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Stability analysis of the Interior Penalty Discontinuous Galerkin method for solving the wave equation coupled with high-order absorbing boundary conditions

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Abstract. In this paper, we study high-order absorbing boundary conditions (ABCs) for the acoustic wave equation the Higdon's one, which only take into account the propagative waves and Hagstrom-Warburton's one, which considers both the evanescent and proagative ones. We discretize the problem by a Discontinuous Galerkin (DG) method. Numerical results illustrate the instability of the method in particular cases.

Keywords: Absorbing boundary conditions, discontinuous Galerkin method, acoustic wave equation.

AMS classification: 65M12,65M60,35L05,35L20.

§1. Introduction

The numerical simulation of wave propagation generally involves boundary conditions which both represent the behavior at infinity and provide a mathematical tool to define a bounded computational domain in which a finite element method (FEM) can be applied. Most of these conditions are derived from the approximation of the Dirichlet-to-Neumann operator and when they both preserve the sparsity of the finite element matrix and enforce dissipation into the system, they are called absorbing boundary conditions. Most of the approximation procedures are justified into the hyperbolic region which implies that only the propagative waves are absorbed. If the exterior boundary is localized far enough from the source field, the approximation is accurate and the absorbing boundary condition is efficient. However, the objective is to use a computational domain whose size is optimized since the solution of wave problems requires to invert matrices whose order is very large and is proportional to the distance between the source field and the exterior boundary. Hence, it is a big deal to derive absorbing boundary conditions which are efficient when the exterior boundary is close to the source field and it is necessary to construct conditions which are efficient not only for propagative waves but both for evanescent and glancing waves. Recently, a new condition has been derived from an approximation of the Dirichlet-to-Neumann operator which is valid both for propagative and evanescent waves and extends the condition which was formerly proposed by Higdon [8]. By using a classical finite element scheme, Hagstrom et al. [7] have shown the improvements induced by the new condition. In this work, we intend to investigate whether the new condition can be introduced into a Interior Penalty Discontinuous Galerkin method [4] which is more accurate to reproduce the propagation of waves into heterogeneous media than standard FEMs. To analyze the impact of the new condition on the accuracy of the numerical solution, we also consider the Higdon condition and we compare the efficiency of the two conditions.

§2. Statement of the problem

In this section, we consider a model problem for the time-dependent wave equation in a two-dimensional domain Ω with a general ABC and we focus on the description of the Interior Penalty Discontinuous Galerkin (IPDG) method ([4]). We have:

$$(\mathcal{S})\begin{cases}\partial_t^2 u - \operatorname{div}\left(c^2\nabla u\right) = f, \quad \text{in } (0,T)\times\Omega,\\ u(0,x) = 0\ ;\ \partial_t u(0,x) = 0, \quad \text{in } \Omega,\\ \partial_{\mathbf{n}} u = 0, \quad \text{on } \Gamma_N,\\ \partial_{\mathbf{n}} u = B(\partial_t, \nabla_\Gamma)u, \quad \text{on } \Gamma_{\text{abs}},\end{cases}$$

where f is the source function, c the velocity of the wave, u the unknown field, T the final time, $\mathbf{n}$ the unit outward normal vector, Γ_N and Γ_{abs} respectively the boundary with the Neumann condition and the ABC which is represented by the operator B. The operator B is differential, for instance, it reads $\frac{1}{c}\partial_t$ which corresponds to the simplest ABC. We refer to [1], where the well-posedness of problem $(\mathcal{S})$ has been established for $f \in L^2(0,T;L^2(\Omega))$ by applying the semi-group theory. More precisely, if $\mathcal{U} = \{u \in H^1(\Omega), \partial_n u \in L^2(\Gamma_{\text{abs}})\}$, $u \in C^0(0,T;\mathcal{U}) \cap C^1(0,T;L^2(\Omega))$.

