

ON AN ANALYTICAL SOLUTION IN THE PLANAR ELLIPTIC RESTRICTED THREE–BODY PROBLEM

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Abstract. Guidelines for an *analytical* treatment of the small–amplitude orbital motion of a particle of negligible (infinitesimal) mass about the libration points in the planar elliptic restricted three–body problem (ERTBP) are outlined.

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§1. Introduction

1.1. Background, Overview and General Facts

A wealth of classical and modern–approach literature, providing the interested reader with information and details concerning diverse mathematical, physical and astronomical aspects of the three–body problem, might be furnished here. Nevertheless, for the purposes of the present study, the list of treatises and journal articles at the end of the paper (and, of course, references therein) should represent helpful reading materials on the subject, even at an introductory level.

The formulation of the so–called (*gravitational*) *Circular (resp. Elliptic) Restricted Three–Body Problem*, CRTBP (resp. ERTBP) for short, can be stated as follows:

Two point masses, called the “primaries”, orbit one another in circular (resp. elliptic) Keplerian motion, each influencing, but not influenced by, a third body of (negligible) infinitesimal mass. The motion of this third (infinitesimal) particle under the gravitational influence of the two other finite masses must be studied.

For our purposes, the ERTBP focusses on the motion of an infinitesimal body attracted –according to the Newtonian gravitational interaction law– by two finite masses revolving in respective ellipses about their centre of mass: accordingly, the relative orbit of two bodies will be an ellipse of eccentricity e , and the third body –of negligible mass– moves in their gravitational attraction field.

It is a well–known fact that there exist *five relative equilibrium solutions*, or libration points, where the gravitational and centrifugal forces balance each other: *three collinear*, and *two triangular*, located in the plane of the motion of the primaries at the vertices of the equilateral triangles which have the primaries as the other vertices.

Lagrange had already shown that three bodies placed at the vertices of an equilateral triangle can be given such initial velocities that they will retain permanently the same configuration: the well-known Lagrangian *equilateral solutions* of the *general problem* of three bodies.

Introducing elliptic orbits for the primaries generalizes the original CRTBP, and improves its applicability, while some outstanding and useful properties of the circular model still hold true or can be adapted to the elliptic case. In particular, possible positions of equilibrium occur when the three bodies form equilateral triangles. An important instance for application of this model is provided by the problem of motion of the Trojan asteroids around the triangular Lagrangian point L_4 : according to this simplified model, the asteroids are only influenced by the gravitational forces of the Sun and Jupiter, and the orbit of Jupiter around the Sun is supposed to be a fixed ellipse.

Although the CRTBP is *not integrable*, there exists a first integral of the system, the so-called *Jacobi integral*: in a rotating reference frame this problem has the property that its Hamiltonian does not depend explicitly on the time; therefore the problem possesses an integral (the Jacobi integral). In addition to the Jacobian integral no other analytical integrals exist with global validity. The existence of this first integral helps to establish the regions of possible motions for given sets of initial conditions, to regularize the problem or to analyse the stability of motions with the help of the method of the Poincaré surface of section in the two-dimensional case.

On the other hand, as far as the ERTBP is concerned, it does not admit this integral of the circular problem, at least not in its usual sense. In the elliptic problem, the energy along any orbit is a time-dependent quantity.

However the equations of motion of the small (infinitesimal) body can be advantageously formulated in a *rotating pulsating coordinate system*: given that the orbits of the primaries are elliptic, the rotating reference frame—in order to maintain the primaries in fixed positions—must be a system which rotates uniformly and with axes which expand and shrink.

Such a coordinate system is defined as follows: the x -axis rotates together with the primaries; the y -axis is obtained by a direct $\pi/2$ rotation of the x -axis. The *independent variable* can be chosen so as to be the true anomaly f of the Keplerian ellipse (with semi-major axis a and numerical eccentricity e) described by the smaller primary, and the instantaneous (variable) distance of the primaries, namely

$$r(t) = \frac{a(1 - e^2)}{1 + e \cos f}, \quad (1)$$

can be taken as the unit of length. Accordingly, in this coordinate system, and with this choice of variables and units, the problem is characterized by a *non-conservative* Lagrangian or Hamiltonian function: the independent variable is explicitly contained in that function. A consequence of this fact is, once again, the non-existence of the Jacobi integral. The elliptic problem is thus fundamentally different from the circular restricted problem: it is not conservative.

