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Singular Perturbations for a Class of Degenerate Inequalities
The Framework of Functions with Bounded Variations

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Abstract. We study the limit as ϵ goes to 0^+ for the sequence $(u_\epsilon)_{\epsilon>0}$ of solutions to the Dirichlet problem for the quasilinear parabolic operators

$$\mathbb{H}_\epsilon(t,x,.) : u \to \partial_t u + \sum_{i=1}^{p} \partial_{x_i}\varphi_i(t,x,u) + \psi(t,x,u) - \epsilon\Delta\phi(u),$$

where ϕ is a nondecreasing function, associated with a positiveness condition in an open bounded domain of $\mathbb{R}^p, 1 \leq p < +\infty$. The positive parameter ϵ being fixed, we first propose the definition of a weak entropy solution, the boundary conditions being expressed through the mathematical framework of the Divergence-Measure fields. Then, the uniqueness proof for refers to the technique of doubling the variables and the existence property is obtained through the artificial viscosity method. Lastly, a $BV \cap L^\infty$-estimate for the sequence $(u_\epsilon)_{\epsilon>0}$ is used to take the limit with ϵ.

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§1. Introduction

1.1. Mathematical framework

Obstacle problems for conservation laws in physics and mechanics have been studied by many authors. A general presentation and a mathematical framework through variational inequalities may be find for example in the book of G.Duvaut & J.L.Lions [3]...

Here, we consider the second-order quasilinear operator

$$\mathbb{H}_\epsilon(t,x,.) : u \to \partial_t u + \sum_{i=1}^{p} \partial_{x_i}\varphi_i(t,x,u) + \psi(t,x,u) - \epsilon\Delta\phi(u),$$

where ϕ is a nondecreasing function (especially ϕ may be constant on some nonempty intervals of $\mathbb{R}$).

Let us consider a measurable and bounded initial datum u_0, satisfying a positiveness condition. Then, the next statement holds whose the proof will be roughly described in the following sections (see [6] for the detailed proofs):

Theorem 1. *(i) For any positive parameter ϵ, the formal free boundary values problem: find a measurable and bounded function u_ϵ on Q such that*

$$0 \leq u_\epsilon \text{ a.e. on }]0,T[\times\Omega, \tag{1}$$

$$u_\epsilon \mathbb{H}_\epsilon(t,x,u_\epsilon) = 0, \mathbb{H}_\epsilon(t,x,u_\epsilon) \geq 0 \text{ on }]0,T[\times\Omega, \tag{2}$$

$$u_\epsilon = 0 \text{ on }]0,T[\times\partial\Omega,\ u_\epsilon(0,.) = u_0 \text{ on } \Omega, \tag{3}$$

has a unique solution.
(ii) When ϵ goes to 0^+, the sequence $(u_\epsilon)_{\epsilon>0}$ gives an L^1-approximation of the weak entropy solution to the corresponding unilateral obstacle problem for the first-order quasilinear operator $\mathbb{H}_0$.

1.2. Notations and Main assumptions on data

• T is a positive finite real, Ω a bounded subset of $\mathbb{R}^p$ with a $\mathcal{C}^2$-class frontier $\Gamma = \partial\Omega$. Thus, we refer to a $\mathcal{C}^2$-covering of Γ with open sets $(B_i)_{i\in I}$, $I \subsetneqq \mathbb{N}$, and to a $\mathcal{C}^2$-local representation of Γ. In the sequel the index i will be dropped and "B belongs to the set $\mathcal{B}$" of all possible recovering of Γ means
$a)$ $\Gamma \subset B$,
$b)$ locally, there exists a $\mathcal{C}^2$-class function f such that $B \cap \Gamma = \{x_p \in B_i, x_p = f(x')\}$, $B \cap \Omega = \{x_p \in B_i, x_p < f(x')\}$ in the local coordinates (x', x_p) introduced by B.

For all s of $[0,T]$, Q_s denotes the cylinder $]0,s[\times\Omega$, $\Sigma_s =]0,s[\times\Gamma$ with the convention $Q = Q_T$ and $\Sigma = \Sigma_T$. The unit outer normal of Ω is denoted ν.

• For any n in $\mathbb{N}^*$, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.

