

MAXIMUM LIKELIHOOD ESTIMATION IN MULTIVARIATE LOGNORMAL DIFFUSION PROCESSES WITH A VECTOR OF EXOGENOUS FACTORS

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Abstract. In this paper we consider a new model of multivariate lognormal diffusion process with a vector of exogenous factors such that each component exclusively affects the respective endogenous variable of the process. Starting from the Kolmogorov differential equations and Ito's stochastic equation of this model, its transition probability density is obtained. A discrete sampling of the process is assumed and the associated conditioned likelihood is calculated. By using matrix differential calculus, the maximum likelihood matrix estimators are obtained and expressed in a computationally feasible form. This model, an extension of previously studied lognormal diffusion processes ([1],[2],[3]), extends the possibility of applications of lognormal dynamic modelling in Economics, Population Growth, Volatility, etc.

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§1. Introduction

Lognormal, logistic and Gompertz stochastic diffusion processes have been widely used to model exponential growth phenomena in economics ([3];[10]), biology ([12]) and other fields.

The lognormal process, moreover, is playing an increasingly important role in fields such as nuclear and mechanical engineering ([9]; [11]) and astrophysics ([15]). Recently, the lognormal process was applied in the modelling of satellite and cellphone-based communication phenomena, in the form of the Nakagami-lognormal process ([1];[14];[2])

In general, these applications consider homogeneous diffusion processes; their infinitesimal moments do not depend on time, but only on the system states. This fact limits their range of applications and prevents the consideration of external influences on the variables that are modelled (endogenous variables). Such external influences can be modelled by exogenous variables or factors that affect the trend and the infinitesimal moments of the process (see [14] with respect to the lognormal case).

Exogenous factors are functions of time that are known and which vary externally to the system. This allows the possibility of controlling the behaviour of the system and of obtaining inferential results to improve the statistical fit of the processes to real data.

The study of lognormal diffusion processes with exogenous factors has been widely addressed in recent years, concerning two main fields of interest, namely first-passage problems and statistical inference. With respect to the first of these, interesting results have been obtained by [5], by [7] and by [8], in the case of univariate processes. In the case of statistical inference, studied at the level of multivariate lognormal processes with multiple exogenous factors common to all the endogenous variables, and for which discrete sampling is utilised, with the corresponding computational treatment, see for example [4] and [6].

In this context of modelling by lognormal diffusions affected by exogenous factors, this paper proposes a new multivariate model in which each endogenous variable can be affected by exogenous factors “ad hoc”, that are different for each endogenous variable. By this procedure we seek to achieve a model that is more flexible for use in real applications and with which the results of statistical inference could be obtained. In the following section, this model is described using the Kolmogorov equations, and the maximum likelihood estimation of its matrix parameters is developed.

§2. Definition of the model

By means of the Kolmogorov equations, we define a model of the multidimensional lognormal diffusion process with two parameters and an exogenous vector, such that each component of this vector affects the corresponding endogenous variable of the infinitesimal trend of the process.

Let $\{X(t); t_0 \leq t \leq T\}$ be a Markov process with values in \mathbb{R}^k , with trajectories that are almost certainly continuous and for which the transition probability is given by

$$P(y, t/x, s) = P\{X(t) = y/X(s) = x\}$$

with $X(t) = (X_{t1}, \dots, X_{tk})$, $X(s) = (X_{s1}, \dots, X_{sk})$
and x and y are two k -dimensional vectors.

Assume the following conditions:

i)

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{|y-x| > \epsilon} P(dy, t+h/x, t) = 0.$$

ii)

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{|y-x| \leq \epsilon} (y-x)P(dy, t+h/x, t) = b(x, t), \quad \text{with}$$

$$b(x, t) = \begin{pmatrix} (\alpha_1 + \gamma_1 g_1(t))x_1 \\ (\alpha_2 + \gamma_2 g_2(t))x_2 \\ \vdots \\ (\alpha_k + \gamma_k g_k(t))x_k \end{pmatrix},$$

where, for $i = 1; \dots, k$, $g_i(t)$ is a continuous function in $[t_0, T]$.

iii)

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{|y-x| \leq \epsilon} (y-x)(y-x)' P(dy, t+h/x, t) = [Diag(X)]A[Diag(X)]',$$

where $A = (a_{ij})_{1 \leq i, j \leq k}$ is a symmetric, non negatively defined matrix with $a_{ij} > 0$, for $i, j=1, 2, \dots, k$; y

$$Diag(X) = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_k \end{pmatrix}$$

iv) The higher order infinitesimal moments are null.

