ON SOME EXTENDED MAXIMUM AND ANTIMAXIMUM PRINCIPLES

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Abstract. We give here new results on eigenvalue problems and on the maximum or the antimaximum principle for some elliptic problems with weights which are either defined on $I\!R^N$ or defined on non smooth domains.

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§1. Recalls: A classical example

Let us first recall some classical results valid for the model case of the Dirichlet Laplacian defined on a smooth bounded domain $\Omega \in R^N$. We assume

(H_1) Ω is a smooth bounded domain in $I\!R^N$.

(H_2) $$f \in L^2(\Omega);\ f(x) > 0\ a.e.\ in\ \Omega.$$

We consider the following Dirichlet boundary problem defined on Ω:

$(E_{a;f})$ $$-\Delta u = au + f\ in\ \Omega;\ u|_{\partial\Omega} = 0.$$

1.1. Eigenvalue Problem

First let us recall some classical results for the associated eigenvalueproblem $(E_{\lambda,0})$:

$(E_{\lambda,0})$ $$-\Delta u = \lambda u\ in\ \Omega;\ u|_{\partial\Omega} = 0.$$

It is well known that there exists an infinite (and countable) number of solutions (eigenpairs) $(\lambda_k;\ \varphi_k)$, $k \in N$, $\|\varphi_k\| = 1$ where $\|.\|$ denotes the L^2 norm and $(.,.)$ the scalar product.

With this normalization, the set of eigenfunctions φ_k is an orthonormal basis of L^2, and

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}; \tag{1}$$

the equality holds in (1) iff $u = c.\varphi_1$.
Also φ_1 does not change sign and we choose

$$\varphi_1(x) > 0, \quad x \in \Omega. \tag{2}$$

Also the second eigenvalue is given by

$$\lambda_2 = \inf_{u \in H_0^1(\Omega); \int u\varphi_1 = 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}. \tag{3}$$

1.2. The Maximum Principle for a Smooth Bounded Domain

We say that $(E_{a,f})$ satisfies the

-*(weak) Maximum Principle* if for $f \geq 0; f \not\equiv 0$ any solution $u \geq 0$.
-*(strong) Maximum Principle* if for $f \geq 0; f \not\equiv 0$ any solution $u > 0$.
-*Hopf Maximum Principle* if for $f \geq 0; f \not\equiv 0$ any solution $u > 0$ and also $\frac{\partial u}{\partial n}|_{\partial\Omega} < 0$, where $\frac{\partial .}{\partial n}|_{\partial\Omega} < 0$ denotes the outward normal derivative.

If (H_1) and (H_2) are satisfied, it is well known that the Hopf Maximum holds for Problem $(E_{a,f})$ iff $a < \lambda_1$. If $a = \lambda_1$, one has the *Fredholm alternative:*
There exists a solution to $(E_{\lambda_1,f})$ iff $\int f\varphi_1 = 0$.

1.3. The antimaximum Principle for a smooth bounded domain

If $\lambda_1 < a < \lambda_2$, $(-\Delta - aI)$ is invertible and hence there exists u solution to $(E_{a,f})$. It has been proved by Clément and Peletier ([ClPe]), in 1979 the *Antimaximum Principle* :

Theorem 1. : *If (H_1) and (H_2) are satisfied and if Ω is smooth enough,*

$$\forall f \in L^2(\Omega), f \geq 0; f \not\equiv 0; \ \exists \delta(f) > 0, \ s.t. \ \forall \lambda_1 < \lambda < \lambda_1 + \delta(f) < \lambda_2, \ \Rightarrow$$

$$u(x) < 0, x \in \Omega; \ \partial u/\partial n|_{\partial\Omega} > 0. \tag{AM'}$$

Several extensions have been done by many authors (see e.g. [H] for problems with indefinite weights and see e.g. [Va] and [FlGoTaTh] for the Dirichlet p-Laplacian). Here we improve some of these results for problem with weights in two directions:
–To problems involving Schrödinger operators on R^2.
–To problems defined on non necessarily smooth domains.

§2. Schrödinger Problems on $\mathbb{R}^2$

We recall first some earlier results of comparison of the solution u with the groundstate φ_1.

