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ANALYTICAL AND NUMERICAL METHODS IN STRATIGRAPHY

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Abstract. In this paper, we are interested in a mathematical problem arising from the modelling of maximal erosion rates in geological stratigraphy. The problem is nonlinear with a diffusion coefficient that is a nonlinear function of u and $\partial_t u$. Moreover, the problem degenerates in order to take implicitly into account a constraint on $\partial_t u$. Our aim in this paper is to present a survey of the results exposed in the oral communication and written in the PhD theses [11] and [13].

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AMS classification: 35K70, 65N30, 65N12..

§1. Introduction and mathematical model

This work deals with the study of a mathematical model arising from the modelling of geological basin formation. It takes into account sedimentation, transport and accumulation, erosion phenomena, and others. The original mathematical aspect of this model is the imposition of a constraint on the time-derivative of the unknown u.

Let us consider a sedimentary basin and denote by Ω its basis. It is assumed to be a fixed bounded domain of $\mathbb{R}^N$ ($N = 1, 2$) with a Lipschitz boundary Γ. As usual, for any positive T, we set $Q =]0, T[\times \Omega$. In the model, u, the sediments thickness, naturally satisfies the mass balance equation

$$\partial_t u + \operatorname{div}(\vec{q}) = f \text{ in } Q, \tag{1}$$

where f denotes a source term (modelling, for example, of suspension matter in the sea that sediments in the domain). According to Darcy-Barenblatt's law (see [8] for example), the flux $\vec{q}$ is given by the relation

$$\vec{q} = -\lambda K(u)\nabla(u + \tau\partial_t u), \tag{2}$$

where λ is a parameter to be defined later and τ is a positive time-scaled parameter.

In a sedimentary basin formation process, sediments must first be produced *in situ* by weathering effects prior to be transported by surfacing erosion. Thus, a constraint on a maximum erosion rate $-\partial_t u \leq E$ in Q has to be introduced (see [9]), where E is non-negative. It takes into account the composition, the structure and the age of the sediments. In their paper, the authors consider a flux limiter λ, $0 \leq \lambda \leq 1$ that satisfies

$$\partial_t u + E \geq 0,\ (1 - \lambda)(\partial_t u + E) = 0,\ \text{a.e in } Q. \tag{3}$$

Following [2], one remarks that, as soon as $f + E \geq 0$, for all $u \in L^2(0, T; H^1_0(\Omega))$ with $\partial_t u \in L^2(0, T; H^1_0(\Omega))$, (1)–(3) is equivalent to the following formulation:

$$\partial_t u - \operatorname{div}[\lambda K(u)\nabla(u + \tau\partial_t u)] = f \text{ in } Q,\ \lambda \in H(\partial_t u + E) \cap L^\infty(Q). \tag{4}$$

Here homogeneous Dirichlet boundary conditions on u and $\partial_t u$ are considered, $u(0,.) = u_0 \in H_0^1(\Omega)$, $E \in L^\infty(0,T;H^1(\Omega))$, $f \in L^\infty(0,T;L^2(\Omega))$ and H denotes the maximal monotone graph of the Heaviside function.

Indeed, if $f + E \geq 0$ is assumed, using the admissible test function $(\partial_t u + E)^-$ (where $x^- = -\min(0,x)$ for $x \in \mathbb{R}$) in (4), we get that $\partial_t u + E \geq 0$ a.e in Q since $\lambda \mathbf{1}_{\{\partial_t u + E < 0\}} = 0$ a.e., and therefore (3) and $\lambda \in H(\partial_t u + E)$ are equivalent assertions. Moreover, using that $\partial_t u + E \geq 0$, one has that

$$\lambda \nabla(u + \tau\partial_t u) = \lambda\nabla[u - \tau E + \tau(\partial_t u + E)] = \lambda\nabla(u - \tau E) + \tau\nabla(\partial_t u + E).$$

Thus, the problem (4) is equivalent to the following one:

$$\partial_t u - \operatorname{div}\left\{\lambda\, K(u)[\nabla u - \tau\nabla E]\right\} - \tau \operatorname{div}\left\{K(u)[\nabla\partial_t u + \nabla E]\right\} = f,\ \lambda \in H(\partial_t u + E) \cap L^\infty(Q). \quad (5)$$