• A boundary-layer sequence in the sense of C.Mascia, A.Porretta & A.Terracina [4] is a sequence $(\rho_\varrho)_{\varrho>0}$ of $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$-class functions such that:

$$\lim_{\rho\to 0^+} \rho_\varrho = 1 \text{ pointwise in } \Omega,\ 0 \leq \rho_\varrho \leq 1 \text{ in } Q,\ \rho_\varrho = 0 \text{ on } \Gamma$$

• The space $\mathcal{DM}_2(Q)$ of the L^2-Divergence Measure fields on Q is given by

$$\mathcal{DM}_2(Q) = \left\{V = (v_0, v_1, .., v_p) \in (L^2(Q))^{p+1}, Div_{(t,x)}V \in \mathcal{M}_b(Q)\right\},$$

were $\mathcal{M}_b(Q)$ denotes the space of Bounded Radon measures on Q. For any V in $\mathcal{DM}_2(Q)$ a linear application Λ_V on $H^1(Q) \cap L^\infty(Q) \cap \mathcal{C}(Q)$ is defined through the next generalized Gauss-Green formula:

$$\Lambda_V(\xi) := \langle V, \xi\rangle_{\partial Q} = \int_Q V.(\partial_t\xi, \nabla\xi)dxdt + \int_Q \xi\, d[Div_{(t,x)}V],$$

and the next property holds (see [4]) for any ξ in $H^1(Q) \cap L^\infty(Q) \cap \mathcal{C}(Q)$ such that $\xi(T,.) \equiv \xi(0,.) \equiv 0$,

$$\lim_{\varrho\to 0^+} \int_Q V\xi.(0, \nabla\rho_\varrho)dxdt = -\langle V, \xi\rangle_{\partial Q},$$

• The functions ψ and φ_i are *smooth* (this point will not be discuss here - see [6]). However, we assume that there exist nonnegative c_1, c_2, $c'_{1,i}$ and $c'_{2,i}$ in $L^\infty(Q)$ satisfying for a.e. (t,x) in Q, $\forall\lambda \in \mathbb{R}$,

$$|\psi(t,x,\lambda)| \leq c_1(t,x)|\lambda| + c_2(t,x),$$
$$|\partial_{x_i}\varphi_i(t,x,\lambda)| \leq c'_{1,i}(t,x)|\lambda| + c'_{2,i}(t,x).$$

Then we set $c_i(\psi) = \|c_i\|_{L^\infty(Q)}$, $c'_i(\varphi) = \max\limits_{j\in\{1,..,p\}} \|c'_{i,j}\|_{L^\infty(Q)}$, $i = 1,2$ and $c_3(\psi) = max(c_1(\psi),$ $c_2(\psi))$.

• The initial data u_0 is a nonnegative element of $H_0^1(\Omega) \cap L^\infty(\Omega) \cap BV(\Omega)$ such that Δu_0 and $\Delta\phi(u_0)$ belong to $\mathcal{M}_b(\Omega)$.

In this context we can define for any t of $[0,T]$, $M(t) = \dfrac{K_1}{K_2}(e^{K_1 t} - 1) + \|u_0\|_{L^\infty(\Omega)}e^{K_1 t}$, where $K_i = c_i(\psi) + c'_i(\varphi)$, $i = 1,2$.

• The diffusive term ϕ is a $W^{1,+\infty}(]-M(T), M(T)[)$-class function such that, by normalization $\phi(0) = 0$. Moreover, we set

$$E = \{l \in \mathbb{R}, \{l\} = \phi^{-1}\{\phi(l)\}\}.$$

• Lastly we will consider "*sgn*$_\lambda$" the Lipschitzian and bounded approximation of the function "sgn" given for any positive parameter λ and nonnegative real x by:

$$sgn_\lambda(x) = \min\left(\frac{x}{\lambda}, 1\right) \textit{ and } sgn_\lambda(-x) = -sgn_\lambda(x).$$

