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On a new conservation law resulting from sedimentary basin dynamics

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Abstract. In this paper, we are interested in the theoretical analysis of a geological stratigraphic model, taking into account a limited weathering condition. Firstly, we present the physical model and the mathematical formulation, which lead to an original conservation law. Then, the definition of a solution and some mathematical tools in order to resolve the problem are given. At last, we treat the $1-D$ case and we present some open problems.

Keywords: stratigraphic models, weather limited, degenerated parabolic - hyperbolic conservation laws.

AMS classification: 35K65, 35K85, 35L80, 35Q35, 49J20

§1. Introduction and modelling presentation

In this paper, we are interested in the mathematical study of a new stratigraphic model, recently developed by the Institut Français du Pétrole (IFP). The model concerns geologic basin formation by the way of sedimentation with a weather limited condition. It leads to mathematical questions which are difficult to answer in the framework of ill-posed and inverse problems.

By taking into account large scale in time and space and by knowing *a priori*, the tectonique, the eustatism and the sediments flux at the basin boundary, the model has to state about the transport of sediments. One may find in D. Greanjeon *and al.* [9] and R. Eymard *and al.* [7] the physical and the numerical modelling of the multi-lithological case. An approach of the mathematical analysis of the mono-lithological case can be found in S.N. Antontsev *and al.* [1] and G. Gagneux *and al.* [3].

Let us consider in the sequel a sedimentary basin (see Figure 1), denoted by Ξ, with base $\Omega \subset \mathbb{R}^N$ ($N = 1, 2$) determined by a known vertical position, given by $H(t, x)$ at each time t and position x. For any positive T, one notes $Q =]0, T[\times\Omega$.

In the sequel, one denotes by u the sediments height, the topography is given by $u + H$ and one is led to consider a gravitational model where:
i) the sediments flux $\overrightarrow{q}$ is assumed to be proportional to $K\nabla h(u + H)$ where K is a viscosity rate and
ii) the erosion speed $\partial_t u$ is underestimated by $-E$ where E is a given non negative bounded measurable function in Q (a weathering limited process): $i.e.\ \partial_t u + E \geq 0$.

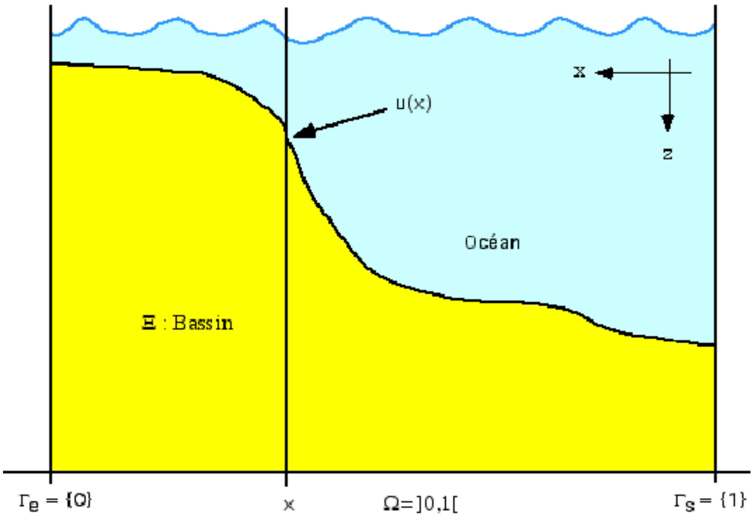

Figure 1:

The original aspect of the model is this weather limited condition on the erosion rate.

Therefore, in order to respect this constraint in a conservative formulation, one has to correct the diffusive flux $-K\,\nabla h(H+u)$ by the introduction of a multiplier λ.

Thus, the real flux is given by $-\lambda K\,\nabla h(H+u)$, where λ is an unknown function with values *a priori* in $[0,1]$.

In the realistic physical problem, $\Gamma = \partial\Omega = \overline{\Gamma_e} \cup \overline{\Gamma_s}$ and one has:
$-\lambda\partial_n h(H+u) = f$ on the inflow boundary and
some unilateral constraints on the outflow boundary: $\partial_n h(H+u) + f \geq 0,\ \partial_t u + E \geq 0$ and $(\partial_n h(H+u)+f)(\partial_t u + E) = 0$ where f is a given bounded measurable function on $\Sigma =]0,T[\times\Omega$.