Results of existence and uniqueness of a solution to such a problem is still an open question. Thus, a modified model where H is replaced by a continuous function a, for example the Yosida approximation of H, will be proposed. Such a problem has been analysed by S. N. Antontsev *et al.* [2] with $K \equiv 1$, $f \equiv 0$ and a constant E. Then, a result of existence and uniqueness of a solution is given in [11, 3] with a null source term and a time dependant function E. While a result of existence of a solution and a numerical scheme based on the discontinuous Galerkin methods (DgFem) are considered in [13, 4] with a source term, a space-time function E and $K = 1$.

Note that the above remark concerning the equivalence between (4) and (5) doesn't hold anymore if one replace λ by $a(\partial_t u+E)$ *i.e.* equivalence between (6) and (8). The two problems have not got the same nature. Indeed, thanks to the localisation methods proposed in the book of S.N. Antontsev *et al.* [1], it has been proved in [11], that under some hypothesis on a, any weak solution u to the 1-D problem:

$$\partial_t u - \partial_x\left\{K(u)a(t,\partial_t u)[\partial_x u + \tau\partial_{xt} u]\right\} = f \quad \text{in } Q =]0,T[\times\Omega \quad \text{with } \Omega =]0,1[\,, \quad (6)$$

with the boundary and initial conditions:

$$\partial_t u_{|_{t=0}} = 0, \quad u(0,x) = u_0(x),\ x \in \Omega, \quad \text{where} \quad u_0(x) = 0,\ x \in]0,\rho_0[\,,\ 0 < \rho_0 < 1, \quad (7)$$

there exist a positive $\delta > 0$ and $\rho(t) \in (0,\rho_0)$, such that, if $f(t,x) = 0$ in $]0,\delta[\times]0,\rho_0[$, then u satisfies the finite speed of propagation property: $u(t,x) = 0$ in $x \in]0,\rho(t)[$, $0 \leq t \leq \delta$.

But this locally hyperbolic behaviour is unknown concerning the pseudo-parabolic problem:

$$\partial_t u - \operatorname{div}\left\{a(\partial_t u + E)\, K(u)[\nabla u - \tau\nabla E]\right\} - \tau\operatorname{div}\left\{K(u)[\nabla\partial_t u + \nabla E]\right\} = f \text{ in } Q. \quad (8)$$

On the one hand, both (6) and (8) are approximations of the same problem when a converges toward the graph of Heaviside. On the other hand, they reveal distinct natures.

From now on, one would consider the pseudo-parabolic problem (8) with an initial height given by: $u(0,\cdot) = u_0$ in Ω, where $u_0 \in H_0^1(\Omega)$, and homogeneous Dirichlet condition for u and $\partial_t u$.

Contrarily to the perturbation (6), the main interest of (8) is that it will be possible to use the theorems of N.G. Meyers and J. Nečas, in order to obtain a more regular solution (i.e $u \in W^{1,\infty}(0,T;W_0^{1,p}(\Omega))$, with $p > 2$ as soon as $u_0 \in W_0^{1,p}$) and thus a uniqueness result.

Let us set the assumptions made on the data and the definition of a solution

$$\textbf{(H)} : \begin{cases} \tau > 0;\ E \in L^\infty(0,T;H^1(\Omega)), E \geq 0;\ f \in L^\infty(0,T;L^2(\Omega)),\ f + E \geq 0 \text{ in } Q; \\ a \in C^{0,\theta}(\mathbb{R}), \text{ with } \theta \geq 1/2,\ 0 \leq a \leq 1,\ \forall x \in]-\infty,0],\ a(x) = 0; \\ K : \mathbb{R} \to \mathbb{R} \text{ is a Lipschitz function, with } 0 < K_{\min} \leq K \leq K_{\max}. \end{cases}$$

Definition 1. Under assumption **(H)**, a solution to problem (8) is any u in $W^{1,2}(0,T;H_0^1(\Omega))$ such that for any v in $H_0^1(\Omega)$ and for a.e. t in $]0,T[$,

$$\int_\Omega \partial_t u v\,dx + \int_\Omega K(u)a(\partial_t u + E)(\nabla u - \tau\nabla E)\cdot\nabla v\,dx + \tau\int_\Omega K(u)(\nabla\partial_t u + \nabla E)\cdot\nabla v\,dx = \int_\Omega f v\,dx$$

with the initial condition $u(t = 0) = u_0$ a.e. in Ω.