2.1. Schrödinger Equations on $I\!R^2$

We consider the equation

$$Lu(x) := (-\Delta + q(x))u(x) = au(x) + f(x),\ x \in I\!R^2, \tag{4}$$

where $q(x) > cst > 0$, and tends to $+\infty$ as $|x| \to \infty$.

Hence $D(L) = \{u \in L^2;\, Lu \in L^2\}$ is compactly embedded in L^2 and L has a discrete spectrum (exactly as the Dirichlet Laplacian). The smallest eigenvalue λ_1 is associated to "the ground-state" $\varphi_1 > 0$. For $a < \lambda_1$, the strong maximum principle is classical (see $e.g.$ [ReSi]): $(L - aI)^{-1}$ "improves positivity" that is $f \in L^2;\ f \geq 0; f \not\equiv 0$ implies $u := (L - aI)^{-1} f > 0$.

This result has been improved for some radial potentials. Assume

$$(H_3) \qquad q : x \to q(|x|) := q(r);\ q(r) = (1 + r^2)^{1+\varepsilon},\ \varepsilon > 0.$$

$f \in L^2(I\!R^2)$, $f \geq 0$, $f > 0$ on an open set with positive measure.

It is shown in [AlTa], that for any $a < \lambda_1$, u solution to (4) is *"φ_1-positive"*, that is, there exists $c(f, a) > 0$ such that

$$u(x) > c(f, a)\varphi_1(x),\ \forall x \in I\!R^2.$$

Now we assume moreover that $f \in X^{1,2}$ with $X^{1,2}$ the Banach space of all the functions $f \in L^2_{\text{loc}}(I\!R^2)$ having the following properties:

$$(\frac{\partial f}{\partial \theta})(r, \bullet) \in L^2(-\pi, \pi) \quad \text{for all } r > 0, \tag{5}$$

and there is a constant $C \geq 0$ such that

$$\|f(r,\theta)\| + \left(\tfrac{1}{2\pi}\int \left|\tfrac{\partial f}{\partial \theta} f(r,\theta)\right|^2 d\theta\right)^{\frac{1}{2}} \leq C u_1(r) \tag{6}$$
$$\text{for almost every } r \geq 0 \text{ and } \theta \in [-\pi, \pi].$$

With these hypotheses, it is shown in [AlFlTa] that there exists $\delta(f) > 0$ such that,

$$\forall a \in]\lambda_1, \lambda_1 + \delta[\subset]\lambda_1, \lambda_2[\exists c_3(f,a)\, s.t.\, u < -c_3\varphi_1;$$

We say that u *is "φ_1-negative".*

Finally, in [AlBe] the constants $\delta(f) > 0,\ c(f, a),\ c_3$ have been computed.

2.2. Some Cooperative Systems of Schrödinger operators

We turn now our attention to cooperative systems as:

$$(S) \begin{cases} Lu_i := -\Delta u_i + q(|x|)u_i = \sum_{j=1}^n a_{ij}u_j + \lambda u_i + f_i & \text{in } \ I\!R^2; \\ i = 1, ..., n \end{cases}$$

We suppose that the potential q is as above, and that the coefficients a_{ij} $(1 \leq i, j \leq n)$ are constants such that $a_{ij} > 0$ for $i \neq j$ (cooperative system). More, we suppose that the matrice $A = (a_{ij})$ has only reals eigenvalues.
Assume that $0 \leq f_i \in L^2(I\!R^2)$ $(1 \leq i \leq n)$.
Denote by $u = (u_1, ..., u_n)$ the weak solution of (S) (when it exists) and denote by Λ_1 the principal eigenvalue of (S) (obtained for $f \equiv 0$). It is associated to $\Phi = (\Phi_1, ..., \Phi_n) > 0$ the "ground state".

Theorem 2. *With the hypotheses and notations above, we have ([Be]):*

(i) if $\lambda < \Lambda_1$, there exists $C = const > 0$ such that $u \geq C\Phi$
(ii) for $f \in Y^n$, with $Y = X^{1,2} \subset L^2(I\!R^2)$, there exists $\delta = \delta(f) > 0$ such that, if $\lambda \in (\Lambda_1, \Lambda_1 + \delta)$, then $u \leq -C'\Phi$, $C' = cst > 0$.