We consider a partition $\mathcal{T}_h$ of Ω composed of triangles K, we denote by Ω_h the set of triangles, by Σ_{abs} the set of the edges on the absorbing boundary, by Σ_{N} the set of the edges on the Neumann boundary and by Σ_{i} the set of the edges in the domain such that $\Sigma_{\text{i}}\cap(\Sigma_{\text{N}}\cup\Sigma_{\text{abs}}) = \emptyset$. For each $\Sigma \in \Sigma_{\text{i}}$, we have to distinguish the two triangles that share Σ: we note them arbitrarily K^+ and K^- . We introduce useful notations to define the jump and the average over edges:

$$[\![v]\!] := v^+\boldsymbol{\nu}^+ + v^-\boldsymbol{\nu}^- \quad \text{and} \quad \{\!\{v\}\!\} := \frac{v^+ + v^-}{2},$$

where v^+ and v^- respectively refers to the restriction of v in K^+ and K^- and $\boldsymbol{\nu}^\pm$ stands for the unit outward normal vector to $K^\pm$.

It is well-known the IPDG formulation of $(\mathcal{S})$ reads as ([4]):

$$\begin{cases}\text{Find } u \in \mathcal{U} \text{ such that } \forall v \in \mathrm{H}^1,\\ \displaystyle\sum_K \int_K \partial_t^2 uv + a(u,v) - \sum_{\Sigma\in\Sigma_{\text{abs}}} \int_\Sigma c^2\partial_{\mathbf{n}} uv = \sum_K \int_K fv,\end{cases}$$

with

$$a(u,v) = \sum_K \int_K c^2\nabla u\nabla v - \sum_{\Sigma\in\Sigma_{\text{i}}} \int_\Sigma \left(\{\!\{v\}\!\}[\![c\nabla u]\!] + \{\!\{u\}\!\}[\![c\nabla v]\!] + \alpha[\![u]\!][\![v]\!]\right).$$

We seek an approximation of the solution in the finite element space V_h^k defined as follows:

$$V_h^k = \left\{ v \in L^2(\Omega); v_{|K} \in P^k, \forall K \right\}, k \in \mathbb{N}$$

where P^k is the set of polynomials of degree at most k on K.

§3. The Higdon's Condition

Here, we are going to study ABCs derived from a transparent boundary condition which only take the propagative waves into account. We will also discuss the implementation of those high-order conditions in the IPDG scheme.

We recall the Higdon's condition of order p, ($p \in \mathbb{N}$) (cf. [8]):

$$\prod_{j=1}^{P} (\cos a_j\, \partial_t + c\, \partial_{\mathbf{n}})\, u = 0, \quad \text{on } \Gamma_{abs}. \tag{1}$$

Remark 1. The Engquist-Majda's condition (cf. [2]), which was one of the first ABCs to be designed, is a particular case of the Higdon one. Indeed, it is obtained by choosing all a_j equal to zero in (1).

To implement this condition in a numerical scheme, we define auxiliary functions u_j, for $1 \leq j \leq P$ on the absorbing boundary (cf. [3]):

$$\begin{cases} (\cos a_1 \partial_t + c\partial_{\mathbf{n}})u = \partial_t u_1, & \\ (\cos a_j \partial_t + c\partial_{\mathbf{n}})u_{j-1} = (\cos a_j \partial_t - c\partial_{\mathbf{n}})u_j, & j = 2, \dots, P, \\ u_j(0, .) = 0, & j = 1, \dots, P. \end{cases}$$

By this way, we avoid to use high-order differential operators into the variational formulation. Indeed, it has been shown in [6] that

$$\prod_{j=1}^{P} (\cos a_j\, \partial_t + c\, \partial_{\mathbf{n}})\, u = 0 \iff u_P = 0$$

and

$$(\partial_t^2 - \Delta)u_j = 0, \ \forall j = 1, \dots, P.$$

Then, thanks to these two properties, we can rewrite the problem including now P differential equations on the boundary which can be easily included and we obtain the following system:

$$\begin{cases} \partial_t^2 u - c^2 \triangle u = f, \quad \text{in } \Omega, \\ \partial_{\mathbf{n}} u = 0, \quad \text{on } \Gamma_N, \\ (\cos a_1 \partial_t + c\partial_{\mathbf{n}}) = \partial_t u_1, \quad \text{on } \Gamma_{\text{abs}}, \\ 2\cos a_2 (1 - \cos^2 a_1)\partial_t^2 u + l_{1,1}\partial_t^2 u_1 + (1 - \cos^2 a_2)\partial_t^2 u_2 \\ \qquad\qquad = c^2 (2\cos a_2 \partial_\tau^2 u + \partial_\tau^2 u_1 + \partial_\tau^2 u_2), \quad \text{on } \Gamma_{\text{abs}}, \\ l_{j,j-1}\partial_t^2 u_{j-1} + l_{j,j}\partial_t^2 u_j + l_{j,j+1}\partial_t^2 u_{j+1} \\ \qquad = c^2 (m_{j,j-1}\partial_\tau^2 u_{j-1} + m_{j,j}\partial_\tau^2 u_j + m_{j,j+1}\partial_\tau^2 u_{j+1}), \quad \text{for } j = 2, \dots, P-1, \ \text{on } \Gamma_{\text{abs}}, \\ u_P = 0, \quad \text{on } \Gamma_{\text{abs}}, \end{cases}$$

where τ is the tangential component such that $(\mathbf{n},\tau)$ is a direct basis and

$$\begin{cases} l_{1,1} = 1 + 2\cos a_2 \cos a_1 + \cos^2 a_2, \\ l_{j,j-1} = \cos a_{j+1}(1-\cos^2 a_j), \\ l_{j,j} = \cos a_{j+1}(1+\cos^2 a_j) + \cos a_j(1+\cos^2 a_{j+1}), \\ l_{j,j+1} = \cos a_j(1-\cos^2 a_{j+1}), \end{cases}$$

and

$$\begin{cases} m_{j,j-1} = \cos a_{j+1}, \\ m_{j,j} = \cos a_{j+1} + \cos a_j, \\ m_{j,j+1} = \cos a_j. \end{cases}$$

Now, let us introduce the approximation space to discretize the ABC. Let W_h^k be defined as

$$W_h^k = \left\{ w \in L^2(\Gamma_{\rm abs});\ w_{|\Sigma} \in P^k(\Sigma), \forall \Sigma \in \Sigma_{\rm abs} \right\}.$$

The equations on $\Gamma_{\rm abs}$ are discretized by a 1D IPDG approximation and we define similar notations to the 2D case. $N_{\rm abs}$ is the set of the vertices of the edges of $\Sigma_{\rm abs}$; for each point p in $N_{\rm abs}$, we arbitrarily denote by Σ^+ and Σ^- the two edges sharing p, and by $\boldsymbol{\nu}^\pm$ the unit tangent vector to $\Sigma^\pm$ in p. The definition of the jumps and the averages are the same as in Section 2.

For a given j, consider the equation

$$l_{j,j-1}\partial_t^2 u_{j-1} + l_{j,j}\partial_t^2 u_j + l_{j,j+1}\partial_t^2 u_{j+1} = c^2(m_{j,j-1}\partial_\tau^2 u_{j-1} + m_{j,j}\partial_\tau^2 u_j + m_{j,j+1}\partial_\tau^2 u_{j+1}),$$

whose variational formulation reads as

$$\begin{aligned} \forall w \in H^1(\Gamma_{\rm abs}),\ \sum_{\Sigma\in\Sigma_{\rm abs}} \int_\Sigma \left(l_{j,j-1}\partial_t^2 u_{j-1} + l_{j,j}\partial_t^2 u_j + l_{j,j+1}\partial_t^2 u_{j+1}\right) w \\ = -m_{j,j-1}a_{j,j-1}(u_{j-1},w) - m_{j,j}a_{j,j}(u_j,w) - m_{j,j+1}a_{j,j+1}(u_{j+1},w), \end{aligned}$$

where

$$a_{i,j}(u,w) = \sum_{\Sigma\in\Sigma_{\rm abs}} \int_\Sigma c^2 \partial_\tau u \partial_\tau w - \sum_{z\in N_{\rm abs}} \left(\{\!\{w\}\!\}[\![u]\!] + \{\!\{u\}\!\}[\![w]\!] - \alpha_{i,j}[\![u]\!][\![w]\!] \right)$$

and $\alpha_{i,j}$ is the penalization term depending on $\cos a_i$ and $\cos a_j$.