Under the above conditions and for certain differentiability conditions of $P = P(y, t/x, s)$, we obtain the backward and forward Kolmogorov equations, namely:

$$\begin{aligned} \frac{\partial p}{\partial t} &= \frac{1}{2} \sum_{i,j=1}^k a_{ij} \frac{\partial^2 (y_i y_j p)}{\partial y_i \partial y_j} - \sum_{i=1}^k (\alpha_i + \gamma_i g_i(t)) \frac{\partial (y_i p)}{\partial y_i} \\ \frac{\partial p}{\partial s} &= -\frac{1}{2} \sum_{i,j=1}^k a_{ij} x_i x_j \frac{\partial^2 p}{\partial x_i \partial x_j} - \sum_{i=1}^k (\alpha_i + \gamma_i g_i(s)) x_i \frac{\partial p}{\partial x_i} \end{aligned}$$

in which $p = p(y, t/x, s)$, is the conditioned transition density, with the initial solution

$$p(y, t/x, t) = \delta(y - x)$$

The common solution to these equations is

$$p(y, t/x, s) = [(\prod_{i=1}^k y_i) (2\pi)^{k/2} (t-s)^{k/2} |A|^{1/2}]^{-1} \exp \left\{ -\frac{1}{2(t-s)} Q \right\}$$

with Q taking the following quadratic form

$$Q = (\log(y) - \log(x) - \beta(t-s) - \Gamma G(t))' A^{-1} \times (\log(y) - \log(x) - \beta(t-s) - \Gamma G(t))$$

where,

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_k \end{pmatrix}$$

$$G(t) = \left(\int_s^t g_1(r)dr, \int_s^t g_2(r)dr, \dots, \int_s^t g_k(r)dr \right)'$$

and

$$\beta = \left(\alpha_1 - \frac{1}{2}a_{11}, \dots, \alpha_k - \frac{1}{2}a_{kk} \right)' = \alpha - \frac{1}{2}diag(A)$$

with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)'$ and $diag(A)$ is the k -vector formed by the elements of the diagonal of A .

§3. Maximum-likelihood estimation of the process parameters

3.1.

Let the following notation describe the transition density function of the above process:

$$B = (\beta; \Gamma) \text{ y } \bar{v}_{t,s} = \begin{pmatrix} t-s \\ G(t,s) \end{pmatrix}; \quad B = \left(\begin{array}{c|ccc} \beta_1 & \gamma_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \beta_k & 0 & 0 & \gamma_k \end{array} \right)$$

In terms of B and $\bar{v}_{t,s}$ the transition density is written as:

$$f(y, t/x, s) = \left[\left(\prod_{i=1}^k y_i \right) (2\pi)^{k/2} (t-s)^{k/2} |A|^{\frac{1}{2}} \right]^{-1} \exp \left\{ \frac{-Q}{2(t-s)} \right\}$$

where

$$Q = (\lg y - \lg x - B\bar{v}_{t,s})' A^{-1} (\lg y - \lg x - B\bar{v}_{t,s})$$

Therefore, this transition density corresponds to a log-normal k -dimensional function with parameters $\lg x - B\bar{v}_{t,s}$ and $(t-s)A$, that is:

$$X_t/X_s = x \rightarrow \Lambda_k [\lg x - B\bar{v}_{t,s}; (t-s)A]$$

The parameters to be estimated are β , Γ and A or in matrix terms, A and B . The problem arising is that matrix B has structural zeros. To avoid this difficulty occurring in the subsequent calculation of the maximum likelihood estimators, it is necessary to introduce a parametric matrix that does not contain any zeros. Let:

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix}; \quad \Delta = (\beta; \gamma) = \begin{pmatrix} \beta_1, & \gamma_1 \\ \vdots & \vdots \\ \beta_k, & \gamma_k \end{pmatrix}$$

such that Δ contains all the non-null parameters of matrix B and contains no structural zeros.

3.2.

It is useful to determine a relation between the matrices B and Δ , one that can be utilised extensively in calculating the estimators. It can be shown that:

$$B = \sum_{j=1}^k E_{jj} \Delta H'_j$$

In fact:

$$vec(B) = \begin{pmatrix} \beta \\ vec(\Gamma) \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & D_{ik} \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & D_{ik} \end{pmatrix} vec(\Delta)$$

where

$$D_{ik} = \sum_{j=1}^k u_j \otimes E_{jj}; \quad \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & D_{ik} \end{pmatrix} = \sum_{j=1}^k (e_1; e_{j+1}) \otimes E_{jj}$$

using H_j to describe $(e_1; e_{j+1})$ we then obtain:

$$vec(B) = \sum_{j=1}^k (H_j \otimes E_{jj}) vec(\Delta) = \sum_{j=1}^k vec(E_{jj} \Delta H'_j) = vec\left(\sum_{j=1}^k E_{jj} \Delta H'_j\right)$$

which proves the relation stated above. Moreover, let:

$$T = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & D_{ik} \end{pmatrix}$$