Therefore, the mathematical modelling is:

The mass balance of the sediment:

$$\partial_t u - div(\lambda\nabla h(H+u)) = 0 \quad in \quad Q \tag{1}$$

The boundary conditions on $\partial\Omega = \overline{\Gamma_e} \cup \overline{\Gamma_s}$:

$$-\lambda\partial_n h(H+u) = f \quad on \quad]0,T[\times\Gamma_e \tag{2}$$

$$\begin{array}{l}\partial_t u + E \geq 0,\\ \lambda\partial_n h(H+u) + f \geq 0\end{array} \quad and \quad (\lambda\partial_n h(H+u)+f)(\partial_t u + E) = 0 \quad on \quad]0,T[\times\Gamma_s \tag{3}$$

The weather limited conditions:

$$\partial_t u \geq -E \quad in \quad Q. \tag{4}$$

And the initial condition:

$$u(0,.) = u_0 \quad in \quad \Omega. \tag{5}$$

In order to simplify, one considers in the sequel:

i) homogeneous Dirichlet conditions on the boundary (The previous authors consider such boundary conditions of unilateral type on Ω. The mathematical analysis is inspired by the chapter 2 of G. Duvaut and J.L. Lions [6] and by the "new problems" of J.-L Lions in [8] dealing of thermic).
and
ii) $H = 0$, $K = Id$ and $h = Id$.

Therefore, the problem becomes : Look for u *a priori* in $H^1(Q) \cap L^2(0,T;H_0^1(\Omega))$, such that,

$$\partial_t u + E \geq 0 \quad in \quad Q, \quad u(0,x) = u_0 \quad in \quad \Omega$$

with u_0 in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$\partial_t u(t,x) - Div\left\{\lambda(t,x)\nabla u(t,x)\right\} = 0 \quad in \quad Q. \tag{6}$$

Remark 1.
If $E = 0$, since $\partial_t u \geq 0$, one has, for a.e. t, $\lambda \nabla u^+ = 0$ a.e. in Ω.
i) Then, if for example $u_0 \geq 0$ in a non empty open subset ω of Ω, $\lambda\nabla u = 0$ a.e. in ω.
Therefore, $\partial_t u = 0$ a.e. in ω and $u(t,.) = u_0$ a.e.in ω.
Thus, if $\nabla u_0 \neq 0$ in ω, the problem must degenerate.
ii) If one assumes that $\nabla u_0 = 0$ in ω, any λ is solution and one needs more information about the modelling of λ.

Th. Gallouët and R. Masson in [7] propose to consider the following global constraint

$$\partial_t u + E \geq 0, \quad 1 - \lambda \geq 0 \quad and \quad (\partial_t u + E)(1 - \lambda) = 0 \quad in\ Q. \tag{7}$$

Note that for the uniqueness of the solution, this constraint may be insufficient. One can see in the above example that there are many possible solutions. In fact, maximal values of λ in $[0,1]$ have to be consider in order to select $\lambda = 1_{\{\nabla u_0 = 0\}}$.

Then S.N. Antontsev, G. Gagneux and G. Vallet propose in [2] a conservative formulation containing (7).

If one denotes by H the maximal monotone graph of the Heaviside function (*i.e.* $H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$ and $H(0) = [0,1]$) then (λ, h) is formally a solution of :

$$0 = \partial_t u - div(\lambda\nabla u) \quad where \quad \lambda \in H(\partial_t u + E) \quad in\ Q. \tag{8}$$

Where from our interest for the study of equations (resp. differential inclusions) of the type

$$0 = \partial_t u - div(a(\partial_t u + E)\nabla u) \quad resp. \quad 0 \in \partial_t u - div(H(\partial_t u + E)\nabla u).$$

In our knowledge, there are no mathematical studies of such equations, while S.N. Antontsev points out in [2] the presence of conservation laws of the shape

$$0 = \partial_t u - div(a(u, \partial_t u)\nabla u)$$

in fluid mechanics: see [10] and [11].

