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# ON THE HELICAL FLOW OF NEWTONIAN FLUIDS INDUCED BY TIME DEPENDENT SHEAR

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**Abstract.** The velocity field and the shear stresses corresponding to the unsteady flow of Newtonian fluids in an infinite circular cylinder are determined by means of the Hankel and Laplace transforms. The motion is produced by the infinite cylinder that at the initial moment is subject to both longitudinal and rotational time dependent shear stresses.

*Keywords:* Newtonian fluids, velocity field, tangential stress, cylindrical domains.
*AMS classification:* 53B25, 53C15.

## §1. Introduction

The study on the flow of a viscous fluid in a circular cylinder is not only of fundamental theoretical interest but it also occurs in many applied problems. The starting solutions for the motion of the second grade fluids due to longitudinal and torsional oscillations of a circular cylinder have been studied by Fetecau in [3]. Vieru et al [6], by means of the Laplace transform and Cauchy's residue theorem, have determined the starting solutions for the oscillating motion of a Maxwell fluid. Akhtar and Nazar [1] have studied the rotational flow of generalized Maxwell fluids in a circular cylinder which rotates around its axis.

The aim of this paper is to study the flow of a Newtonian fluid in an infinite circular cylinder of radius R. The motion is produced by the cylinder that at the initial moment is subjected to longitudinal and torsional time dependent shear stresses. The exact solutions of the problems with initial and boundary conditions are determined by means of the finite Hankel and Laplace transforms. The solutions obtained in this paper can be used to make a comparison between flows of Newtonian and non-Newtonian fluids.

## §2. Governing equations

The Cauchy stress in an incompressible Newtonian fluid is characterized by the next constitutive equation [5]:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}, \tag{1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericksen tensor, $\mathbf{L}$ is the velocity gradient, $\mu$ is the dynamic viscosity, the superscript $T$ denotes the transpose operator.

In cylindrical coordinates $(r, \theta, z)$, the velocity of the flow is given by

$$\mathbf{v} = \mathbf{v}(r, t) = w(r, t)\mathbf{e}_\theta + v(r, t)\mathbf{e}_z, \tag{2}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are the unit vectors in the $\theta$ and $z$ directions respectively. For such flows the constraint of incompressibility is automatically satisfied.

Introducing (2) into constitutive equation (1), we find

$$\tau_1(r,t) = \mu \frac{\partial v(r,t)}{\partial r}, \tag{3}$$

$$\tau_2(r,t) = \mu\Big(\frac{\partial w(r,t)}{\partial r} - \frac{1}{r} w(r,t)\Big), \tag{4}$$

where $\tau_1(r,t) = S_{rz}(r,t)$ and $\tau_2(r,t) = S_{r\theta}(r,t)$ are the shear stress which is different of zero. The last equations together with the equations of motion leads to the governing equations [4]

$$\frac{\partial v(r,t)}{\partial t} = \nu\Big(\frac{\partial^2 v(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,t)}{\partial r}\Big), \quad r \in (0,R), \quad t > 0, \tag{5}$$

$$\frac{\partial w(r,t)}{\partial t} = \nu\Big(\frac{\partial^2 w(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,t)}{\partial r} - \frac{1}{r^2} w(r,t)\Big), \quad r \in (0,R), \quad t > 0, \tag{6}$$

where $\nu = \mu/\rho$ is the kinematic viscosity and $\rho$ is the constant density of the fluid.

## §3. Helical flow through an infinite circular cylinder

Let us consider an incompressible Newtonian fluid at rest in an infinite circular cylinder of radius $R$. At time zero, the cylinder suddenly begins to rotate and move along its axis due to time dependent shear stress. Owing to the shear, the fluid is gradually moved, its velocity being given by Eq.(2) and the governing equations are (5) and (6). The appropriate initial and boundary conditions are

$$v(r,0) = 0, \quad w(r,0) = 0; \quad r \in [0,R], \tag{7}$$

$$\tau_1(R,t) = \mu \frac{\partial v(R,t)}{\partial r} = f.t; \quad t \geq 0, \tag{8}$$

$$\tau_2(R,t) = \mu\Big(\frac{\partial w(R,t)}{\partial r} - \frac{1}{R} w(R,t)\Big) = f.t; \quad t \geq 0. \tag{9}$$

To solve this problem we shall use as in [1, 2] the Laplace and Hankel transforms.