This box matrix with dimensions $(k \times k)$ for the two in the first row and $(k^2 \times k)$ for those in the second, is full rank by columns, such that:

$$T_g = (T'T)^{-1}T' = T'$$

and T can be expressed as:

$$T = \left(\begin{array}{c|c} \sum_{j=1}^k E_{jj} & \mathbf{0} \\ \hline \mathbf{0} & \sum_{j=1}^k u_j \otimes E_{jj} \end{array} \right) = \sum_{j=1}^k \left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & u_j \end{array} \right) \otimes E_{jj} = \sum_{j=1}^k H_j \otimes E_{jj}$$

3.3.

Parameters A and B of the process must now be estimated by maximum likelihood. The objective is to construct the likelihood associated with the diffusion process, which is achieved, in Markov processes, by means of transition densities.

1. The process is considered to be observed by discrete sampling, that is, we observe the process at instants t_1, t_2, \dots, t_n , thus obtaining the sample $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ of values of the process at these instants. The process is k -dimensional, and therefore, $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ are k -dimensional vectors:

$$X_{t_1} = (X_{t_1,1}; \dots; X_{t_1,k})'$$

and we write x_1, x_2, \dots, x_n and in general x_α , for $\alpha = 1, \dots, n$ with x_α representing the k -dimensional vector of the values observed.

2. We consider the transition density values of the process between each two consecutive instants, assuming that, with a probability of 1, the value at t_1 is $X_{t_1} = x_1$, that is:

$$\begin{aligned} P[X_{t_1} = x_1] &= 1 \\ P[X_{t_2} = x_2/X_{t_1} = x_1] & \\ P[X_{t_3} = x_3/X_{t_2} = x_2] & \\ \vdots & \\ P[X_{t_\alpha} = x_\alpha/X_{t_{\alpha-1}} = x_{\alpha-1}] & \\ \vdots & \\ P[X_{t_n} = x_n/X_{t_{n-1}} = x_{n-1}] & \end{aligned}$$

The conditioned likelihood function takes the form

$$\begin{aligned} L(x_1, \dots, x_n/B, A) &= P[X_{t_1} = x_1]P[X_{t_2} = x_2/X_{t_1} = x_1] \dots P[X_{t_n} = x_n/X_{t_{n-1}} = x_{n-1}] = \\ & \prod_{\alpha=1}^n \left[\left(\prod_{i=1}^k x_{\alpha i} \right) (2\pi)^{k/2} (t_\alpha - t_{\alpha-1})^{k/2} |A|^{1/2} \right]^{-1} \times \\ & \exp \left[-\frac{1}{2(t_\alpha - t_{\alpha-1})} [\lg x_\alpha - \lg x_{\alpha-1} - B\bar{v}_\alpha]' A^{-1} [\lg x_\alpha - \lg x_{\alpha-1} - B\bar{v}_\alpha] \right] \end{aligned}$$

NOTE: $P[X_{t_\alpha} = x_\alpha/X_{t_{\alpha-1}} = x_{\alpha-1}]$ is obtained from $P[X_t = y/X_s = x]$ with $t \rightarrow t_\alpha$, $s \rightarrow t_{\alpha-1}$; $y \rightarrow x_\alpha$, $x \rightarrow x_{\alpha-1}$ and moreover $\bar{v}_\alpha = \bar{v}_{\alpha,\alpha-1}$ and $x_{\alpha i}$ is the i -th component of x_α ,

$$x_\alpha = \begin{pmatrix} x_{\alpha 1} \\ \vdots \\ x_{\alpha k} \end{pmatrix} \bullet$$

The above conditioned likelihood can, in turn, be written as:

$$\begin{aligned} L(x_1, \dots, x_n/B, A) &= (2\pi)^{-\frac{(n-1)k}{2}} |A|^{-\frac{n-1}{2}} \prod_{\alpha=2}^n \left[\left(\prod_{i=1}^k x_{\alpha i} \right) (t_\alpha - t_{\alpha-1})^{k/2} \right]^{-1} \times \\ & \exp \left[-\frac{1}{2} [(t_\alpha - t_{\alpha-1})^{-1/2} (\lg x_\alpha - \lg x_{\alpha-1}) - Bv_\alpha]' A^{-1} [\quad] \right] \end{aligned}$$

where $v_\alpha = (t_\alpha - t_{\alpha-1})^{-1/2} \bar{v}_\alpha$, for $\alpha = 2, \dots, n$.