Since a vanishes on $\mathbb{R}^-$, as previously shown with λ, the constraint $\partial_t u + E \geq 0$ in Q is implicitly satisfied.

The sequel of this paper is organised as follows: in Section 2, we present the result of existence and uniqueness of the solution to the problem (8). Then, in Section 3, the DgFem for the model is introduced. It is construct in order to satisfy implicitly the constraint (3) in the lowest-order case. A last section is concerned by numerical results.

§2. Existence and uniqueness result

In this section, we present the result of existence and uniqueness of a solution to problem (8).

Theorem 1. *Assume* (**H**). *For any* $u_0 \in H_0^1(\Omega)$, *a solution u to the problem* (8) *exists in the sense of the definition* 1 *and is given in the space* $W^{1,\infty}(0,T;H_0^1(\Omega))$. *Moreover, the constraint* $\partial_t u + E \geq 0$ *a.e. in* Q *is implicitly satisfied.*

Following [4, 11, 13], the result is based on the implicit time discretization:

$$\int_\Omega \frac{u^k - u^{k-1}}{h} v + K(u^k)\Big[a\Big(\frac{u^k - u^{k-1}}{h} + E^k\Big)(\nabla u^k - \tau\nabla E^k) + \tau\Big(\nabla\frac{u^k - u^{k-1}}{h} + \nabla E^k\Big)\Big]\cdot\nabla v\,dx = \int_\Omega f^k v\,dx.$$

The existence of u^k comes from Schauder's fixed point theorem. Then, one notes that it is the unique solution in $H_0^1(\Omega)$ to the elliptic problem: $-\operatorname{div}[\alpha\nabla u^k] = f_0 - \operatorname{div}\vec{f}$, with non constant and symmetrical coefficients α and suitable f_0 and $\vec{f}$ (see [3] for more details).

By applying the theorem of Meyers [12], we get, following [11], that

Lemma 2. *Assuming that* $u^{k-1} \in W_0^{1,p_0}(\Omega)$ *and that* $f^k \in L^{p_0}(\Omega)$ *with* $p_0 > 2$, *there exists a real* $\overline{p}(p_0) > 2$, *depending on* p_0 *and* $\frac{K_{\max}(a_{\max}+\tau/h)}{(\tau/h)K_0}$, *and a positive constant* $C(\overline{p}(p_0))$ *such that*

$$u^k \in W_0^{1,\overline{p}(p_0)}(\Omega) \quad \text{and} \quad \|\nabla u^k\|_{L^{\overline{p}(p_0)}(\Omega)^N} \leq C(\overline{p}(p_0))\Big(\|u_0\|_{W_0^{1,p_0}(\Omega)}, \|f\|_{L^{p_0}(\Omega)}\Big).$$

Indeed, $K_0\tau \leq \alpha \leq K_{\max}[a_{\max}h + \tau]$ with $h << 1$. Since the dimension N is 1 or 2, estimation

$$\|\nabla \frac{u^k - u^{k-1}}{h}\|_{L^2(\Omega)^N} \leq C\Big(K_0, K_{\max}, a_{\max}, \tau, T, \|\nabla u_0\|_{L^2(\Omega)^N}, \|f\|_{L^2(\Omega)}\Big)$$

yields that $f_0 \in L^{p_0}(\Omega)$. Moreover, $\vec{f_1} \in L^{p_0}(\Omega)^N$, and Meyers' theorem [5] leads to the assertion.

Still following [11], by using the theorem of Nečas, the regularity $W_0^{1,p_0}(\Omega)$ can be obtained.