We obtain then,

$$\begin{cases} M\dfrac{d^2U}{dt^2} + C\dfrac{dU}{dt} + KU = F + G\dfrac{dU^1}{dt}, \quad \text{in } \Omega, \\ B_1\dfrac{d^2U}{dt^2} + l_{1,1}B_2\dfrac{d^2U^1}{dt^2} + (1-\cos^2 a_2)B_2\dfrac{d^2U^2}{dt^2} + EU + DU^1 + DU^2 = 0, \quad \text{on } \Gamma_{\rm abs}, \\ l_{j,j-1}B_2\dfrac{d^2U^{j-1}}{dt^2} + l_{j,j}B_2\dfrac{d^2U^j}{dt^2} + l_{j,j+1}B_2\dfrac{d^2U^{j+1}}{dt^2} \\ \quad + m_{j,j-1}DU^{j-1} + m_{j,j}DU^j + m_{j,j+1}DU^{j+1} = 0, \quad \text{for } j = 2,\dots,P-1, \text{ on } \Gamma_{\rm abs}, \\ U^P = 0, \quad \text{on } \Gamma_{\rm abs}, \end{cases}$$

where U is the solution vector, U^j the auxiliary functions, M the mass matrix, K the stiffness matrix, F the source vector and all the other matrices come from the ABC.

To simplify, we rewrite this system. We have:

$$R\frac{d^2X}{dt^2} + S\frac{dX}{dt} + TX = \begin{pmatrix} F \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where X is the vector of all the unknowns (U and U^j).

Next, we apply a time-discretization using a second-order finite difference scheme:

$$\left(R + \frac{\Delta t}{2}S\right)X^{n+1} = \Delta t^2 \begin{pmatrix} F(n\Delta t, .) \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \Delta t^2 TX^n + 2RX^n - RX^{n-1} + \frac{\Delta t}{2}SX^{n-1},$$

with $X^n = X(n\Delta t)$ and Δt is the time step. Note that, since M, B_1, B_2, C and G are block-diagonal matrices, $\left(R + \frac{\Delta t}{2}S\right)$ is easily invertible.

§4. The Hagstrom-Warburton's condition

In this section, we study a new condition proposed by T. Hagstrom and T. Warburton [7] which takes into account not only propagative waves but also evanescent waves. More accuracy is then expected.

The Hagstrom-Warburton's ABC (H-W ABC) of order $P + Q$, ($P, Q \in \mathbb{N}$) is given by

$$\left[\prod_{j=1}^{Q}(\sigma_j + \partial_{\mathbf{n}})\right]\left[\prod_{j=1}^{P}(\cos a_j\,\partial_t + c\partial_{\mathbf{n}})\right]u = 0. \tag{2}$$

For the same reasons as for the Higdon's ABC, we introduce auxiliary functions defined on the absorbing boundary:

$$\begin{cases} (\cos a_1\partial_t + c\partial_{\mathbf{n}})u = \cos a_1\partial_t u_1, \\ (\cos a_j\partial_t + c\partial_{\mathbf{n}})u_{j-1} = (\cos a_j\partial_t - c\partial_{\mathbf{n}})u_j, \quad \text{for } 2 \le j \le P, \\ (\sigma_j + \partial_{\mathbf{n}})u_{P+j-1} = (\sigma_j - \partial_{\mathbf{n}})u_{P+j}, \quad \text{for } 1 \le j \le Q, \\ u_j((x,y),0) = 0, \quad \text{for } 1 \le j \le P+Q. \end{cases}$$