§2. The Degenerate Parabolic-Hyperbolic Problem

2.1. Mathematical formulation

Definition 1. A measurable and bounded function u_ϵ is called a weak solution to $(1,2,3)$ if:

$$u_\epsilon \geq 0 \textit{ a.e. in } Q, \quad (4)$$
$$\partial_t u_\epsilon \in L^2(0,T;H^{-1}(\Omega)), \phi(u_\epsilon) \in L^2(0,T;H_0^1(\Omega)), \quad (5)$$
$$ess\lim_{t\to 0^+} \int_\Omega |u_\epsilon(t,x) - u_0(x)|dx = 0, \quad (6)$$

and for any v in $H_0^1(\Omega)$, $v \geq 0$ a.e. in Ω, for a.e. t of $]0,T[$,

$$\langle\partial_t u_\epsilon, v - \phi(u_\epsilon)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega \varphi(t,x,u_\epsilon).\nabla(v - \phi(u_\epsilon))dx +$$
$$+ \int_\Omega \psi(t,x,u_\epsilon)(v - \phi(u_\epsilon))dx + \epsilon\int_\Omega \nabla\phi(u_\epsilon).\nabla(v - \phi(u_\epsilon))dx \geq 0. \quad (7)$$

In a strongly degenerate framework, the previous definition is not sufficient to ensure the uniqueness. We need an additional entropy criterium inside Q connected with a suitable formulation for boundary conditions that controls some possible boundary layers. So we state:

Definition 2. A measurable and bounded function u_ϵ is called a weak entropy solution to (1,2,3) if:
i) the relations (4),(5) and (6) hold,
ii) $\forall k \in \mathbb{R}^+$, $\forall B \in \mathcal{B}$, $\forall \zeta \in \mathcal{D}_+(B)$,

$$U_k^\epsilon \zeta \in \mathcal{DM}_2(Q), \tag{8}$$

iii) $\forall k \in \mathbb{R}^+$, $\forall \xi \in H_0^1(Q) \cap L^\infty(Q)$, $\xi \geq 0$

$$\int_Q U_k^\epsilon . \overline{\nabla} \xi dxdt - \int_Q sgn(u_\epsilon - k) G(u_\epsilon, k) \xi dxdt \geq 0, \tag{9}$$

iv) $\forall B \in \mathcal{B}$, $\forall \zeta \in \mathcal{D}_+(B)$,

$$\int_\Sigma \mathbf{F}(k,0).\nu \xi \zeta d\mathcal{H}^p \leq \langle U_k^\epsilon \zeta, \xi \rangle_{\partial Q} + \langle U_0^\epsilon \zeta, \xi \rangle_{\partial Q}, \tag{10}$$

for all ξ in $L^\infty(Q) \cap H^1(Q) \cap \mathcal{C}_+(Q)$, $\xi(T,.) = \xi(0,.) = 0$ and any k in $\mathbb{R}^+$ where

$$\mathbf{F}(u,k) = sgn(u-k)\{\varphi(t,x,u) - \varphi(t,x,k)\}, G(u,k) = Div_x \varphi(t,x,k) + \psi(t,x,u),$$
$$U_k^\epsilon = (|u-k|, -\epsilon \nabla |\phi(u) - \phi(k)| + \mathbf{F}(u,k)), \overline{\nabla} \zeta = (\partial_t \zeta, \nabla \zeta),$$

the dependence of **F** and G on (t,x) being not essential to comprehension.

2.2. The Uniqueness Result

The proof essentially relies on a comparison theorem which is a J.Carrillo's extension to second-order equations of the usual hyperbolic method based on a doubling of the time and space variables (*cf.* J.Carrillo [2]) and the demonstration presented here follows C.Mascia, A.Porretta & A.Terracina's [4] especially for the treatment of the *boundary* terms. However, numerous adaptations are necessary due to the presence of an obstacle condition.

Let us start with the following lemma that plays an important part in the comprehension of mechanisms which lead to the uniqueness property. At this point stands one of the main differences between the context of obstacle problems and the general setting for parabolic degenerate equations. Namely, this lemma can be viewed as an *inequality version* of the standard *energy equality* due to J.Carrillo [2] and satisfied by any weak solution:

Lemma 2. *Let u be a weak solution to (1,2,3). Then, for any ξ of $\mathcal{D}_+(Q)$, k of E, $k \geq 0$:*

$$\int_Q \left(U_k^\epsilon . \overline{\nabla} \xi - sgn(u_\epsilon - k) G(u_\epsilon, k) \xi \right) dxdt \geq \epsilon \limsup_{\lambda \to 0^+} \int_Q sgn'_\lambda(\phi(u_\epsilon) - \phi(k)) (\nabla \phi(u_\epsilon))^2 \xi \, dxdt. \tag{11}$$