§2. Définition of a solution and existence

Definition 1. A solution to (8) is a couple (λ, u) of $L^\infty(Q) \times [H^1(Q) \cap L^2(0,T;H_0^1(\Omega))]$ such that:

$$\lambda \in H(\partial_t u + E), \quad u(t=0) = u_0,$$

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \partial_t uv + \lambda \nabla u \nabla v \, dx = 0$$

and λ is maximal in the sense: if (μ, w) is another solution then $\mu \leq \lambda$.

In order to prove the existence of such a solution, one proposes a method of time - discretization, with a technique of artificial viscosity. Finally, one supposes that E is a regular function of time variable.

Thus, considering two real positive parameters ε and h, one notes $E_k = E(kh)$, for any real x,

$$H_\varepsilon(x) = \max[\varepsilon, \min(\frac{x}{\varepsilon} + \varepsilon, 1)] \quad \textit{and} \quad F_\varepsilon^k(x) = \int_0^x \frac{1}{H_\varepsilon(t + E_k)} \, dt.$$

Proposition 1. *There exists a unique sequence $(u_\varepsilon^k)_k$ in $H_0^1(\Omega)$ such that $u_\varepsilon^0 = u_0$ with $\forall v \in H_0^1(\Omega)$,*

$$\int_\Omega \frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h} v + H_\varepsilon(\frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h} + E_k) \nabla u_\varepsilon^k . \nabla v \, dx = 0.$$

Moreover, $\inf_\Omega \text{ess } u_0 \leq u_\varepsilon^k \leq \sup_\Omega \text{ess } u_0$.

The proof comes from Schauder-Tykonov fixed point theorem and the maximum principle.

Lemma 2. *Independently of ε and h, sequence $(u_\varepsilon^k)_k$ is bounded in $L^\infty(\Omega)$ and for any integer n,*

$$\frac{2}{h} \sum_{k=1}^n ||u_\varepsilon^k - u_\varepsilon^{k-1}||_{L^2(\Omega)}^2 + ||u_\varepsilon^n||_{H_0^1(\Omega)}^2 + \sum_{k=1}^n ||u_\varepsilon^k - u_\varepsilon^{k-1}||_{H_0^1(\Omega)}^2 \leq ||u_0||_{H_0^1(\Omega)}^2.$$

One has just to use the test-function $v = F_\varepsilon^k(\dfrac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h})$.

First of all, one has to pass to the limit over ε to 0:

Proposition 3. *There exists a sequence $(\lambda_k, u^k)_k$ in $L^\infty(\Omega) \times H_0^1(\Omega)$ such that*
$\lambda_k \in H(\dfrac{u^k - u^{k-1}}{h} + E_k)$, $u^0 = u_0$ *and* $\forall v \in H_0^1(\Omega)$,

$$\int_\Omega \frac{u^k - u^{k-1}}{h} v + \lambda_k \nabla u^k . \nabla v \, dx = 0. \tag{9}$$

Moreover, $\inf_\Omega \text{ess } u_0 \leq u^k \leq \sup_\Omega \text{ess } u_0$ *and* $u^k \geq u^{k-1} - hE_k$ *a.e. in* Ω.

Let us note: $\hat{u}_h(t,x) = \sum_{k=0}^N [\dfrac{u^k - u^{k-1}}{h}(t - kh) + u^{k-1}] 1_{[kh,(k+1)h]}$ where $u^{-1} = u^0$ and $h = \frac{T}{N}$.
So, one gets

Proposition 4. *The sequence* $(\hat{u}_h)$ *is bounded in* $H^1(Q) \cap L^\infty(Q) \cap L^\infty(0, T; H_0^1(\Omega))$. *Thus, it is relatively compact in* $C([0, T], L^2(\Omega))$.
Moreover,

$$\lambda_h = \sum_{k=0}^{N} \lambda_k I_{[kh,(k+1)h[} \in H(\partial_t \hat{u}_h + \sum_{k=0}^{N} E_k I_{[kh,(k+1)h[}),\ \partial_t \hat{u}_h + \sum_{k=0}^{N} E_k I_{[kh,(k+1)h[} \geq 0 \ a.e\ .in\ Q$$

and for any v *in* $L^2(0, T, H_0^1(\Omega))$, *one has:*

$$\int_Q \partial_t \hat{u}_h v + \lambda^h \nabla \hat{u}_h . \nabla v \, dxdt = o(h). \tag{10}$$