To sum up: The two-dimensional ERTBP is an important instance of a *non-autonomous non-integrable* dynamical system with *two degrees of freedom*.

On the other hand, and in spite of its intrinsic difficulty, certain adequate *changes of variables* bring the analytical form of the equations for the orbit of the third body into formulae which are similar to those governing the CRTBP, and then important conclusions can be drawn

from *analogies inspired by this reduction* into a situation that has been (and is still) studied in great detail.

No exact, complete and general solution to this problem can be obtained *in finite terms*, but this mathematical inconvenience is usually overcome –or, at least, softened– by concentrating on the thorough investigation of significant features and properties of certain *special cases* of the problem (established under severe restrictions and simplifying hypotheses concerning the mathematical model), but also on the construction of *analytical* expressions and formulae providing *approximate particular solutions* to the system at issue.

1.2. Aims and Scope of the Paper

An *analytical* treatment of the small–amplitude orbital motion of a particle of negligible mass about the libration points in the planar ERTBP constitutes the main purpose of the study.

Small motions close to the equilibrium points (more specifically, infinitesimal deviations from the libration centres) are considered: the objective of this work is to sketch a procedure to produce *explicit* analytical expressions for the small–amplitude motion of the infinitesimal mass about the equilibrium points in the ERTBP. To handle this dynamical problem, it is formulated and approached within the framework of *Hamiltonian Mechanics*.

The (approximate) integration of the model problem is undertaken by means of a perturbation technique based on *Lie series* developments, which leads to an approximate solution to the differential system of canonical equations of motion derived from the chosen Hamiltonian function (expanded in powers of the numerical eccentricity e of the elliptic orbit of the two primaries, which plays the role of the *small parameter* of the perturbation).

Concerning the fundamental guidelines, approach and solution method, the essential “intellectual roots” of this research can be found in the articles by D. Şelaru & C. Cucu–Dumitrescu ([9], [10]), and Delva [3].

Şelaru and Cucu–Dumitrescu performed an analytical investigation concerning the construction of asymptotic perturbative approximations for small–amplitude motions of the (infinitesimal) third point mass in the neighbourhood of a Lagrangian equilateral libration position in the planar, elliptic restricted problem of three bodies. To this end, after a sequence of canonical transformations introducing successive adequate sets of phase variables, they formulated the Hamiltonian governing the motion of the negligible–mass body, using the eccentric anomaly of the primaries’ elliptic Keplerian orbit as the independent variable, and then studied the linearized system of differential equations of motion as obtained from the quadratic part of the Taylor expansion (with respect to the canonical variables) of that Hamiltonian around a Lagrange solution. At a later stage, in [9] they developed their theory and calculations of an asymptotical solution up to the first order in the orbital eccentricity of the primaries taken as the perturbation parameter: terms up to the first power of the eccentricity of the relative orbit of the primaries were retained both in the Hamiltonian and in the approximate solution. An extension of these considerations to a similar second–order theory was presented in [10].

In the present paper *we re-examine their analytical treatment and amend some errors and misapplications of certain canonical transformations*. In particular, we have detected *a mistake in the Hamiltonian function* taken by these authors as the starting point for their expansions leading to the construction of the linearized system and the subsequent perturbative treatment in terms of powers of the small parameter e . In fact [see below, Eq.(38)], *a first–order term*

in e , proportional to $\cos E$ and explicitly containing a contribution of the eccentric anomaly (which, remember, is just the independent variable in this *non-autonomous* dynamical system as considered in this formulation) *seems to be missing* in these authors' basic Hamiltonian.

With the necessary precautions, the analysis leading to the construction of the Hamiltonian function and its truncated Taylor expansion (about the libration points of interest) closely follows, in principle, the considerations due to Şelaru & Cucu–Dumitrescu, according to whose proposal the Hamiltonian for small motions in the neighbourhood of the equilibria can be established.

On the contrary (see below), the direct approximate analytical integration of the resulting linearized equations of motion around the equilibrium points can be undertaken according to a Lie series formalism, using an eccentric anomaly as the independent variable. To this end, the Hamiltonian and the canonical equation should be explicitly arranged according to ascending powers of e , the eccentricity of the relative elliptic orbit of the two primaries.