Lemma 3. *If $u_0 \in W_0^{1,p_0}(\Omega)$ with $p_0 > 2$, then, $u^k \in W_0^{1,p_0}(\Omega)$, for any $k = 1, \dots, N$, and there exists a positive constant $C(p_0)$ such that*

$$\|\nabla u^k\|_{L^{p_0}(\Omega)^N} \leq C(p_0)\Big(\|u_0\|_{W_0^{1,p_0}(\Omega)}, \|f\|_{L^{p_0}(\Omega)}\Big).$$

Indeed, since $N \leq 2$, the Sobolev embedding ensures that α is a continuous function on $\overline{\Omega}$ and the expected regularity comes from Nečas theorem [7, 12]. Thus, the assertion yields by using the discrete Gronwall's lemma.

Thanks to those lemmata, following [13, 11], suitable *a priori* estimates hold to get that

Theorem 4. *If $u_0 \in W_0^{1,p_0}(\Omega)$ and $f \in L^2(0, T; L^{p_0}(\Omega))$, for a given $p_0 > 2$, then, any limit point u in $W^{1,\infty}(0, T; H_0^1(\Omega))$ of the implicit time discretized scheme satisfies moreover that $u \in W^{1,\infty}(0, T; W_0^{1,p_0}(\Omega))$.*

Then, one is able to adapt the demonstrations involving tri-linear terms and based on Hölder type inequalities to prove that

Theorem 5. *There exists a real $\tau^* \geq 0$, such that for any $\tau > \tau^*$, the problem (8) has a unique solution in the space $W^{1,\infty}(0, T; W_0^{1,p_0}(\Omega))$ with $p_0 > 2$. Moreover, the application: $u_0 \mapsto \partial_t u$ is a locally Lipschitz continuous function in the space $H_0^1(\Omega)$ to the space $L^\infty(0, T; H_0^1(\Omega))$.*

§3. Space DgFem discretization

In this section, we consider a numerical scheme for the computation of the semi-discretized problem. Our approach is based on the discontinuous Galerkin finite element method (DgFem). For convenience, we assume in the sequel that E is a non-negative constant, and that the function a is Lipschitz-continuous. Before discretizing the problem, some notations are collected.

We suppose that $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and that h is a regular triangular mesh in a family of shape-uniform meshes [6].

We denote by $\mathcal{K}_h$ the set of triangles and by $\mathcal{S}_h$ the set of edges, divided into interior edges $\mathcal{S}_h^{int}$ and boundary edges $\mathcal{S}_h^{\partial}$. An interiori edge $S \in \mathcal{S}_h^{int}$ is shared by two triangles, we arbitrarily chose a normal n_S pointing from K^+ to K^-. For $p \in \mathbb{N}$, we define the discontinuous finite element space:

$$V_h^p = \Big\{v_h \in L^2(\Omega) \ : \ v_h|_K \in P^p \text{ for all } K \in \mathcal{K}_h\Big\},$$

where P^p denotes the space of polynomials functions of maximal degree p. Due to the discontinuity of the approximation space, the weak formulation reveals terms of jumps through the cell interfaces. We make use of the standard notations concerning the jumps and averages for $v_h \in V_h^p$, $S \in \mathcal{S}_h^{int}$, and $x \in S$:

$$v_h^{\pm}(x) = \lim_{\varepsilon \to 0^+} v_h(x \pm \varepsilon n_S) \quad \text{and} \quad [v_h]_S = v_h^+ - v_h^-.$$

For a boundary edge we set $[v_h]_S := v_h^-$.

In addition, for any bounded positive piece-wise continuous function κ with respect to h, we define the weighted average of $v_h \in V_h$ on an interior edge S as

$$\left\{\frac{\partial v_h}{\partial n}\right\}_{S,\kappa} = \frac{\kappa^- \kappa^+}{\kappa^+ + \kappa^-}\left(\frac{\partial v_h^+}{\partial n_S}\bigg|_S + \frac{\partial v_h^-}{\partial n_S}\bigg|_S\right).$$

Let us now consider the following time semi-discrete problem which reads: Find $u^{k+1} \in H_0^1(\Omega)$ such that for all v in $H_0^1(\Omega)$

$$\begin{cases} \dfrac{1}{\Delta t}\displaystyle\int_\Omega u^{k+1} v\, dx + \int_\Omega D(u^{k+1})\nabla u^{k+1} \cdot \nabla v\, dx + \frac{\tau}{\Delta t}\int_\Omega K(u^{k+1})\nabla u^{k+1} \cdot \nabla v\, dx \\ \qquad = \displaystyle\int_\Omega f^{k+1} v\, dx + \frac{1}{\Delta t}\int_\Omega u^k v\, dx + \frac{\tau}{\Delta t}\int_\Omega K(u^{k+1})\nabla u^k \cdot \nabla v\, dx, \\ \qquad u^0 = u_0 \ \text{ in } \Omega, \end{cases} \tag{9}$$

where $D(w) := a(\frac{w-u^k}{\Delta t} + E)K(w)$.