On the one hand, each accumulation point is a ”mild solution” in the sense of Ph. Bénilan *and al.* [4] ; on the other hand, the double weak convergence does not allow us to pass to the limit in the term $\int_Q \lambda^h \nabla \hat{u}_h . \nabla v \, dxdt$.

In the forthcoming paragraph, one proposes cases where such a convergence can be proven.

First, observe some obvious cases:
i) if $u_0 \geq 0$ in Ω then the solution is $(1_{\{\nabla u_0 = 0\}}, u_0)$.
ii) if u_0 is non positive with $\Delta u_0 \geq 0$, the solution is $(1, w)$ where w is the solution of the heat equation.

§3. The 1-D case

In this section, $\Omega =]-1, 1[$ and one assumes that $E = 0$.

Let us start with this essential remark for the sequel:

Remark 2. Since sequence $(u^k)_k$ is non decreasing, function $x \mapsto (\lambda^1 u^{k\prime})(x)$ is continuous and non decreasing in $[-1, 1]$.

This allows us to treat the following examples.

3.1. Between two hills

If $u_0 \geq 0$ in $]a, b[\cup]c, d[$ for given $-1 \leq a < b \leq c < d \leq 1$, then $\lambda^1 u^{1\prime} = 0$ and $u^1 = u_0$ in $]a, d[$.

By induction, the looking for solution for problem (8) is $(1_{\{\nabla u_0 = 0\}}, u_0)$ in $]a, d[$.

In particular, if $u_0 \geq 0$ in $]-1, -1 + \varepsilon[\cup]1 - \varepsilon, 1[$ for a given positive ε, the looking for solution for problem (8) is $(1_{\{\nabla u_0 = 0\}}, u_0)$ in Ω.

3.2. A convex sea against a hill

Assume now that $u_0 \geq 0$ in $]-1, 0]$, $u_0 \leq 0$ and convex in $[0, 1[$.

Then u_0 is decreasing in $[0, \alpha]$, constant in $[\alpha, \beta]$ (if needed, otherwise $\alpha = \beta$) and increasing in $[\beta, 1]$.

According to what precedes, $u^1 = u_0$ in $[-1, 0]$, and since $u_0 \le 0$ in $]0, 1[$, thanks to the maximum principle, one has $u^1 \le 0$ in $]0, 1[$.

Let us note $x(h) = \sup\{x \in [-1, 1],\ \lambda^1 u^{1\prime}(x) = 0\}$.

In order to have a non trivial solution, one is looking for $x(h) < 1$.

Thanks to the above proposition, for any $x > x(h)$, one has $\lambda^1 u^{1\prime}(x) > 0$. In particular, u^1 is an increasing function in $]x(h), 1[$. Since $u^1(0) \ge 0$, one notices that inevitably $x(h) > 0$, otherwise, one would have a contradiction with $u^1(1) = 0$.

One is so returned to look for $x(h) > 0,\ u^1 \in H^1(x(h), 1)$ and $\lambda^1 \in H(\dfrac{u^1 - u_0}{h})$ with $\lambda^1 > 0$, such that

$$\begin{aligned} u^1 - h(\lambda^1 u^{1\prime})' &= u_0 \ in\ ]x(h), 1[, \\ &\quad with \\ u^1(x(h)) &= u_0(x(h)),\ u^{1\prime}(x(h)) = 0 \ and\ u^1(1) = 0. \end{aligned}$$

As one speculates to have a maximal value of λ^1, one assumes that $\lambda^1 = 1$ in $]x(h), 1[$ and then, u^1 is given by:

$$u^1(x) = u_0(x) - \int_{x(h)}^{x} u_0'(y) ch(\frac{y - x}{\sqrt{h}})\, dy$$

where the unique point $x(h)$ is defined by:

$$\int_{x(h)}^{1} u_0'(y) ch(\frac{y - 1}{\sqrt{h}})\, dy = 0.$$

Moreover, one obviously notes that $x(h) \in]0, \alpha[$ and that $u^1 \ge u_0$.