In their approximate analytical integration of the problem, Şelaru & Cucu–Dumitrescu ([9], [10]) applied a *von Zeipel* type perturbation method, which produces a near-identity canonical transformation derived from a generating function *in mixed variables* (the old coordinates and the new canonically conjugate momenta). This leads to *implicit transformation equations* derived from a generating function in which phase variables belonging to *different canonical sets* are mixed.

An advantage of our option over the von Zeipel procedure is that the differential equations of motion are treated *directly*, avoiding the cumbersome reckoning work required in the determination of the von Zeipel generating function and the subsequent formation of partial derivatives to express the implicit transformation equations and the elimination process leading to explicit relations between the canonical sets of old and new variables.

An eccentric-like anomaly is chosen as the independent variable, in terms of which the solution to the canonical system of differential equations of motion governing the problem can be expanded (Delva [3] based her expansions on a true-like anomaly). The final results for the canonical solution should be *in explicit form for the coordinates and conjugate momenta* as functions of the eccentric anomaly as the independent variable.

In adapting Delva's approach, the linear differential Lie operator for the motion of the third (massless) body is to be obtained in terms of the canonical variables and the eccentric anomaly of the primaries. Successive terms of the Lie series are then obtained by repeating the action of this operator, that is, by constructing "powers" of the operator. The Lie series solution to the canonical equations in the chosen eccentric anomaly is developed by propagating initial conditions with the help of the operator.

Another alternative treatment of the problem might be based on judicious modifications and adaptations of canonical perturbation methods combining ideas of averaging and Lie transform technique (e.g., variants of the Hori [6] and Deprit [4] type procedures).

§2. Statement of the Problem and Basic Notations

We consider the problem (see, e.g. Brouwer & Clemence [2], Ch. X, §3) of motion of three point particles, with masses m_0 , m_1 and m_2 , in which the mass m_2 (that of the infinitesimal body) is negligibly small in comparison with m_0 and m_1 (those of the finite bodies, primary bodies or, simply, "primaries"). In this case the motions of the finite-mass bodies are not sen-

sibly affected by the gravitational attraction due to the infinitesimal particle: they are precisely the Keplerian motions of the pure, unperturbed two–body problem, and can be contemplated as completely known and given in advance. This simplification gives rise to the *restricted three–body problem*, say RTBP.

In what follows, we concentrate on the problem of motion of the infinitesimal body, assuming that this motion takes place in the common orbital plane of the primaries (*planar RTBP*), and that their Keplerian orbits are known *ellipses* around their centre of mass (*planar elliptic RTBP*). Consequently, the relative orbit of one of the primaries with respect to the other one is also an ellipse. All these ellipses share a common numerical value for the eccentricity e .

Details concerning the treatment of the CRTBP can be found, e.g. in Boccaletti & Pucacco ([1], vol. 1, Ch. 4), Brouwer & Clemence ([2], Ch. X, §3; Ch. II, §10), Marchal [7], Szebehely ([8], Ch. 1).

As for the elliptic problem, see Boccaletti & Pucacco ([1], vol. 1, Ch. 4, §4.7), Marchal ([7], Ch. 9, §9.3), Szebehely ([8], Ch. 10, §10.3), and some of the journal articles in the section of References at the end of this contributed paper.

For future calculations, fix some *notations* and the *value of some parameters* occurring in the problem. According to the usual practice and without loss of generality, after appropriate choice of the system of units, we are allowed to take the value of the gravitational constant equal to 1, and put for the finite masses $m_0 + m_1 = 1$, from which we choose $m_1 \equiv \nu$, $m_0 = 1 - \nu$, with $0 \leq \nu \leq 1/2$. If, in addition to this, the value of the orbital angular momentum of the relative motion of the primaries is unity, then the semi–latus rectum of that elliptic relative orbit will be 1, and the polar equation of this ellipse will read

$$r = 1 / (1 + e \cos f) . \tag{2}$$

Let us coordinatize the orbital plane of the primaries by means of a Cartesian reference frame with the origin placed at the centre of mass of m_0 and m_1 . Let (ξ, η) denote the coordinates of the infinitesimal mass m_2 in this configuration space, and consider the corresponding phase space, by introducing the corresponding canonically conjugate momenta $p_\xi = d\xi/dt$, $p_\eta = d\eta/dt$. In this *fixed reference system*, the Hamiltonian function of the problem of motion of the infinitesimal body (per unit of mass), with time t as the independent variable, can be expressed as