Performing the change of variable

$$z_1 = x_1$$

$$z_2 = (t_2 - t_1)^{-1/2}(\lg x_2 - \lg x_1)$$

⋮

$$z_n = (t_n - t_{n-1})^{-1/2}(\lg x_n - \lg x_{n-1})$$

and applying the theorem of the change of variable:

$$L(x_1, \dots, x_n/B, A) \Rightarrow L(z_1, \dots, z_n/B, A)$$

where:

$$\begin{aligned} L(z_1, \dots, z_n/B, A) &= (2\pi)^{-\frac{(n-1)k}{2}} |A|^{-\frac{n-1}{2}} \exp \left[-\frac{1}{2} \sum_{\alpha=2}^n (z_\alpha - Bv_\alpha)' A^{-1} (z_\alpha - Bv_\alpha) \right] = \\ &= (2\pi)^{-\frac{(n-1)k}{2}} |A|^{-\frac{n-1}{2}} \exp \left[-\frac{1}{2} \text{tr} \left[A^{-1} \sum_{\alpha=2}^n (z_\alpha - Bv_\alpha)(z_\alpha - Bv_\alpha)' \right] \right] \end{aligned}$$

but

$$\sum_{\alpha=2}^n (z_\alpha - Bv_\alpha)(z_\alpha - Bv_\alpha)' = (Z - Bv)(Z - Bv)'$$

with $Z = (z_2, \dots, z_n)$ of dimension $k \times (n-1)$ and $v = (v_2, \dots, v_n)$ of dimension $(k+1) \times (n-1)$; therefore:

$$\begin{aligned} L(x_1, \dots, x_n/B, A) &\Rightarrow L(z_1, \dots, z_n/B, A) = \\ &(2\pi)^{-\frac{(n-1)k}{2}} |A|^{-\frac{n-1}{2}} \exp \left[-\frac{1}{2} \text{tr} [A^{-1}(Z - Bv)(Z - Bv)'] \right] \end{aligned}$$

The next step is to differentiate with respect to parameters A and B. Calculating the differential of the logarithm of the likelihood function:

$$\begin{aligned} \lg L(z_2, \dots, z_n/B, A) &= -\frac{(n-1)k}{2} \lg(2\pi) - \frac{n-1}{2} \lg |A| - \frac{1}{2} \text{tr} [A^{-1}(Z - Bv)(Z - Bv)'] \\ d \lg L(z_2, \dots, z_n/B, A) &= -\frac{n-1}{2} \text{tr} [A^{-1}(dA)] - \frac{1}{2} \text{tr} [-A^{-1}(dA)A^{-1}(Z - Bv)(Z - Bv)'] - \\ &\quad - \frac{1}{2} \text{tr} [-A^{-1}(dB)v(Z - Bv)' - A^{-1}(Z - Bv)v'(dB)'] \end{aligned}$$

expressions that are obtained by applying the following rules of derivation:

1. $d_A \lg |A| = \text{tr} [A^{-1}(dA)]$
2. $d_A \text{tr} [A^{-1}(Z - Bv)(Z - Bv)'] = \text{tr} [-A^{-1}(dA)A^{-1}(Z - Bv)(Z - Bv)']$
3. $d_B \text{tr} [A^{-1}(Z - Bv)(Z - Bv)'] = \text{tr} [-A^{-1}(dB)v(z - Bv)'] + \text{tr} [-A^{-1}(Z - Bv)v'(dB)']$

Thus we have:

$$\begin{aligned} d \lg L(z_2, \dots, z_n/B, A) &= \frac{1}{2} \text{tr} [A^{-1}(dA)A^{-1}(Z - Bv)(Z - Bv)' - (n - 1)A^{-1}(dA)] + \\ &\quad + \frac{1}{2} \text{tr} [2v(Z - Bv)'A^{-1}(dB)] = \\ &= \frac{1}{2} \text{Vec}' [A^{-1}[(Z - Bv)(Z - Bv)' - (n - 1)A]A^{-1}] d\text{Vec}(A) + \\ &\quad + \text{Vec}' [A^{-1}(Z - Bv)v'] T d\text{Vec}(\Delta) \end{aligned}$$

as $\text{Vec}(B) = T\text{Vec}(\Delta)$, where

$$T = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & D_{ik} \end{array} \right)$$

as shown above. Finally, the maximum likelihood method requires $d \lg L = 0$, which implies:

$$\begin{aligned} \text{Vec}' [A^{-1}[(Z - Bv)(Z - Bv)' - (n - 1)A]A^{-1}] d\text{Vec}(A) &= 0 \\ \text{Vec}' [A^{-1}(Z - Bv)v'] T d\text{Vec}(\Delta) &= 0 \end{aligned}$$