Assume now that there exist a solution (μ, v) such that $\mu \ne 0$ in $]0, x(h)]$.

As $u_0' < 0$ in $]0, x(h)]$, inevitably $v \ne u_0$ in $]0, x(h)]$. Thus, there exists a in $]0, x(h)[$ and $\varepsilon > 0$ such that $v > u_0$ in $]a, a + \varepsilon[$. Therefore, $\mu = 1$ in $]a, a + \varepsilon[$, $\mu v' > 0$ in $]a, 1[$ and v is an increasing function in $]a, 1[$. As u_0 is non increasing in $]0, \alpha[$, $\mu = 1$ in $]a, \alpha[$.

Remark that $v > u$ in $]a, x(h)[$ and denote by $b = \inf\{x \in]a, 1],\ v(x) = u(x)\}$ (Remind that $v(1) = u(1) = 0$).

Since $v \ge u > u_0$ in $]x(h), b[$, one has $\mu = 1$ and $u - v$ is a solution to:

$$\begin{aligned} u - v - h(u - v)'' &= 0 \ in\ ]x(h), b[\\ &\quad with \\ (u - v)(b) &= 0, \quad (u - v)(x(h)) = u_0(x(h)) - v(x(h)) < 0 \\ &\quad and \\ (u - v)'(x(h)) &= -v'(x(h)) < 0. \end{aligned}$$

Thus, $u - v$ is concave on $[x(h), b]$ with $(u - v)'(x(h)) < 0$ and $(u - v)(x(h)) < 0$.

Therefore, $(u - v)(b) < 0$ and one has a contradiction. So, $\lambda^1 = 1_{]x(h),1[}$ is the only maximal solution.

That allows us to build explicitly iteration u^1 and one is able to remark that u^1 is non positive and convex over $]0, 1[$; decreasing on $]0, x(h)[$ and increasing on $]x(h), 1[$.

Moreover, as u_0 is a convex function, one has $u^1 \geq u_0$ and this constructed solution is the maximal solution with respect to any possible value of λ^1 in $H(\dfrac{u^1 - u_0}{h})$.

So, it is possible to pursue the construction of u^k and λ^k by induction, in the following way : there exists a non increasing sequence $x^k(h)$ in $[0, \alpha]$ such that

$$\lambda^k = 1_{]x^k(h),1[} \quad and \quad u^k = u_0 1_{]-1,x^k(h)]} + w^k 1_{]x^k(h),1[}$$

where w^k is the solution to :

$$\begin{array}{rcl} w^k - hw^{k"} &=& u_0 \quad in \quad]x^k(h), 1[\\ && with \\ w^k(x^k(h)) &=& u_0(x^k(h)),\ w^{k\prime}(x^k(h)) = 0 \quad and \quad w^k(1) = 0. \end{array}$$

So, according to the notation of Property 4, $(\lambda_h)_h$ is a bounded sequence in $BV(Q) \cap L^\infty(Q)$ and in particular $var(\lambda_h) \leq T + 1$.

It is therefore possible to extract from (λ_h) a sub-sequence that converges a.e. in Q and in any $L^p(Q)$ (for any finite p) towards λ with $0 \leq \lambda \leq 1$ a.e. in Q.

Furthermore, by a monotone argument, one has $(\lambda - 1)\partial_t u = 0$ a.e. in Q.

Then, one has : $\lambda \in H(\partial_t u)$ and as the limit in (10) is then possible, one constructs a solution to problem (8), with the supplementary information, appropriate for the 1 - D case: Thanks to Ascoli's theorem, $u \in C^0(\overline{Q})$.

Let us note the that a.e. convergence with values of $(\lambda_h)_h$ in $\{0, 1\}$ implies that $\lambda(t, x) \in \{0, 1\}$ and that $a.e.\ \lambda = 1_\omega$ where $\omega \subset Q$ is a finite perimeter set.