Since it is fulfilled by any weak solution and is only true for a range of nonnegative parameters k, Energy inequality (11) is not sufficient to ensure the uniqueness. We complement it with (9) which is available for any nonnegative k. This technique, adapted from J.Carrillo's one [2], leads to a comparison result between two weak entropy solutions. With this view, we consider a $\mathcal{C}^\infty_+$-function Ψ such that:

$$(\tilde{t},\tilde{x}) \longmapsto \Psi(\tilde{t},\tilde{x},t,x) \in \mathcal{C}^\infty_c(Q) \text{ for every } (t,x) \in Q,$$
$$(t,x) \longmapsto \Psi(\tilde{t},\tilde{x},t,x) \in \mathcal{C}^\infty_c(Q) \text{ for every } (\tilde{t},\tilde{x}) \in Q,$$

with formally $dp = dxdt$, $d\tilde{p} = d\tilde{x}d\tilde{t}$ and we add a "tilde" superscript to any function in "tilde" variables.

Proposition 3. *Let u_1 and u_2 be two bounded measurable functions satisfying (4,5,7) and (9). Then:*

$$-\int_{Q\times Q} (|u_1-\tilde{u}_2|(\partial_t\Psi_t+\partial_{\tilde{t}}\Psi) - \epsilon\, sgn(\phi(u_1)-\phi(\tilde{u}_2))(\nabla_x\phi(u_1)-\nabla_{\tilde{x}}\phi(\tilde{u}_2)).(\nabla_x\Psi+\nabla_{\tilde{x}}\Psi))\, dpd\tilde{p}$$
$$-\int_{Q\times Q}\left(\boldsymbol{F}(u_1,\tilde{u}_2).\nabla_x\Psi + \tilde{\boldsymbol{F}}(\tilde{u}_2,u_1).\nabla_{\tilde{x}}\Psi\right)\, dpd\tilde{p}+$$
$$\int_{Q\times Q} sgn(u_1-\tilde{u}_2)(G(u_1,\tilde{u}_2)-\tilde{G}(\tilde{u}_2,u_1))\Psi\, dpd\tilde{p} \le 0.$$

Proof. Let us describe briefly the main outlines: on the one hand we may choose in (11) written in the variables (t,x) for the solution u_1,

$$k = u_2(\tilde{t},\tilde{x}) \textit{ for a.e. } (\tilde{t},\tilde{x}) \in Q\setminus Q_0^{\tilde{u}_2} \equiv \{(\tilde{t},\tilde{x})\in Q, \tilde{u}_2 \in E\}.$$

On the other hand we choose in (9) written in variables (t,x) for the solution u_1,

$$k = u_2(\tilde{t},\tilde{x}) \textit{ for a.e. } (\tilde{t},\tilde{x}) \in Q_0^{\tilde{u}_2} \equiv \{(\tilde{t},\tilde{x})\in Q, \tilde{u}_2 \notin E\}.$$

Each inequality obtained by this way may be integrated with respect to the variables $\tilde{t}$ and $\tilde{x}$ on the corresponding domain. By it comes for u_1:

$$\begin{aligned}
&-\int_{Q\times Q}|u_1-\tilde{u}_2|\partial_t\Psi\, dpd\tilde{p} + \int_{Q\times Q}\left[\epsilon\nabla_x|\phi(u_1)-\phi(\tilde{u}_2)| - \mathbf{F}(u_1,\tilde{u}_2)\right].\nabla_x\Psi\, dpd\tilde{p}\\
&+\int_{Q\times Q} sgn(u_1-\tilde{u}_2)G(u_1,\tilde{u}_2)\Psi\, dpd\tilde{p}\\
&\le -\limsup_{\lambda\to 0^+}\int_{Q\times Q\setminus Q_0^{\tilde{u}_2}} \epsilon\, sgn'_\lambda(\phi(u_1)-\phi(\tilde{u}_2))(\nabla\phi(u_1))^2\Psi\, dpd\tilde{p}\\
&\le -\limsup_{\lambda\to 0^+}\int_{Q\setminus Q_0^{u_1}\times Q\setminus Q_0^{\tilde{u}_2}} \epsilon\, sgn'_\lambda(\phi(u_1)-\phi(\tilde{u}_2))(\nabla\phi(u_1))^2\Psi\, dpd\tilde{p},
\end{aligned}$$