As for the Higdon's ABC, we have (cf. [6]):

$$\left[\prod_{j=1}^{Q}(\sigma_j + \partial_{\mathbf{n}})\right]\left[\prod_{j=1}^{P}(\cos a_j\,\partial_t + c\partial_{\mathbf{n}})\right]u = 0 \iff u_{P+Q} = 0$$

and

$$\forall j \in 1, \ldots, P+Q,\ (\partial_t^2 - \Delta)u_j = 0.$$

Hence, the system can be rewritten in a more convenient way (cf. [5]). The approach is the same as before except when j is equal to P. For $j < P$ or $j > P$, we get:

$$\begin{cases} 2\cos a_2(1-\cos^2 a_1)\partial_t^2 u + l_{1,1}\cos a_1 \partial_t^2 u_1 + \cos a_1(1-\cos^2 a_2)\partial_t^2 u_2 \\ \qquad\qquad = 2c^2\cos a_2 \partial_\tau^2 u + c^2(\cos a_1 \partial_\tau^2 u_1 + \cos a_1 \partial_\tau^2 u_2), \quad \text{on } \Gamma_{\text{abs}}, \\ l_{j,j-1}\partial_t^2 u_{j-1} + l_{j,j}\partial_t^2 u_j + l_{j,j+1}\partial_t^2 u_{j+1} \\ \qquad = c^2(m_{j,j-1}\partial_\tau^2 u_{j-1} + m_{j,j}\partial_\tau^2 u_j + m_{j,j+1}\partial_\tau^2 u_{j+1}), \quad \text{for } j = 2,\dots,P-1, \text{ on } \Gamma_{\text{abs}}, \\ \bar{l}_{j,j-1}\partial_t^2 u_{P+j-1} + \bar{l}_{j,j}\partial_t^2 u_{P+j} + \bar{l}_{j,j+1}\partial_t^2 u_{P+j+1} \\ \qquad = c^2(\bar{m}_{j,j-1}\partial_\tau^2 u_{P+j-1} + \bar{m}_{j,j}\partial_\tau^2 u_{P+j} + \bar{m}_{j,j+1}\partial_\tau^2 u_{P+j+1}) \\ \qquad\quad + c^2(\bar{s}_{j,j-1}u_{P+j-1} + \bar{s}_{j,j}u_{P+j} + \bar{s}_{j,j+1}u_{P+j+1}), \quad \text{for } j = 2,\dots,P-1, \text{on } \Gamma_{\text{abs}}, \end{cases}$$

where l, m are the coefficients defined in Section 3 and $\bar{l}$, $\bar{m}$ and $\bar{s}$ are given by:

$$\begin{cases} \bar{l}_{j,j-1} = \bar{m}_{j,j-1} = \dfrac{1}{\sigma_j}, \\ \bar{l}_{j,j} = \bar{m}_{j,j} = \dfrac{1}{\sigma_j} + \dfrac{1}{\sigma_{j+1}}, \\ \bar{l}_{j,j+1} = \bar{m}_{j,j+1} = \dfrac{1}{\sigma_{j+1}}, \end{cases} \qquad \text{and} \qquad \begin{cases} \bar{s}_{j,j-1} = \sigma_j, \\ \bar{s}_{j,j} = -(\sigma_j + \sigma_{j+1}), \\ \bar{s}_{j,j+1} = \sigma_{j+1}. \end{cases}$$

When $j = P$, we have to introduce a seam function ψ which makes the link between the two kinds of auxiliary functions: those defined for the propagative waves (using cos) and those for the evanescent ones (using σ). Hence, we get two equations for $j = P$ which are:

$$\begin{cases} (1-\cos^2 a_P)\partial_t^2 u_{P-1} + (\cos^2 a_P + 1)\partial_t^2 u_P + \cos^2 a_P \partial_t^2 \psi = c^2(\partial_\tau^2 u_{P-1} + \partial_\tau^2 u_P), \quad \text{on } \Gamma_{\text{abs}}, \\ \partial_t^2 u_P + \partial_t^2 u_{P+1} - \cos a_P \sigma_1 c \partial_t \psi = \sigma_1^2 c^2(u_P + u_{P+1}) + c^2(\partial_\tau^2 u_P + \partial_\tau^2 u_{P+1}), \quad \text{on } \Gamma_{\text{abs}}. \end{cases}$$

For the space-discretization, we use a similar method to the one described in Section 3 and we finally get:

$$\left(R_2 + \frac{\Delta t}{2}S_2\right)X^{n+1} = \Delta t^2 \begin{pmatrix} F(n\Delta t, .) \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \Delta t^2 T_2 X^n + 2R_2 X^n - R_2 X^{n-1} + \frac{\Delta t}{2}S_2 X^{n-1},$$

where X is the vector of all the unknowns: u, u_j and ψ.