The discrete DgFem formulation of problem (9) reads: Find $u_h^{k+1} \in V_h^p$ such that for all $v_h \in V_h^p$

$$A(u_h^{k+1})(u_h^{k+1}, v_h) = L^k(u_h^{k+1})(v_h), \tag{10}$$

where the bilinear form A and the linear form L are given for $\rho_h \in V_h^p$ by

$$\begin{aligned} A(\rho_h)(u_h, v_h) = {} & \frac{1}{\Delta t}\int_\Omega u_h v_h\, dx + \sum_{K \in \mathcal{K}_h}\int_K\left(D(\rho_h) + \frac{\tau}{\Delta t}K(\rho_h)\right)\nabla u_h \cdot \nabla v_h\, dx \\ & + \sum_{S \in \mathcal{S}_h}\int_S\left(\frac{\gamma}{h_S}\gamma_S[u_h][v_h] - \left\{\frac{\partial u_h}{\partial n_S}\right\}_{S,D}[v_h]_S - [u_h]_S\left\{\frac{\partial v_h}{\partial n_S}\right\}_{S,D}\right)ds, \\ & - \frac{\tau}{\Delta t}\sum_{S \in \mathcal{S}_h}\int_S\left(\left\{\frac{\partial u_h}{\partial n_S}\right\}_{S,K}[v_h]_S + [u_h]_S\left\{\frac{\partial v_h}{\partial n_S}\right\}_{S,K}\right)ds, \end{aligned}$$

and

$$\begin{aligned} L^k(\rho_h)(v_h) = {} & \int_\Omega f^{k+1} v_h\, dx + \frac{1}{\Delta t}\int_\Omega u_h^k v_h\, dx + \frac{\tau}{\Delta t}\sum_{K \in \mathcal{K}_h}\int_K K(\rho_h)\nabla u_h^k \cdot \nabla v_h\, dx \\ & + \sum_{S \in \mathcal{S}_h}\int_S\left(\frac{\gamma}{h_S}\beta_S[u_h^k][v_h] - \frac{\tau}{\Delta t}\left\{\frac{\partial u_h^k}{\partial n_S}\right\}_{S,K}[v_h]_S - \frac{\tau}{\Delta t}[u_h^k]_S\left\{\frac{\partial v_h}{\partial n_S}\right\}_{S,K}\right)ds, \end{aligned}$$

where $\beta_S = \frac{2K^+K^-}{K^- + K^+}$, $\gamma_S = \beta_S + \frac{2D^+D^-}{D^+ + D^-}$ and $\gamma > 0$ has to be chosen large enough.

In the lowest-order case ($p = 0$), we assume that the triangulation h satisfies the classical angle condition given in [10]. In this case, the bilinear form A reduces to

$$A(\rho_h)(u_h, v_h) = \frac{1}{\Delta t} \int_\Omega u_h v_h \, dx + \sum_{S \in \mathcal{S}_h} \frac{1}{h_S} \int_S \gamma_S [u_h][v_h] \, ds.$$

As well as for the continuous formulation, we are able to say

Proposition 6. *Assume* (**H**)*, the problem* (10) *has at least one solution. In addition, if* τ *sufficiently large, the solution is unique and if* $p = 0$*, then, we have for* $k = 0, \ldots, N-1$*,*

$$\frac{u_h^{k+1} - u_h^k}{\Delta t} + E^{k+1} \geq 0 \text{ a.e in } \Omega.$$

Proof. The proof of this result is given with more detail in [4] in the case $K \equiv 1$, using hypothesis on K this result can be generalised to the case $K := K(u)$. □

§4. Numerical results

In the numerical example, we consider Problem (8) in the domain $\Omega =]-1, 1[^2$, for $0 \leq t \leq T$, with homogenous Dirichlet condition, a null source term and we consider the initial height $u_0(x, y) = -\sin(\pi x)\sin(\pi y)$ and $a(z) = a_\varepsilon(z) = \inf\left(1, [\frac{3z^2}{\varepsilon^2}(1 - \frac{2z}{3\varepsilon})]^+\right)$, $\varepsilon > 0$.