Moreover, we integrate over Q the Gauss-Green formula:

$$\int_Q \nabla_x\phi(u_1).\nabla_{\tilde{x}}[sgn_\lambda(\phi(u_1)-\phi(\tilde{u}_2))\Psi]\,d\tilde{p} = 0 \text{ a.e. on } Q.$$

By developing the partial derivatives and taking into account that $\phi(\tilde{u}_2)$ belongs to $L^2(0,T;H_0^1(\Omega))$, the λ-limit provides the next equality:

$$\begin{aligned}&\int_{Q\times Q} sgn(\phi(u_1)-\phi(\tilde{u}_2))\nabla_x\phi(u_1).\nabla_{\tilde{x}}\Psi\,dpd\tilde{p}\\ &= \lim_{\lambda\to 0^+}\int_{Q\setminus Q_0^{u_1}\times Q\setminus Q_0^{\tilde{u}_2}} sgn'_\lambda(\phi(u_1)-\phi(\tilde{u}_2))\nabla_x\phi(u_1).\nabla_{\tilde{x}}\phi(\tilde{u}_2)\Psi\,dpd\tilde{p}.\end{aligned}$$

In the right-hand side, the integral over $Q\times Q$ has been turned into an integral over $Q\setminus Q_0^{u_1}\times Q\setminus Q_0^{\tilde{u}_2}$. We apply the same reasoning with the weak entropy solution $\tilde{u}_2$ and group all together the results to obtain the next Kruskov-type relation between two weak entropy solutions:

$$\begin{aligned}&-\int_{Q\times Q}|u_1-\tilde{u}_2|(\partial_t\Psi+\partial_{\tilde{t}}\Psi)-\epsilon\, sgn(\phi(u_1)-\phi(\tilde{u}_2))(\nabla_x\phi(u_1)-\nabla_{\tilde{x}}\phi(\tilde{u}_2)).(\nabla_x\Psi+\nabla_{\tilde{x}}\Psi)dpd\tilde{p}\\ &-\int_{Q\times Q}\Big(\mathbf{F}(u_1,\tilde{u}_2).\nabla_x\Psi+\tilde{\mathbf{F}}(\tilde{u}_2,u_1).\nabla_{\tilde{x}}\Psi\Big)-sgn(u_1-\tilde{u}_2)(G(u_1,\tilde{u}_2)-\tilde{G}(\tilde{u}_2,u_1))\Psi\,dpd\tilde{p}\\ &\le -\limsup_{\lambda\to 0^+}\int_{Q\setminus Q_0^{u_1}\times Q\setminus Q_0^{\tilde{u}_2}}\epsilon\, sgn'_\lambda(\phi(u_1)-\phi(\tilde{u}_2))\,[\nabla_x\phi(u_1)-\nabla_{\tilde{x}}\phi(\tilde{u}_2)]^2\,\Psi\,dpd\tilde{p}\le 0.\end{aligned}$$

□

Now Proposition 3 allows us to state that if $u_{\epsilon,1}$ and $u_{\epsilon,2}$ are two weak entropy solutions to the positiveness problem for $\mathbb{H}_\epsilon$ respectively associated with initial data $u_{0,1}$ and $u_{0,2}$ then,

$$\text{for a.e. } t \text{ in }]0,T[,\ \int_\Omega |u_{\epsilon,1}(t,x)-u_{\epsilon,2}(t,x)|\,dx \le e^{c_3(\psi)\,t}\int_\Omega |u_{0,1}(x)-u_{0,2}(x)|\,dx,$$

with the notations of Subsection 1.2. As a consequence,

Theorem 4. *For a fixed ϵ, unilateral Problem (1,2,3) admits at most one weak entropy solution.*