§5. Numerical results

We have considered the square $[-2; 2] \times [-2; 2]$ and the following Ricker-type source:

$$f(x, y, t) = \delta(x - x_0, y - y_0)2\lambda(2\lambda(t - t_0)^2 - 1)e^{-\lambda(t-t_0)^2},$$

(x_r, y_r)	Higdon				H-W
	$a_1 = 0$	$a_1 = \frac{\pi}{6}$	$a_1 = a_2 = 0$	$a_1 = 0, a_2 = \frac{\pi}{6}$	$a_1 = a_2 = 0$ $\sigma_1 = 10$
$(0, -1.8)$	1.98	9.14	0.54	0.54	0.54
$(0.7, -1.8)$	7.3	4.06	0.61	0.60	0.60
$(1.8, 1.8)$	18.0	12.0	1.02	0.82	0.90

Table 1: Relative L^2 error for Higdon's and H-W conditions

where $\lambda = (5\pi)^2$, $t_0 = 0.2$, $(x_0, y_0) = (0, -1)$ and δ denotes the Dirac distribution. The penalization coefficient in the domain is $\alpha = 8$. We have computed the solution U^{app} near the absorbing boundary at three different points (x_r, y_r) equal to $(0, -1.8)$, $(0.7, -1.8)$ and $(1.3, -1.8)$ for different values of the coefficients a_j and σ_j and we have compared it to the exact solution U (i.e. the solution of the wave equation in $\mathbb{R}^2$). On Tab.1, we represent the relative $L^2([0,T])$ error, $err = \frac{\|U^{\text{app}} - U\|_{L^2([0,T])}}{\|U\|_{L^2([0,T])}} * 100$ for four Higdon's conditions ($a_1 = 0$, $a_1 = \pi/6$, $a_1 = 0$ and $a_2 = 0$,$a_1 = 0$ and $a_2 = \pi/6$) and one H-W condition ($a_1 = a_2 = 0$ and $\sigma = 10$).

For the first two tests, we have no auxiliary functions since we consider first-order conditions. For the three other tests we have imposed the same penalization coefficient $\alpha_j = 16$ for all the auxiliary equations on Γ. We remark that, as expected, the second-order Higdon's condition performs better than the first-order one. However, the third-order H-W condition does not improve the error as compared to the second-order Higdon's condition. This is due to the discretization method, since the accuracy of the ABC can be improved by decreasing the penalization coefficient but if this coefficient is too small the scheme becomes unstable. We have observed the same problem with thethird-order Higdon's condition and for higher-order conditions too. Moreover, for some particular coefficients, for instance $a_1 = 0$, $a_2 = \pi/6$ and $a_3 = \pi/4$, the scheme is unconditionnaly unstable (i.e. there is no penalization parameters that stabilize the scheme). In Fig. 1, we have represented the solution U^{app} for these coefficients at point $(0, -1.8)$.

Therefore, the method of auxiliary functions proposed in [3] to implement Higdon's and H-W conditions is not adapted to an IPDG approximation and we are now considering other type of ABC compatible with the IPDG method. In the same time, we are looking for an enriched IPDG scheme which is able to use the ABCs we consider in this work.

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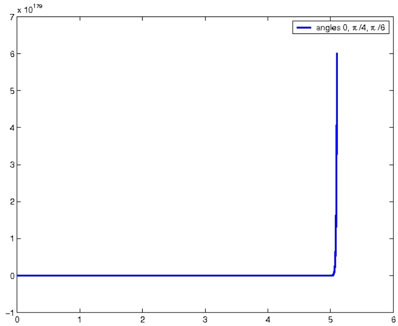

Figure 1: The solution U^{app} at point (0,-1.8) for $a_1 = 0$, $a_2 = \pi/6$ and $a_3 = \pi/4$

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