$$\mathcal{H} = \frac{1}{2} (p_\xi^2 + p_\eta^2) - \left(\frac{1 - \nu}{r_0} + \frac{\nu}{r_1} \right) , \tag{3}$$

$$r_j^2 = \text{dist}^2(m_2, m_j) = (\xi - \xi_j)^2 + (\eta - \eta_j)^2 , \quad j = 0, 1 , \tag{4}$$

where, obviously, here r_0 and r_1 denote the distances of the infinitesimal body to the primaries.

Transition to a *non–uniformly rotating pulsating system* with coordinates (X, Y) is considered. Furthermore, one wishes the new X–axis to coincide –at each instant of time– with the straight line connecting the primaries, and the scale of distances to be such that the unit of length should be equal to that in the fixed reference frame divided by the distance r between the primaries. Accordingly, the separation between the primaries will remain constant and equal to 1, and the position of the finite masses (fixed on the new, moving abscissa–axis) will be determined by the coordinates

$$(X_0, Y_0) = (-\nu, 0) , \quad (X_1, Y_1) = (1 - \nu, 0) . \tag{5}$$

This is accomplished by a point transformation in configuration space that is the composition of a *rotation*

$$\xi = X \cos f - Y \sin f, \quad \eta = X \sin f + Y \cos f, \quad (6)$$

and a *homotecy* with similitude ratio r . This transformation can be *extended* to a (time-dependent) completely *canonical transformation*

$$(\xi, \eta, p_\xi, p_\eta; t) \longrightarrow (X, Y, P_X, P_Y; t),$$

derived from a generating function of the third type (Goldstein [5], Ch. 9, §9.1 and §9.2; Szebehely [8], Ch. 7, §7.2)

$$S_3 \equiv S_3(X, Y, p_\xi, p_\eta; f(t)) = -r [p_\xi (X \cos f - Y \sin f) + p_\eta (X \sin f + Y \cos f)], \quad (7)$$

from which the following transformation equations are obtained,

$$\begin{aligned} \xi &= -\partial S_3 / \partial p_\xi = r (X \cos f - Y \sin f), \quad \eta = -\partial S_3 / \partial p_\eta = r (X \sin f + Y \cos f), \quad (8) \\ P_X &= -\partial S_3 / \partial X = r (p_\xi \cos f + p_\eta \sin f), \quad P_Y = -\partial S_3 / \partial Y = r (-p_\xi \sin f + p_\eta \cos f) \quad (9) \end{aligned}$$

and then

$$p_\xi = (p_X \cos f - p_Y \sin f) / r, \quad p_\eta = (p_X \sin f + p_Y \cos f) / r. \quad (10)$$

The rule for the transformation of the Hamiltonian into a new function in the new variables is

$$\begin{aligned} \mathcal{H} &\rightarrow \tilde{\mathcal{H}} : \mathcal{H}(\xi, \eta, p_\xi, p_\eta; t) \longrightarrow \tilde{\mathcal{H}}(X, Y, P_X, P_Y; t), \\ \tilde{\mathcal{H}} &= \mathcal{H}^*(X, Y, P_X, P_Y; t) + \frac{\partial S_3}{\partial t}(X, Y, P_X, P_Y; t) \end{aligned} \quad (11)$$

$$\partial S_3 / \partial t = (\partial S_3 / \partial f) (df / dt). \quad (12)$$

From the well-known geometrical and dynamical properties of the (Keplerian) relative orbit, our previous choice of units and Eq. (2), the law of areas yields

$$r^2 (df / dt) = 1 \implies df / dt = 1 / r^2 = (1 + e \cos f)^2, \quad dt = \tilde{f} df, \quad \tilde{f} = r^2, \quad (13)$$

which also expresses a differential transformation of the independent variable $t \rightarrow f$. From the polar equation of the relative orbit, Eq. (2),

$$dr / df = (e \sin f) / (1 + e \cos f)^2 = (e \sin f) r^2. \quad (14)$$

Accordingly,

$$\begin{aligned} \partial S_3 / \partial f &= -r [p_\xi (-X \sin f - Y \cos f) + p_\eta (X \cos f - Y \sin f)] + r S_3 e \sin f \\ &= Y P_X - X P_Y - r e \sin f (X P_X + Y P_Y), \quad \partial S_3 / \partial t = (1 / r^2) \partial S_3 / \partial f \end{aligned} \quad (15)$$