2.3. Existence property *via* the vanishing viscosity method

We introduce some diffusion in the whole domain *via* a positive parameter δ destined to tend to 0^+. So we define $\phi_\delta = \phi + \delta Id_{\mathbb{R}}$ a bilipschitzian function so as to obtain the nondegenerate parabolic operator

$$\mathbb{H}_{\epsilon,\delta}(t,x,.) : u \to \partial_t u + \sum_{i=1}^p \partial_{x_i}\varphi_i(t,x,u) + \psi(t,x,u) - \epsilon\Delta\phi_\delta(u),$$

and the corresponding unilateral obstacle problem formally described by:
find a measurable and bounded function $u_{\epsilon,\delta}$ such that

$$u_{\epsilon,\delta} \geq 0 \textit{ a.e. in } Q, \tag{12}$$

$$\mathbb{H}_{\epsilon,\delta}(t,x,u_{\epsilon,\delta}) \geq 0,\ u_{\epsilon,\delta}\mathbb{H}_{\epsilon,\delta}(t,x,u_{\epsilon,\delta}) = 0 \textit{ on } Q, \tag{13}$$

$$u_{\epsilon,\delta} = 0 \textit{ on } \Sigma,\ u_{\epsilon,\delta}(0,.) = u_0 \textit{ on } \Omega. \tag{14}$$

2.3.1. A priori estimates

To begin with, we remind the existence and uniqueness property obtained in [5]:

Theorem 5. *The nondegenerate unilateral obstacle problem (12,13,14) has a unique solution $u_{\epsilon,\delta}$ which is an element of $L^\infty(Q) \cap H^1(Q) \cap L^\infty(0,T;H_0^1(\Omega)) \cap \mathcal{C}^0([0,T];L^q(\Omega))$, $1 \leq q < +\infty$, and is such that $\phi_\delta(u_{\epsilon,\delta})$ belongs to $L^2(0,T;H^2(\Omega))$. Furthermore, $u_{\epsilon,\delta}$ is characterized through the strong variational inequality, for all v in $L^2(\Omega)$, $v \geq 0$, and a.e. on $]0,T[$,*

$$\int_\Omega \mathbb{H}_{\epsilon,\delta}(t,x,u_{\epsilon,\delta})(v-u_{\epsilon,\delta})dx \geq 0. \tag{15}$$

Moreover, by using the method of penalization, the next estimates hold:

$$\forall t \in [0,T],\ |u_{\epsilon,\delta}(t,.)| \leq M(t) \textit{ a.e. in } \Omega,\ \|\partial_t u_{\epsilon,\delta}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_1,$$
$$\forall s \in [0,T],\ \epsilon\|\partial_t F_\delta(u_{\epsilon,\delta})\|^2_{L^2(Q_s)} + \epsilon^2\|\phi_\delta(u_{\epsilon,\delta})(s,.)\|^2_{H_0^1(\Omega)} \leq C_2,$$

where $F_\delta(x) = \int_0^x (\phi_\delta{}'(\tau))^{1/2}\, d\tau$ and C_1 and C_2 are constants independent from any parameter.

Besides, essentially linked to the framework of a constant obstacle function and of a sufficiently smooth initial data, there exist positive constants A_1 and A_2 such that: $\forall h \in]0,T[, \forall t \in]0,T-h[$,

$$\tfrac{1}{h}\|u_{\epsilon,\delta}(t+h,.) - u_{\epsilon,\delta}(t,.)\|_{L^1(\Omega)} \leq A_1,$$
$$\|\partial_t u_{\epsilon,\delta}\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla u_{\epsilon,\delta}\|_{L^\infty(0,T;L^1(\Omega)^p)} \leq A_2.$$

2.3.2. The degenerate problem: existence of a weak entropy solution

Theorem 5 ensures that the sequence $(u_{\epsilon,\delta})_{\delta>0}$ remains in a fixed bounded of $W^{1,1}(Q) \cap L^\infty(Q)$. Thus, a compactness argument and a generalization of Ascoli's lemma prove the existence of a function u_ϵ of $BV(Q) \cap L^\infty(Q) \cap \mathcal{C}^0([0,T],L^1(\Omega))$ such that up to a subsequence when δ goes to 0^+,

$$u_{\epsilon,\delta} \to u_\epsilon \text{ in } \mathcal{C}^0([0,T];L^q(\Omega)),\ 1 \leq q < +\infty, \tag{16}$$

and $\partial_t u_\epsilon$ belongs to $L^2(0,T;H^{-1}(\Omega))$. Furthermore,

$$\phi_\delta(u_{\epsilon,\delta}) \to \phi(u_\epsilon) \begin{cases} \text{in } H^1(Q) \text{ weak },\ (17.a) \\ \text{in } \mathcal{C}^0([0,T];L^2(\Omega)).\ (17.b) \end{cases} \tag{17}$$

• The function u_ϵ clearly satisfies (4), (5) and the initial condition holds a.e. in Ω.