Assume that $K \equiv 1$, the meshes are obtained by uniform refined from a coarse mesh h_0, verifying the angle condition required for $p = 0$. In the practice, we use the algorithm of Newton with line search to solve the nonlinear system of equations. Numerical experiments show that, the convergence of Newton algorithm is very slow if the parameter τ is very small.

4.1. Mesh stability study

In this section, we study the stability behaviour of the mesh in the L^2 norm. Since there is no exact known solution for this problem, nor benchmark, the "error" would be understood by the L^2-difference between a calculate solution u_h and a reference solution, denote by u_h^*, obtained by solving the problem (8) using quadratic DgFem scheme in a fine mesh with 57344 elements.

For numerical runs, we choose $T = 1$, $\varepsilon = 10^{-1}$, $\Delta t = 10^{-1}$ and $\tau = 10^{-1}$. We represent in Table 1 the L^2-norm of this so called error as a function of h, at time $t = T$. This table confirms the $p + 1$-order behaviour of the scheme, excepted when $p = 2$ which may depends of the regularity of the solution.

4.2. Numerical simulations

In this section, we present the numerical solution obtained by using DgFem(0) scheme. Our attention is to test numerically the discrete version of the constraint. For numerical runs, we choose $\varepsilon = E = 10^{-1}$, $\Delta t = \tau = 10^{-1}$. Figure 1 represents the numerical solution at different time t and the corresponding discrete constraint

This numerical simulations show that, the constraint is satisfied in all the domain, which confirms our theoretical result.

Ne	$\|\|u_h^* - u_h\|\|_{L^2(\Omega)}$			convergence rate		
	$p = 0$	$p = 1$	$p = 2$	$p = 0$	$p = 1$	$p = 2$
896	$1.51e-1$	$3.57e-2$	$1.48e-2$	–	–	–
3584	$6.95e-2$	$9.47e-3$	$1.48e-3$	1.11	1.91	2.90
14336	$3.32e-2$	$2.42e-3$	$2.25e-4$	1.06	1.96	2.70

Table 1: L^2 norm of the "error" with respect to h and p.

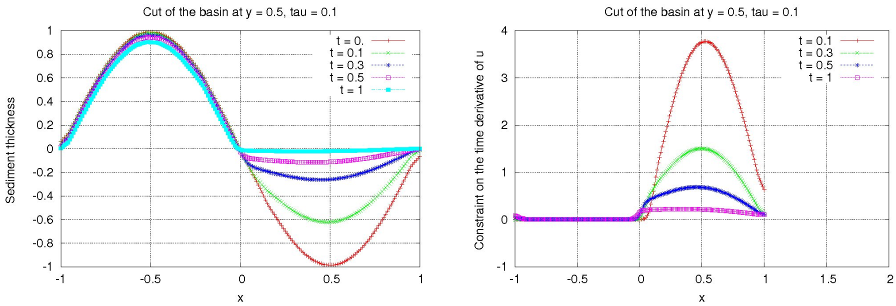

Figure 1: Vertical 1D cut at $y = 0.5$ of the numerical solution (left) and the constraint (right) with $\tau = 0.1$ and $p = 0$.

§5. Conclusion

In this survey, we have presented a result of existence and uniqueness of the solution to a realistic problem where the diffusion coefficient depends of the unknown u. However, many open questions still have to be treated, as to deal with the physics boundary conditions, *i.e.* nonhomogenuous Neumann boundary conditions on the inward part and unilateral boundary conditions on the outward one. Concerning the numerical aspect, we have presented a numerical scheme that implicitly takes into account the constraint on the time-derivative of the unknown. It's well known that a higher order scheme doesn't keep this important property, an adaptive algorithm that combines h-refinement with $p = 0$ and p-refinement has to be proposed in order to get a more accuracy, while still verifying the the monotonocity.

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