$$\begin{aligned} \tilde{\mathcal{H}} &= \frac{1}{r^2} \left[\frac{1}{2} (P_X^2 + P_Y^2) + (Y P_X - X P_Y) \right] - \frac{e \sin f}{r} (X P_X + Y P_Y) - \\ &\quad \left(\frac{1 - \nu}{\tilde{r}_0} + \frac{\nu}{\tilde{r}_1} \right), \end{aligned} \quad (16)$$

$$\tilde{r}_j^2 = r^2 [(X - X_j)^2 + (Y - Y_j)^2], \quad j = 0, 1, \quad (17)$$

where $\tilde{\mathcal{H}}$ stands for the *Hamiltonian* of the problem when expressed *in terms of the new canonical set*, using t as the independent variable.

§3. A Modified Hamiltonian

The Hamiltonian can be *further modified* by changing the independent variable from t to a *new pseudo-time*. If the *eccentric anomaly* E of the *relative orbit* of the primaries is introduced as the new independent variable, a reparametrization of motion $t \rightarrow E$ is performed, for instance by means of a differential time transformation $dt = \bar{f} dE$. The corresponding Hamiltonian, with this fictitious time E as the independent variable, will be $\overline{\mathcal{H}} = \bar{f} \tilde{\mathcal{H}}$. The usual basic formulae pertaining to the mathematical description of elliptic Keplerian motion, along with the choice of our system of units and the values previously fixed for some parameters, allow us to conclude that

$$\bar{f} = r/\sqrt{1-e^2}, \quad \text{where } r = a(1 - e \cos E). \quad (18)$$

Thus, the Hamiltonian for the motion of the infinitesimal body, in canonical variables (X, Y, P_X, P_Y) , with the *eccentric anomaly* E of the primaries' relative orbit as the time parameter, reads

$$\begin{aligned} \overline{\mathcal{H}} \equiv \overline{\mathcal{H}}(X, Y, P_X, P_Y; E) &= \frac{\sqrt{1-e^2}}{1-e \cos E} \left[\frac{1}{2} (P_X^2 + P_Y^2) + (Y P_X - X P_Y) \right] \\ &\quad - \frac{\Phi(X, Y; \nu)}{\sqrt{1-e^2}} - \frac{e \sin E}{1-e \cos E} (X P_X + Y P_Y), \end{aligned} \quad (19)$$

$$\Phi(X, Y; \nu) = r \left(\frac{1-\nu}{\tilde{r}_0} + \frac{\nu}{\tilde{r}_1} \right) = \frac{1-\nu}{\bar{r}_0} + \frac{\nu}{\bar{r}_1}, \quad r = \text{dist}(m_0, m_1), \quad (20)$$

$$\bar{r}_j^2 = \tilde{r}_j^2/r^2 = (X - X_j)^2 + (Y - Y_j)^2, \quad j = 0, 1, \quad (21)$$

$$(X_0, Y_0) = (-\nu, 0), \quad (X_1, Y_1) = (1-\nu, 0), \quad (22)$$

where, clearly, \bar{r}_0 and \bar{r}_1 represent the distances of the infinitesimal mass to the primaries, r is the distance between the primaries. while the position of the primaries is given by coordinates (X_j, Y_j)

§4. Fixing the Origin of Phase Space at a Libration Point

In order to study the small-amplitude orbits of the infinitesimal particle in the neighbourhood of a libration point, we must obtain an adequate expression for the Hamiltonian governing such motions.

Let $L(a, b)$ be a *Lagrangian point*, which in the *phase space* of the canonical variables (X, Y, P_X, P_Y) is located at $L \equiv L(\hat{X}_0, \hat{Y}_0, P_{\hat{X}_0}, P_{\hat{Y}_0})$, with

$$\hat{X}_0 = a, \quad P_{\hat{X}_0}(E) = -b + a \frac{e \sin E}{\sqrt{1-e^2}}, \quad (23)$$

$$\hat{Y}_0 = b, \quad P_{\hat{Y}_0}(E) = a + b \frac{e \sin E}{\sqrt{1-e^2}}, \quad (24)$$

and constitutes a stationary solution of the problem. The origin of the phase space will be transferred to point $L(\hat{X}_0, \hat{Y}_0, P_{\hat{X}_0}, P_{\hat{Y}_0})$, by means of a canonical translation defining a new canonical set (x, y, p_x, p_y) .

