Analytical techniques to solve numerically linear initial–value problems

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Abstract. The authors provide a numerical method in order to approximate the solution of a linear initial–value problem, by means of von Neumann series and Faber–Schauder systems.

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§1. Preliminaries

In this work some results are discussed in order to approximate the solution of the following initial–value problem: given $x_0 \in \mathbb{R}^n$, $a \in C([\alpha, \alpha+\beta], \mathcal{M}_n(\mathbb{R}))$ ($\mathcal{M}_n(\mathbb{R})$ is the set of all $n \times n$ real matrices) and $b \in C([\alpha, \alpha+\beta], \mathbb{R}^n)$, find $x \in C^1([\alpha, \alpha+\beta], \mathbb{R}^n)$ such that

$$\left\{ \begin{array}{ll} x'(t) = a(t)x(t) + b(t), & t \in [\alpha, \alpha+\beta] \\ x(\alpha) = x_0 & \end{array} \right. . \tag{0.1}$$

For the sake of simplicity we shall assume that $\alpha = 0$ and $\beta = 1$.

It is an elementary and well–known fact that the unique solution u of the initial–value problem above is characterised by the equality

$$u = f + Lu, \tag{0.2}$$

where

$$f := x_0 + \int_0^t b(s)ds$$

and L is the bounded and linear operator defined on the Banach space $C([0,1], \mathbb{R}^n)$, endowed with its usual sup–sup norm ($\|x\|_\infty := \sup_{t\in[0,1]} \|x(t)\|_\infty, \quad (x \in C([0,1], \mathbb{R}^n))$) by

$$Lx(t) := \int_0^t a(s)x(s)ds, \qquad (x \in C([0,1], \mathbb{R}^n)),\ 0 \le t \le 1).$$

Form now on, we shall understand f and L to be such function and operator. On the other hand, since operator L satisfies that for all $x \in C([0,1],\mathbb{R}^n))$ and for all $m \geq 1$

$$\|L^m x\|_\infty \leq \frac{1}{m!} M^m \|x\|_\infty, \tag{0.3}$$

where $M := \max_{0\leq t\leq 1} \|a(t)\|_\infty$, we have that the series $\sum_{m\geq 0} L^m$ converges. Then, it follows from the geometric series theorem [2] that operator $I-L$ is one–to–one and onto and its inverse operator $(I-L)^{-1}$ is bounded and linear. In fact

$$(I-L)^{-1} = \sum_{m\geq 0} L^m,$$

the so–called von Neumann series of L. Hence, in view of (0.2) we deduce that the unique solution of problem (0.1) is given by

$$u = \sum_{m\geq 0} L^m f. \tag{0.4}$$

Hence, letting

$$s_0 = f$$

and for $m \geq 1$

$$s_m = \sum_{k=0}^{m} L^k f = f + L s_{m-1},$$

then the sequence $\{s_m\}_{m\geq 0}$ converges uniformly to u. Moreover, we derive from (0.3) and (0.4) that

$$\|u - s_m\|_\infty \leq \|f\|_\infty \sum_{k\geq m+1} \frac{M^k}{k!}. \tag{0.5}$$

In order to obtain the sequence $\{s_m\}_{m\geq 0}$ we shall make of the usual Faber–Schauder systems in the space $C([0,1])$.

Let us recall ([3]) that a sequence $\{x_j\}_{j\geq 1}$ in a Banach space X is said to be a *Schauder basis* provided that for all $x \in X$ there exists a unique sequence of scalars $\{\lambda_j\}_{j\geq 1}$ in such a way that $x = \sum_{j\geq 1} \lambda_j x_j$. The j^{th} (continuous and linear) *biorthogonal functional* x_j^* is defined at such an x as $x_j^*(x) = \lambda_j$, and the j^{th} (continuous and linear) *projection* Q_j by $Q_j(x) = \sum_{i=1}^{j} \lambda_i x_i$.