### 3.1. Calculation of the velocity field

Applying the Laplace transform to Eqs. (5), (6), (8) and (9) and using Eq. (7) we obtain the following problems with boundary conditions

$$q\bar{v}(r,q) = \nu\Big(\frac{\partial^2 \bar{v}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}(r,q)}{\partial r}\Big), \tag{10}$$

$$\frac{\partial \bar{v}(R,q)}{\partial r} = \frac{f}{\mu q^2}, \tag{11}$$

$$q\overline{w}(r,q) = \nu\Big(\frac{\partial^2 \overline{w}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{w}(r,q)}{\partial r} - \frac{1}{r^2}\overline{w}(r,q)\Big), \tag{12}$$

$$\frac{\partial \overline{w}(R,q)}{\partial r} - \frac{1}{R}\overline{w}(R,q) = \frac{f}{\mu q^2}, \tag{13}$$

where

$$\bar{v}(r,q)=\int_0^\infty v(r,t)e^{-qt}dt,\quad \bar{w}(r,q)=\int_0^\infty w(r,t)e^{-qt}dt$$

are the Laplace transforms of $v(r,t)$ and $w(r,t)$ respectively. In the following we denote by

$$\bar{v}_H(r_{0n},q)=\int_0^R r\bar{v}(r,q)J_0(rr_{0n})\,dr,\quad \bar{w}_H(r_{1n},q)=\int_0^R r\bar{w}(r,q)J_1(rr_{1n})\,dr, \tag{14}$$

the finite Hankel transforms of $\bar{v}(r,q)$ and $\bar{w}(r,q)$ respectively, where $J_0(\cdot)$ and $J_1(\cdot)$ are the Bessel functions of first kind of order zero and one and $r_{0n}$ and $r_{1n}$, for $n=1,2,3,\ldots$, are the positive roots of the transcendental equations $J_1(Rr)=0$ and $J_2(Rr)=0$ respectively.

Multiplying both sides of Eq. (10) by $rJ_0(rr_{0n})$, integrating with respect to $r$ from 0 to $R$ and taking into account the condition (11) and the equality

$$\int_0^R r\Big[\frac{\partial^2\bar{v}(r,q)}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{v}(r,q)}{\partial r}\Big]J_0(rr_{0n})\,dr=\frac{RfJ_0(Rr_{0n})}{\mu q^2}-r_{0n}^2\bar{v}_H(r_{0n},q), \tag{15}$$

we find that

$$\bar{v}_H(r_{0n},q)=\frac{Rf}{\rho}J_0(Rr_{0n})\frac{1}{q^2(q+\nu r_{0n}^2)}. \tag{16}$$

Multiplying both sides of Eq. (12) by $rJ_1(rr_{1n})$, integrating with respect to $r$ from 0 to $R$ and taking into account the condition (13) and the equality

$$\int_0^R r\Big[\frac{\partial^2\bar{w}(r,q)}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{w}(r,q)}{\partial r}-\frac{1}{r^2}\bar{w}(r,q)\Big]J_1(rr_{1n})\,dr=\frac{RfJ_1(Rr_n)}{\mu q^2}-r_{1n}^2\bar{w}_H(r_{1n},q), \tag{17}$$

we find that

$$\bar{w}_H(r_{1n},q)=\frac{Rf}{\rho}J_1(Rr_{1n})\frac{1}{q^2(q+\nu r_{1n}^2)}. \tag{18}$$

Now, for a more suitable presentation of the final results, we rewrite Eqs. (16) and (18) in the following equivalent forms

$$\bar{v}_H(r_{0n},q)=\bar{v}_{1H}(r_{0n},q)+\bar{v}_{2H}(r_{0n},q), \tag{19}$$

where

$$\bar{v}_{1H}(r_{0n},q)=\frac{RfJ_0(Rr_{0n})}{r_{0n}^2}\frac{1}{\mu q^2},\quad \bar{v}_{2H}(r_{0n},q)=-\frac{RfJ_0(Rr_{0n})}{\mu r_{0n}^2}\frac{1}{q(q+\nu r_{0n}^2)} \tag{20}$$

and

$$\bar{w}_H(r_{1n},q)=\bar{w}_{1H}(r_{1n},q)+\bar{w}_{2H}(r_{1n},q), \tag{21}$$

where

$$\bar{w}_{1H}(r_{1n},q)=\frac{RfJ_1(Rr_{1n})}{r_{1n}^2}\frac{1}{\mu q^2},\quad \bar{w}_{2H}(r_{1n},q)=-\frac{RfJ_1(Rr_{1n})}{\mu r_{1n}^2}\frac{1}{q(q+\nu r_{1n}^2)} \tag{22}$$

A straightforward calculus deals to the following function-Hankel transform pairs

$$f(r)=\frac{fr^2}{2R},\quad f_H(r_{0n})=\frac{RfJ_0(Rr_{0n})}{r_{0n}^2},\quad g(r)=\frac{fr^3}{2R^2},\quad g_H(r_{1n})=\frac{RfJ_1(Rr_{1n})}{r_{1n}^2}. \tag{23}$$

The inverse Hankel transforms of the functions $\bar{v}_{2H}(r_{0n}, q)$ and $\overline{w}_{2H}(r_{1n}, q)$ are [2]

$$\bar{v}_2(r,q) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{J_0^2(Rr_{0n})}\bar{v}_{2H}(r_{0n},q),$$
$$\overline{w}_2(r,q) = -2\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_1(rr_{1n})}{[(r_{1n}^2+h^2)R^2-1]J_1^2(Rr_{1n})}\overline{w}_{2H}(r_{1n},q), \tag{24}$$

where $h = -\frac{1}{R}$.