A point transformation representing a mere translation

$$X = x + X_0 \iff x = X - X_0, \quad Y = y + Y_0 \iff y = Y - Y_0, \quad (25)$$

can be *extended* to a canonical transformation (Goldstein [5], §9.2, pp. 386–387; see also Marchal [7], pp. 197–198; Szebehely [8], §7.4, pp. 352–354) whose generating function and equations may be assumed to have the form

$$S_2 \equiv S_2(X, Y, p_x, p_y) = (X - X_0)p_x + (Y - Y_0)p_y + g(X, Y), \quad (26)$$

$$P_X = \frac{\partial S_2}{\partial X} = p_x + \frac{\partial g}{\partial X} = p_x + P_{X_0} \implies g = X P_{X_0} + \varphi(Y), \quad (27)$$

$$P_Y = \frac{\partial S_2}{\partial Y} = p_y + \frac{\partial g}{\partial Y} = p_y + P_{Y_0} \implies g = Y P_{Y_0} + \psi(X), \quad (28)$$

where g is any differentiable function of the old coordinates, whereas φ and ψ are arbitrary functions of their respective arguments. For our purposes, they can be chosen in such a way that $g = X P_{X_0} + Y P_{Y_0}$,

$$S_2 = (X - X_0)p_x + (Y - Y_0)p_y + X P_{X_0} + Y P_{Y_0} \quad (29)$$

$$= X(p_x + P_{X_0}) - X_0 p_x + Y(p_y + P_{Y_0}) - Y_0 p_y, \quad (30)$$

In our case, we perform a canonical transformation

$$(X, Y, P_X, P_Y; E) \longrightarrow (x, y, p_x, p_y; E),$$

derived from a generating function, in which the independent variable E explicitly occurs,

$$S_2(X, Y, p_x, p_y; E) = X(p_x + P_{\hat{X}_0}) - \hat{X}_0 p_x + Y(p_y + P_{\hat{Y}_0}) - \hat{Y}_0 p_y \quad (31)$$

$$= (X - a)p_x + (Y - b)p_y + X P_{\hat{X}_0}(E) + Y P_{\hat{Y}_0}(E). \quad (32)$$

The transformation equations derived from S_2 are

$$P_X = \frac{\partial S_2}{\partial X} = p_x + P_{\hat{X}_0}, \quad x = \frac{\partial S_2}{\partial p_x} = X - a, \quad (33)$$

$$P_Y = \frac{\partial S_2}{\partial Y} = p_y + P_{\hat{Y}_0}, \quad y = \frac{\partial S_2}{\partial p_y} = Y - b, \quad (34)$$

and the rule for the transformation of the Hamiltonian yields

$$\overline{\mathcal{H}} \rightarrow \mathcal{K} : \overline{\mathcal{H}}(X, Y, P_X, P_Y; E) \longrightarrow \mathcal{K}(x, y, p_x, p_y; E), \quad (35)$$

$$\mathcal{K} = \overline{\mathcal{H}}^*(x, y, p_x, p_y; E) + \frac{\partial S_2}{\partial E}(x, y, p_x, p_y; E), \quad (36)$$

$$\frac{\partial S_2}{\partial E} = (ax + by) \frac{e \cos E}{\sqrt{1 - e^2}} + (a^2 + b^2) \frac{e \cos E}{\sqrt{1 - e^2}}. \quad (37)$$

Accordingly, the final expression for the Hamiltonian in the new canonical variables takes on the form:

$$\begin{aligned} \mathcal{K} &= \frac{\sqrt{1 - e^2}}{1 - e \cos E} \left[\frac{1}{2} (p_x^2 + p_y^2) + (yp_x - xp_y) \right] - \frac{\oplus(x, y; \nu)}{\sqrt{1 - e^2}} \\ &\quad - \frac{e \sin E}{1 - e \cos E} (xp_x + yp_y) + \frac{e \cos E}{2\sqrt{1 - e^2}} (a^2 + b^2), \end{aligned} \quad (38)$$

$$\oplus = \Phi^*(x, y; \nu) + (ax + by) + \frac{1}{2} (a^2 + b^2), \quad (39)$$

where we have *underlined* the “missing” term [cf Şelaru & Cucu–Dumitrescu [9], Eq. (9); [10], Eq. (9)].