Now we introduce the classical Schauder basis for the space $C([0,1])$, endowed with its usual sup–norm, the so–called Faber–Schauder system. Suppose that $\{t_j\}_{j\geq 1}$ is a dense sequence of distinct points in $[0,1]$ such that $t_1 = 0$ and $t_2 = 1$. The classical Faber–Schauder system $\{\Gamma_j\}_{j\geq 1}$ (associated with $\{t_j\}_{j\geq 1}$) for the Banach space $C([0,1])$ is defined as follows:

$$\Gamma_1(t) = 1, \qquad (0 \leq t \leq 1)$$

and for all $j > 1$, Γ_j is the piecewise linear continuous function with nodes at $t_1, \ldots, t_j$, such that

$$\text{for all } 1 \leq i < j, \quad \Gamma_j(t_i) = 0$$

while

$$\Gamma_j(t_j) = 1.$$

In what follows, $\{\Gamma_j\}_{j\geq 1}$ will denote such basis and $\{\Gamma_j^*\}_{j\geq 1}$ and $\{Q_j\}_{j\geq 1}$, respectively, the associated sequences of biorthogonal functionals and projections. In the next statement we collect some basic elementary facts that will play a fundamental role in our results. For a proof, see [3] or [4].

Theorem 1. *Let* $x \in C([0,1])$. *Then*

$$\Gamma_1^*(x) = x(t_1)$$

and for all $j > 1$,

$$\Gamma_j^*(x) = x(t_j) - \sum_{i=1}^{j-1} \Gamma_i^*(x)\Gamma_i(t_j).$$

In particular, for all $j \geq 1$ *and for all* $i \leq j$,

$$(Q_j x)(t_i) = x(t_i). \tag{1.1}$$

§2. The results

Let us point out that it is possible to obtain the image under operator L of any continuous function in terms of certain sequences of scalars, sequences which are obtained just by evaluating some functions at adequate points. More precisely; we shall consider the sup–sup norm on the space $C([0,1],\mathcal{M}_n(\mathbb{R}))$:

$$\|a\|_\infty := \sup_{t\in[0,1]} \|a(t)\|_\infty, \qquad (a \in C([0,1],\mathcal{M}_n(\mathbb{R}))).$$

Let $n \geq 1$ and assume that $a = (a_{ij})_{i,j=1,\ldots n} \in C([0,1],\mathcal{M}_n(\mathbb{R}))$, $b = (b_j)_{j=1,\ldots,n} \in C([0,1],\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. Given $1 \leq j,k \leq n$ let $\{a_{jk}^{(i)}\}_{i\geq 1}$ and $\{b_j^{(i)}\}_{i\geq 1}$ be the sequences of scalars satisfying

$$a_{jk} = \sum_{i\geq 1} a_{jk}^{(i)}\Gamma_i \qquad \text{and} \qquad b_j = \sum_{i\geq 1} b_j^{(i)}\Gamma_i.$$

Then, for all $x = (x_j)_{j=1,\ldots,n} \in C([0,1],\mathbb{R}^n)$ and for all $t \in [0,1]$ it is not difficult to obtain, integrating, that

$$f(t) + (Lx)(t) = x_0 + \left(\sum_{i\geq 1} c_j^{(i)} \int_\alpha^t \Gamma_i(s)ds\right)_{j=1,\ldots,n},$$

where for $j = 1,\ldots,n$,

$$\begin{cases} c_j^{(1)} = & b_j^{(1)} + \displaystyle\sum_{k=1}^n a_{jk}^{(1)} x_k(t_1) \\ c_j^{(i)} = & \displaystyle\sum_{l=1}^i \left(b_j^{(l)} + \sum_{k=1}^n a_{jk}^{(l)} x_k(t_i)\right)\Gamma_k(t_i) - \sum_{l=1}^{i-1} c_j^{(l)}\Gamma_l(t_i), \quad \text{if } i \geq 2 \end{cases}.$$

In the following result we replace the sequence $\{s_m\}_{m\geq 0}$ by another $\{y_m\}_{m\geq 0}$ which can be calculated explicitly:

Theorem 2. *Let* $n \geq 1$ *and suppose that* $a = (a_{ij})_{i,j=1,...n} \in C([0,1], \mathcal{M}_n(\mathbb{R}))$, $b = (b_j)_{j=1,...,n} \in C([0,1], \mathbb{R}^n)$ *and* $x_0 \in \mathbb{R}^n$. *Let* $m \geq 1$ *and* $n_1, \ldots, n_m \geq 1$. *Consider the continuous function*

$$y_0(t) := x_0(t), \qquad (t \in [0,1])$$

and for $r = 1, \ldots, m$ *the continuous functions*

$$\varphi_{r-1}(t) := a(t)y_{r-1}(t) + b(t), \qquad (t \in [0,1]),$$

and

$$y_r(t) := x_0 + \int_0^t (Q_{n_r}(\varphi_{r-1}(s))_k)_{k=1,...,n} ds, \qquad (t \in [0,1]).$$

Assume in addition that certain positive numbers $\varepsilon_1, \ldots, \varepsilon_m$ *satisfy*

$$\|f + Ly_{r-1} - y_r\|_\infty < \varepsilon_r,$$

where L *is the linear integral operator on* $C([0,1], \mathbb{R}^n)$ *associated with the initial–value problem (0.1) Then, if* u *is the solution of such problem, we have that*

$$\|u - y_m\|_\infty \leq \|f\|_\infty \sum_{r \geq m} \frac{M^r}{r!} + \|x_0\|_\infty \frac{M^m}{m!} + \sum_{r=1}^{m} \varepsilon_r \frac{M^{m-r}}{(m-r)!},$$

where $M = \max_{0 \leq t \leq 1} \|a(t)\|_\infty$.

Proof. Since

$$\|u - y_m\|_\infty \leq \|u - (s_{m-1} + L^m x_0)\|_\infty + \|y_m - (s_{m-1} + L^m x_0)\|_\infty, \tag{2.1}$$

we shall separately obtain upper bounds for both terms on the left hand side in (2.1). On the one hand, inequalities (0.5) and (0.3) give

$$\|u - (s_{m-1} + L^m x_0)\|_\infty \leq \|u - s_{m-1}\|_\infty + \|L^m x_0\|_\infty \leq \|f\|_\infty \sum_{r \geq m} \frac{M^r}{r!} + \|x_0\|_\infty \frac{M^m}{m!}. \tag{2.2}$$

On the other hand, the hypothesis on the ε_r's and inequality (0.3) give

$$\begin{gathered}
\|y_m - s_{m-1} - L^m x_0\|_\infty = \|y_m - f - Lf - L^2 f - \ldots - L^{m-1} f - L^m y_0\|_\infty \leq \\
\|y_m - f - Ly_{m-1}\|_\infty + \|Ly_{m-1} - Lf - L^2 y_{m-2}\|_\infty + \\
\|L^2 y_{m-2} - L^2 f - L^3 y_{m-3}\|_\infty + \cdots + \\
\|L^{m-1} y_1 - L^{m-1} f - L^m y_0\|_\infty \leq \\
\varepsilon_m + \|L\| \varepsilon_{m-1} + \|L^2\| \varepsilon_{m-2} + \cdots + \|L^{m-1}\| \varepsilon_1 \leq \\
\sum_{r=1}^{m} \varepsilon_r \frac{M^{m-r}}{(m-r)!}.
\end{gathered} \tag{2.3}$$

Finally, the proof is complete in view of (2.1), (2.2) and (2.3). □

Note that given $\varepsilon_1, \dots, \varepsilon_m > 0$ we can find positive integers $n_1, \dots, n_m$ such that $\|f + Ly_{r-1} - y_r\|_\infty < \varepsilon_r$, since for all $x \in C([0,1])$, $\lim_{j\geq 1} \|Q_j x - x\|_\infty = 0$. However, if we wish to find the integers $m, n_1, \dots, n_m$ from the positive numbers $\varepsilon_1, \dots, \varepsilon_m$, we can use this easy and well–known consequence of the mean value theorem and the interpolating property (1.1) of the basis for $C([0,1])$: suppose that $x \in C^1([0,1])$ (in fact, we can assume that x is a continuous and C^1 class function on $[0,1]$, except perhaps for a finite number of points), $j \geq 2$ and