• For variational Inequality (7), we write (15) under the equivalent form

$$\int_\Omega \mathbb{H}_{\epsilon,\delta}(t,x,u_{\epsilon,\delta})(v-\phi_\delta(u_{\epsilon,\delta}))dx \geq 0, \tag{18}$$

and we integrate by parts the diffusive and convective terms in order to take the δ-limit.

• Concerning inner entropy Inequality (9) we choose k in $\mathbb{R}^+$ and a nonnegative function ξ in $H_0^1(Q)\cap L^\infty(Q)$ so as to consider in (15) the test-function $v = u_{\epsilon,\delta} - \frac{\lambda}{\|\xi\|_\infty} sgn_\lambda(u_{\epsilon,\delta}-k)\xi$. We integrate over $]0,T[$. After an integration by parts with respect to t and by using the Green formula to transform the convective term we may pass to the limit with respect to λ to obtain the desired inequality.

• For (8) the demonstration is inspired from the one presented in [4]. However, it cannot be developed directly from viscous Problem (12,13,14) and we come back to the penalized problem associated with (12,13,14). Moreover we take advantage of the homogeneous boundary conditions to remark that for any nonnegative real k, $(U_k^\epsilon)_{\epsilon>0}$ is a bounded sequence in $\mathcal{DM}_2(Q)$.

• Lastly (10) is proved by following the C.Mascia, A.Porretta & A.Terracina's reasoning and the calculuses can be developed directly from (12,13,14). First of all, we fix ζ in $\mathcal{D}(\mathbb{R}^p)$ such that $0\leq\zeta\leq 1$ and $supp\zeta \in B' \subset\subset B$, with $B\in\mathcal{B}$. In (18) we consider the test-function

$$v = \phi_\delta(u_{\epsilon,\delta}) - \frac{\lambda}{B} sgn_\lambda(\phi_\delta(u_{\epsilon,\delta}) - \phi_\delta(k))\zeta\xi\chi,$$

where ξ and χ are nonnegative functions respectively of $H^1(Q)\cap L^\infty(Q)$ and $H_0^1(\Omega)\cap L^\infty(\Omega)$ and $B = \|\zeta\chi\|_{L^\infty(\Omega)}\|\xi\|_{L^\infty(Q)}$, $k\geq 0$. The λ-limit and (8) are used to provide the result.

As a consequence of Theorem 4 and *a priori* estimates of Theorem 5, we claim:

Corollary 6. *When δ goes to 0^+, the whole sequence $(u_{\epsilon,\delta})_{\delta>0}$ strongly converges to u_ϵ in $\mathcal{C}^0([0,T];L^q(\Omega))$, $1\leq q<+\infty$. Besides, there exists a constant C, independent from ϵ such that:*

$$\|u_\epsilon\|_{W^{1,1}(Q)\cap L^\infty(Q)} \leq C,\ \epsilon\|\phi(u_\epsilon)\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C,$$
$$\forall h\in]0,T[, \forall t\in]0,T-h[,\ \|u_{\epsilon,\delta}(t+h,.)-u_{\epsilon,\delta}(t,.)\|_{L^1(\Omega)} \leq A_1\, h.$$

§3. The Singular Perturbations Property

Obstacle problems for first-order hyperbolic operators were introduced by A.Bensoussan & J.L.Lions [1] in 1973, as part of the study of cost-functions associated with deterministic processes. Since then numerous works have been carried out on this matter.

The feature of this work is to specify the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ when ϵ tends to 0^+. This kind of singular perturbations property for some inequalities has already been obtained by F.Mignot & J.P.Puel [8] for linear operators and by M.Madaune-Tort [7] for nonlinear parabolic degenerate equations in one space dimension associated with a positiveness constraint on the boundary; the case of a unilateral positiveness condition inside the studied field and in a multidimensional framework has been achieved in [5] for *weakly degenerate* operators $\mathbb{H}_{\epsilon,\delta}$, that means when we are able to ensure the existence of ϕ^{-1} as a function.