Moreover, one proves that the free boundary $\partial\omega \cap Q$ is the graph of a continuous, non increasing function $t \mapsto \xi(t)$.

3.3. A convexo-concave sea against a hill

Let us have a look now to the case: $u_0 \geq 0$ in $]-1, 0]$, $u_0 \leq 0$ in $[0, 1]$, convex in $[0, \beta[$ with u_0 decreasing in $[0, \alpha]$, non decreasing in $[\alpha, \beta]$, increasing and concave in $[\beta, 1]$.

Due to the ideas developed above, one proposes the following algorithm to build (λ^1, u^1).

Consider $x_1(h)$ in $]\alpha, 1[$ and denote by $x_0(h)$ the unique point in $]0, \alpha[$ such that

$$\int_{x_0(h)}^1 u_0'(y) ch(\frac{y - x_1(h)}{\sqrt{h}})\, dy = 0.$$

Therefore, $u^1(x) = u_0(x) - \int_{x(h)}^x u_0'(y) ch(\frac{y-x}{\sqrt{h}})\, dy$ is the unique solution to

$$\begin{array}{rcl} u^1 - hu^{1"} &=& u_0 \ in \ ]x_0(h), x_1(h)[, \\ && with \\ u^1(x_0(h)) &=& u_0(x_0(h)), \quad u^{1\prime}(x_0(h)) = 0, \quad u^1(x_1(h)) = u_0(x_1(h)), \\ && and \\ u^1 &=& u_0 \quad in \quad]-1, 1[\backslash]x_0(h), x_1(h)[. \end{array}$$

At first, assume that $x_1(h)$ is in $]\alpha, \beta[$ such that $u^1 \geq u_0$. Note that since u_0 is not convex on $]\beta, 1[$, $x_1(h) = 1$ is not obvious.

Remark that $\lambda^1 = 0$ in $]-1, x_0(h)[$, $\lambda^1 = 1$ in $]x_0(h), x_1(h)[$ and $h(\lambda^1 u^{1\prime})' = u^1 - u_0 = 0$ in $]x_1(h), 1[$.

As, $\lambda^1 u^{1\prime}$ is a non decreasing continuous function, for any $x \geq x_1(h)$, one has :

$$u^{1\prime}(x_1(h)^-) = (\lambda^1 u^{1\prime})(x_1(h)^-) = (\lambda^1 u^{1\prime})(x_1(h)^+) = \lambda^1(x)u_0'(x) \leq u_0'(x).$$

As u_0 is concave in $]\beta, 1[$, $\lambda^1(x)u_0'(x) \leq u_0'(1)$ and it remains only to consider $\lambda^1(x) = \frac{u_0'(1)}{u_0'(x)}$ in order to construct a solution (λ^1, u^1).

Note that, if $u_0'(1) = 0$, then $\lambda^1 = 0$ in $]x_1(h), 1[$. Thus, $x_0(h) = x_1(h)$ and $u^1 \equiv u_0$.

At last, one only has to choose $x_1(h)$ as close as possible to 1 within the above constraints.

§4. Conclusion and open problems

One presents here a new conservation law of which general study remains still opened. In particular, the way to understand the forming of the hyperbolic zone and the parabolic one. As an example in $1 - D$, the previous study shows that if the sign of u_0 changes and if it remains non negative locally around -1 and 1, then the zone where u_0 is negative is outside influence.

For the autonomous problem, one succeeds however in presenting some results and illustrations in one space dimension for simple initial topography.

These illustrations seem to confirm the following guess: when $\mathcal{L}^2 - mes\{x \in \Omega,\ u_0(x) < 0\} > 0$, is there a set $\omega \subset Q$ such that $u = u_0$ in $Q \backslash \omega$ and u is the solution of the heat equation in ω?
Is the free boundary $\partial\omega \cap Q$ characterized, if one notes $\widetilde{u} = u_{|\omega}$, by a double condition of type Dirichlet - Neuman on $\partial\omega \cap Q$?
One finds under a generalized shape Bernoulli's problem as presented for example by A. Beurling in [5].

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