$$h := \max_{i=2,\dots,j} (s_i - s_{i-1}),$$

where $\{s_1 = 0 < s_2 < \cdots < s_{j-1} < s_j = 1\}$ is the set $\{t_1, \dots, t_j\}$ ordered in a increasing way. Then

$$\|x - Q_j x\|_\infty \leq 2\|x'\|_\infty h. \tag{2.4}$$

If one assumes in the initial–value problem that a and b are functions of C^1 class on $[0,1]$ then the norm appearing in Theorem 2, $\|f + Ly_{r-1} - y_r\|_\infty$ can be estimated as follows: $\|f + Ly_{r-1} - y_r\|_\infty \leq \|\varphi_r - (Q_{n_r}(\varphi_r)_k)_{k=1,\dots,n}\|_\infty$ and then above applies.

Remark 1. The Faber–Schauder system has also been used in [1] for solving numerically the linear Volterra integro–differential equation.

Remark 2. Although our numerical method works for any Faber–Schauder system in the Banach space $C([0,1])$, we have chosen the classical one because the biorthogonal functionals and the projections associated have an easy expression.

§3. A numerical example

Finally we exhibit an example which shows the behaviour of our results. To this end, we fix the data's initial–value problem: $x_0 \in \mathbb{R}^n$, $a = (a_{ij})_{ij=1,\dots,n} \in C^1([0,1], \mathcal{M}_n(\mathbb{R}))$ and $b = (b_j)_{j=1,\dots,n} \in C^1([0,1], \mathbb{R}^n)$. We choose an $n \in \mathbb{N}$ with $n = 2^k + 1$, $k \in \mathbb{N}$, and thus

$$h = \max_{2 \leq i \leq n} (s_i - s_{i-1}) = \frac{1}{2^k}.$$

Then we calculate the sequences of coefficients $\{a_{jk}^{(i)}\}_{i=1}^n$ and $\{b_j^{(i)}\}_{i=1}^n$ and obtain recursively the functions y_r in Theorem 2, taking $n_1 = \cdots = n_r = n$. We determine the errors

$$E_{nr} = \max_i |y_r(s_i) - u(s_i)|,$$

where u is the exact solution. We have considered the approximation of the exact solution y_m in such a way that

$$\left| \frac{E_{nm}}{E_{nm+1}} \right| < 1 + 10^{-2}.$$

Let us point out that we do not need to solve systems of algebraical linear equations –collocation methods– or to use quadrature formulas.

Example 1. The function $y(t) = \arctan t$ is the analytical solution of the second order equation

$$\begin{cases} y''(t) + \dfrac{2t}{1+t^2} y(t) = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}.$$

If one associates, in the usual way, this problem with an initial–value problem in the form (0.1) and applies the above results, he obtains the following table. In its columns we give the absolute errors E_{nm} in nine representative points of the approximations y_m, obtained with different values of n.

	$(n = 9, m = 4)$	$(n = 17, m = 6)$	$(n = 33, m = 6)$
0	0	0	0
0.125	3.01×10^{-4}	7.61×10^{-5}	1.90×10^{-5}
0.250	4.98×10^{-4}	1.25×10^{-4}	3.14×10^{-5}
0.375	5.30×10^{-4}	1.33×10^{-4}	3.33×10^{-5}
0.500	4.05×10^{-4}	9.89×10^{-5}	2.47×10^{-5}
0.625	1.91×10^{-4}	3.38×10^{-5}	8.55×10^{-6}
0.750	1.01×10^{-5}	4.77×10^{-5}	1.13×10^{-5}
0.875	3.94×10^{-5}	1.31×10^{-4}	3.02×10^{-5}
1	5.32×10^{-4}	2.04×10^{-4}	4.05×10^{-5}

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