We first remind the definition of the weak entropy solution to the positiveness problem for $\mathbb{H}_0$ inspired from the one given in [5] - where the uniqueness property is established - and that corresponds to the special situation of Definition 2 when $\epsilon = 0$.

Definition 3. A function u of $BV(Q) \cap L^\infty(Q)$ is the weak entropy solution to the positiveness obstacle problem for $\mathbb{H}_0$ if
i) $u \geq 0$ a.e. on Q,
ii) for any k in $\mathbb{R}^+$ and any nonnegative ξ of $H_0^1(Q) \cap L^\infty(Q)$,

$$\int_Q U_k^0.\overline{\nabla}\xi dxdt - \int_Q sgn(u-k)G(u,k)\xi dxdt \geq 0, \tag{19}$$

iii) for all nonnegative ξ in $L^\infty(Q) \cap H^1(Q)$, $\xi(T,.) = \xi(0,.) = 0$, for any k in $\mathbb{R}^+$,

$$\int_\Sigma \mathbf{F}(k,0).\nu\zeta d\mathcal{H}^p \leq \int_\Sigma \mathbf{F}(\gamma_u,k).\nu\zeta d\mathcal{H}^p + \int_\Sigma \mathbf{F}(\gamma_u,0).\nu\zeta d\mathcal{H}^p, \tag{20}$$

γ_u denoting the trace of u along Σ in the sense of functions with bounded variations on Q.

iv) $u(0,x) = u_0(x)$ for a.e. x in Ω.

Due to Corollary 6 and to the compactness embedding of $BV(Q)$ into $L^1(Q)$, there exists a function u in $BV(Q) \cap L^\infty(Q)$ such that up to a subsequence when ϵ goes to 0^+,

$$u_\epsilon \to u \textit{ in } \mathcal{C}^0([0,T]; L^q(\Omega)),\ 1 \leq q < +\infty,$$

and the next property holds:

Proposition 7. *The function u is the weak entropy solution to the positiveness problem for $\mathbb{H}_0$.*

Proof. The positiveness constraint for u follows from the one for u_ϵ and the initial condition is due to the convergence property of $(u_\epsilon)_{\epsilon>0}$ toward u.

Besides, inner entropy Inequality (19) is obtained in the same manner as (9) and we observe that as a consequence of the third estimate in Corollary 6, the diffusive term goes to 0 with ϵ.

Lastly, boundary Condition (20) can be viewed as the singular perturbation of (10). Indeed, since $(U_k^\epsilon)_{\epsilon>0}$ is a bounded sequence in $\mathcal{DM}_2(Q)$, $U_k^\epsilon \geq 0$ in $\mathcal{M}_b(Q)$, when ϵ goes to 0^+

$$Div_{(t,x)}(U_k^\epsilon\zeta) \rightharpoonup Div_{(t,x)}(U_k\zeta) \textit{ in } \mathcal{M}_b(Q) \textit{ weak } * .$$

As a result, $\forall \xi \in H^1(Q) \cap L^\infty(Q) \cap \mathcal{C}_+(Q)$, with $\xi(T,.) = \xi(0,.) = 0$, $\forall k \in \mathbb{R}^+$ and $\forall \zeta \in \mathcal{D}_+(B)$,

$$\limsup_{\epsilon \to 0^+} \langle U_k^\epsilon \zeta, \xi \rangle_{\partial Q} \leq \langle U_k \zeta, \xi \rangle_{\partial Q}.$$

It comes, for any boundary layer sequence $(\rho_\varrho)_{\varrho > 0}$,

$$\int_\Sigma \mathbf{F}(k,0).\nu \xi \zeta d\mathcal{H}^p \leq - \lim_{\varrho \to 0^+} \int_Q U_k^0 \zeta \xi (0, \nabla \rho_\varrho) dxdt - \lim_{\varrho \to 0^+} \int_Q U_0^0 \zeta \xi (0, \nabla \rho_\varrho) dxdt.$$

The fact that u belongs to $BV(Q)$ permits to take the ϱ-limit thanks to an integration by parts formula. Inequality (20) follows, that completes the proof. □

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