The canonical system of differential equations of motion derived from \mathcal{K} will account for the orbital evolution of the infinitesimal particle around the considered libration point. Remember that the canonical variables (x, y, p_x, p_y) measure “deviations” with respect to the origin of the phase space (which is, precisely, an equilibrium solution).

§5. Outline of the Procedure

If, to start with, “deviations” are supposed to be small, one may undertake the study of the dynamical system generated by the linearized equations of motion. To this end, one performs a Taylor expansion of \mathcal{K} , in terms of the canonical variables, around the origin (i.e., around the libration point at issue). Given that, in this reference framework, the origin is an equilibrium position, the gradient of function \mathcal{K} vanishes when evaluated at the origin. The canonical equations derived from the “Hamiltonian” constructed with terms up to the quadratic part of that expansion will give rise to the linearized equations of motion.

Next, a perturbative treatment of that system will use a power series expansion of the preceding truncated “Hamiltonian”, arranged in ascending powers of the eccentricity e of the relative orbit of the primaries. Let $\hat{\mathcal{K}}(x, y, p_x, p_y; E; e)$ denote the resulting expression after these steps, truncated at the desired order in e . The corresponding canonical equations and the linear differential Lie operator are

$$x' \equiv \frac{dx}{dE} = \frac{\partial \hat{\mathcal{K}}}{\partial p_x}, \quad p'_x \equiv \frac{dp_x}{dE} = -\frac{\partial \hat{\mathcal{K}}}{\partial x}, \quad (40)$$

$$y' \equiv \frac{dy}{dE} = \frac{\partial \hat{\mathcal{K}}}{\partial p_y}, \quad p'_y \equiv \frac{dp_y}{dE} = -\frac{\partial \hat{\mathcal{K}}}{\partial y}, \quad (41)$$

$$D \equiv \frac{d}{dE} = \frac{\partial}{\partial x} \frac{dx}{dE} + \frac{\partial}{\partial y} \frac{dy}{dE} + \frac{\partial}{\partial p_x} \frac{dp_x}{dE} + \frac{\partial}{\partial p_y} \frac{dp_y}{dE} + \frac{\partial}{\partial E}, \quad (42)$$

which allows us to obtain the solution in the form of Lie series, once we have given initial conditions (cf Delva [3]).

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References

- [1] BOCCALETTI, D., AND PUCACCO, G. *Theory of Orbits* (2 volumes). Springer, Berlin and Heidelberg, 1996.
- [2] BROUWER, D., AND CLEMENCE, G. M. *Methods of Celestial Mechanics*. Academic Press, New York and London, 1961.

- [3] DELVA, M. Integration of the Elliptic Restricted Three–Body Problem with Lie Series. *Celestial Mechanics* 34, 1–4 (1984), 145–154.
- [4] DEPRIT, A. Canonical Transformations Depending on a Small Parameter. *Celestial Mechanics* 1, 1 (1969), 12–30.
- [5] GOLDSTEIN, H. *Classical Mechanics (Second Edition)*. Addison–Wesley, 1980.
- [6] HORI, G.–I. Theory of General Perturbations with Unspecified Canonical Variables. *Publications of the Astronomical Society of Japan* 18, 4 (1966), 287–296.
- [7] MARCHAL, CH. *The Three–Body Problem*. Elsevier, Amsterdam, 1990.
- [8] SZEBEHELY, V. *Theory of Orbits (The Restricted Problem of Three Bodies)*. Academic Press, New York, 1967.
- [9] ŞELARU, D., AND CUCU–DUMITRESCU, C. An Analytic Asymptotical Solution in the Three–Body Problem. *Romanian Astronomical Journal* 4, 1 (1994), 59–67.
- [10] ŞELARU, D., AND CUCU–DUMITRESCU, C. Infinitesimal Orbits Around Lagrange Points in the Elliptic, Restricted Three–Body Problem. *Celestial Mechanics and Dynamical Astronomy* 61, 4 (1995), 333–346.

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