Applying the inverse Hankel transform to Eqs. (19)-(22) and using (23) and (24) we obtain the following form of the Laplace transforms of the functions $v(r,t)$ and $w(r,t)$

$$\bar{v}(r,q) = \frac{fr^2}{2R}\frac{1}{\mu q^2} - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r_{0n}^2 J_0(Rr_{0n})}\frac{1}{q(q+\nu r_{0n}^2)}, \tag{25}$$

$$\overline{w}(r,q) = \frac{fr^3}{2R}\frac{1}{\mu q^2} - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r_{1n}^2 J_1(Rr_{1n})}\frac{1}{q(q+\nu r_{1n}^2)}. \tag{26}$$

We denote by

$$h_i(r_{in},q) = \frac{1}{q+\nu r_{in}^2},\ i = 0,1,$$

and have [2]

$$L^{-1}\{h_i(r_{in},q)\} = \exp(-\nu r_{in}^2).$$

The inverse Laplace transform of the function $g_i(r_{in},q) = \frac{1}{q}h_i(r_{in},q)$ is

$$g_i(r_{in},t) = \int_0^t h_i(r_{in},u) = \frac{1}{\nu r_{in}^2}[1-\exp(-\nu r_{in}^2 t)]. \tag{27}$$

Applying inverse Laplace transform to Eqs. (25) and (26) and using (27) we find the following forms of the velocity fields:

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r_{0n}^4 J_0(Rr_{0n})}[1-\exp(-\nu r_{0n}^2 t)], \tag{28}$$

and

$$w(r,t) = \frac{fr^3}{2\mu R^2}t - \frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r_{1n}^4 J_1(Rr_{1n})}[1-\exp(-\nu r_{1n}^2 t)]. \tag{29}$$

## 3.2. Calculation of the shear stresses

Applying the Laplace transform to Eqs. (3) and (4) we find that

$$\overline{\tau}_1(r,q) = \mu\frac{\partial\bar{v}(r,q)}{\partial r}, \tag{30}$$

$$\overline{\tau}_2(r,q) = \mu\Big(\frac{\partial\overline{w}(r,q)}{\partial r} - \frac{1}{r}\overline{w}(r,q)\Big). \tag{31}$$

Differentiating Eqs. (25) and (26) with respect to $r$ we get

$$\frac{\partial\overline{v}(r,q)}{\partial r}=\frac{rf}{R}\frac{1}{\mu q^2}+\frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r_{0n}J_0(Rr_{0n})}\frac{1}{q(q+\nu r_{0n}^2)},\tag{32}$$

respectively

$$\frac{\partial\overline{w}(r,q)}{\partial r}-\frac{1}{r}\overline{w}(r,q)=\frac{fr^2}{R^2}\frac{1}{\mu q^2}+\frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}J_1(Rr_{1n})}\frac{1}{q(q+\nu r_{1n}^2)}.\tag{33}$$

Introducing (32) into (30) and (33) into (31) we find that

$$\overline{\tau}_1(r,q)=\frac{rf}{R}\frac{1}{q^2}+\frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r_{0n}J_0(Rr_{0n})}\frac{1}{q(q+\nu r_{0n}^2)},\tag{34}$$

$$\overline{\tau}_2(r,q)=\frac{r^2f}{R^2}\frac{1}{q^2}+\frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}J_1(Rr_{1n})}\frac{1}{q(q+\nu r_{1n}^2)}.\tag{35}$$

Inverting Eqs. (34) and (35) and using (27), we find the following forms of the shear stresses

$$\tau_1(r,t)=\frac{rft}{R}+\frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r_{0n}^3J_0(Rr_{0n})}[1-\exp(-\nu r_{0n}^2t)],\tag{36}$$

$$\tau_2(r,t)=\frac{r^2ft}{R^2}+\frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}^3J_1(Rr_{1n})}[1-\exp(-\nu r_{1n}^2t)].\tag{37}$$

From (36) and (37) it is easy to verify that $\tau_1(R,t)=ft$ and $\tau_2(R,t)=ft,\quad t\geq 0$.

## §4. Conclusion

In this note, the velocity field and the resulting shear stresses corresponding to the helical flow induced by an infinite circular cylinder in an incompressible Newtonian fluid have been determined using the finite Hankel and Laplace transforms. The motion is produced by the cylinder that at the initial moment is subjected to both rotation and translation by time dependent shear. The solutions that have been obtained satisfy all imposed initial and boundary conditions and can be used to make a comparison between flows of Newtonian and non-Newtonian fluids. For $t\to\infty$, the solutions (28) and (29) reduce to the steady-state solutions

$$v(r,t)=\frac{fr^2}{2\mu R}t-\frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r_{0n}^4J_0(Rr_{0n})},$$

and

$$w(r,t)=\frac{fr^3}{2\mu R^2}t-\frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r_{1n}^4J_1(Rr_{1n})}.$$

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