We now analyse the solutions to these equations, starting with the second of them. We have:

$$\begin{aligned} \text{Vec}' [A^{-1}(Z - Bv)v'] T &= 0; & T' \text{Vec} [A^{-1}(Z - Bv)v'] &= 0 \\ \sum_{j=1}^k (H_j \otimes E_{jj}) \text{Vec} (A^{-1}(Z - Bv)v') &= 0; & \sum_{j=1}^k \text{Vec} [E_{jj} (A^{-1}(Z - Bv)v') H_j] &= 0 \\ \text{Vec} \left[\sum_{j=1}^k E_{jj} (A^{-1}(Z - Bv)v') H_j \right] &= 0; & \sum_{j=1}^k E_{jj} A^{-1} Z v' H_j &= \sum_{j=1}^k E_{jj} A^{-1} B v v' H_j \end{aligned}$$

The final expression implies that:

$$E_{ll} \sum_{j=1}^k E_{jj} A^{-1} Z v' H_j = E_{ll} \sum_{j=1}^k E_{jj} A^{-1} B v v' H_j$$

and there only remains the summand of $j=1$, from the properties of $E_{ll} E_{jj}$. Thus

$$E_{ll} A^{-1} Z v' H_l = E_{ll} A^{-1} B v v' H_l; \quad l = 1; \dots; k \quad y$$

Taking into account the relation shown above between B and Δ

$$B = \sum_{n=1}^k E_{nn} \Delta H'_n$$

we obtain:

$$E_{ll} A^{-1} \left(\sum_{n=1}^k E_{nn} \right) Z v' H_l = E_{ll} A^{-1} \left(\sum_{n=1}^k E_{nn} \Delta H'_n \right) v v' H_l; \quad l = 1; \dots; k.$$

which implies

$$\sum_{n=1}^k u_l u_l' A^{-1} u_n u_n' Z v' H_l = \sum_{n=1}^k u_l u_l' A^{-1} u_n u_n' \Delta H_n' v v' H_l; \quad l = 1; \dots; k.$$

from which

$$\sum_{n=1}^k a'_{ln} E_{ln} Z v' H_l = \sum_{n=1}^k a'_{ln} E_{ln} \Delta H_n' v v' H_l$$

that is

$$a'_{ll} E_{ll} Z v' H_l = a'_{ll} E_{ll} \Delta H_l' v v' H_l$$

where $H_l' v v' H_l$ is 2×2 and full range. Therefore we verify:

$$E_{ll} Z v' H_l (H_l' v v' H_l)^{-1} = E_{ll} \Delta; \quad u_l' E_{ll} Z v' H_l (H_l' v v' H_l)^{-1} = u_l' E_{ll} \Delta$$

>From which it can be deduced that, considering all the $l = 1; \dots; k$

$$\widehat{\Delta} = \sum_{l=1}^k E_{ll} Z v' H_l (H_l' v v' H_l)^{-1}; \quad \widehat{\Delta}' = \sum_{l=1}^k (H_l' v v' H_l)^{-1} H_l' v Z' E'_{ll}$$

$$Vec(\widehat{\Delta}') = \sum_{l=1}^k [E_{ll} \otimes (H_l' v v' H_l)^{-1} H_l' v] Vec(Z')$$

Thus, to eliminate the differential, the above-described relation must be fulfilled, this relation being independent of A (it only depends on Z and v; z depends on the observations and v does not depend on the random values observed but on exogenous factors). The only random value is $Vec(Z')$. Therefore, this expression enables us to study the distribution of the estimators. With respect to the first equation, in $dVec(A)$, we have:

$$Vec' [A^{-1}[(Z - Bv)(Z - Bv)' - (n - 1)A]A^{-1}] dVec(A) = 0$$

which implies

$$\widehat{A} = \frac{1}{n - 1} (Z - Bv)(Z - Bv)'$$

In conclusion, the maximum likelihood estimators calculated are:

$$\widehat{A} = \frac{1}{n - 1} (Z - Bv)(Z - Bv)'; \quad Vec(\widehat{\Delta}') = \sum_{l=1}^k [E_{ll} \otimes (H_l' v v' H_l)^{-1} H_l' v] Vec(Z')$$

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