C_0 - \alpha^* B_0)}{(1 - \alpha^*)}.$$

This formula introduces a slight modification to the quadrature C_0 corresponding to P.

Example 1. Let us consider the original function $X(t) = \frac{1}{1+25t^2}$ in the interval $[-1, 1]$ with the partition $\Delta : -1 < \frac{-2}{3} < \frac{-1}{3} < 0 < \frac{1}{3} < \frac{2}{3} < 1$. The value obtained for α^* in the optimization method described in the previous section is $\alpha^* = 0.04$. The polynomial quadrature gives $C_0 = 0.77407$, with an error of -0.224729. The correction M_0 provided by our procedure is $M_0 = 0.45459$, obtaining an error of 0.0947702, what improves the sought scalar.

Exact Value	C_0	Error C_0	α^*	M_0	Error M_0
0.54936	0.77409	-0.224729	0.04	0.45459	0.0947702

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