2.3. Schrödinger Equations on $I\!R^2$ with a positive weight

It is also possible to consider the case of a Schrödinger equation with a positive weight (as in [AC]).
Let us consider the following equation

$$Lu = (-\Delta + q)u = \lambda mu + f \text{ in } I\!R^2,$$

where q is a radial positive potential satisfying (H_3) and m is a radially symmetric positive and bounded weight such that $0 < m_1 < m(r) < m_2$ for $r \geq 0$, with m_1 and m_2 two positive constants.
Of course for such a potential there exists a principal eigenpair $(\lambda_1, \varphi_1 > 0)$. Then we obtain the following result:

Theorem 3. *Assume that $u \in \mathcal{D}(L)$, $Lu = \lambda mu + f \in L^2(I\!R^2)$, $\lambda \in I\!R$, and $f \geq 0$ a.e. in $I\!R^2$ with $f > 0$ in some set of positive Lebesgue measure. Then, for every $\lambda \in (-\infty, \lambda_1)$, there exists a constant $c > 0$ (depending upon f and λ) such that*

$$u \geq c\varphi_1 \quad \text{in } I\!R^2.$$

Moreover, if also $f \in X^{1,2}$, then there exists a positive number δ (depending upon f) such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the inequality

$$u \leq -c\varphi_1 \quad \text{in } I\!R^2$$

is valid with a constant $c > 0$ (depending upon f and λ).

§3. Non Smooth Domains

We consider now an eigenvalue problem with indefinite weight defined on "any bounded domain" Ω (that is a domain which is not necessarily smooth); we extend to problems with indefinite weight some of earlier results (valid for positive weights) by Berestycki, Nirenberg, Varadhan ([BNV]).

Assume that the weight function m is such that:

$$m \in L^{\infty}(\Omega)$$

and $|\Omega_+| > 0$, $|\Omega_-| > 0, \ (H_4)$ where $|.|$ denotes the Lebesgue measure and where

$$\Omega_+ := \{x \in \Omega : m(x) > 0\}; \ \Omega_- := \{x \in \Omega : m(x) < 0\}.$$

We consider the following eigenvalue problem :

$$(REVP) \qquad \begin{cases} -\Delta u = \lambda m(x) u \ \ on \ \Omega \\ u \overset{u_0}{=} 0 \ \ on \ \partial\Omega \end{cases}$$

where the "refined boundary conditions" $u \overset{u_0}{=} 0 \ \ on \ \partial\Omega$ is defined in [BNV].
Before going further, let us first recall it.

3.1. Refined Dirichlet Boundary Condition ([BNV])

We do not assume here that $\partial\Omega$ is smooth. The classical Dirichlet boundary condition : $u = 0$ at every point of $\partial\Omega$ is too strong. It has to be replaced by the "refined" ones introduced in Berestycki, Nirenberg, Varadhan ([BNV]).
We introduce first several definitions

Definition 1. ("strong barrier") A point $y \in \partial\Omega$ is said to admit a "strong barrier" if for some ball $B_r(y) = \{|x-y| < r\}$ there is in $B_r(y) \cap \Omega = U$ a positive function $h \in W^{2,n}_{loc}(U)$ satisfying $-\Delta h \geq 1$ which can be extended continuously to the point y with $h(y) = 0$.

Note that every point $y \in \partial\Omega$ where $\partial\Omega$ statisfies an exterior cone condition admits a strong barrier.

We define now as in [BNV] *"the boundary function"* u_0 *associated to the (refined) Dirichlet Laplacian defined on this* Ω. It plays a crucial role.
Let $(H_j)_{j \in I\!N^*}$ be a sequence of open subsets of Ω having smooth boundaries, and such that

$$H_j \subset \overline{H_j} \subset H_{j+1}, \ \ \cup_j H_j = \Omega.$$

Let us denote by $u_j \in W^{2,p}(H_j)$ the solution of the following (classical) Dirichlet boundary value problem:

$$\begin{cases} -\Delta u_j = 1 \ \ on \ H_j \\ u_j = 0 \ \ on \ \partial H_j \end{cases}$$

As $j \to \infty$, the sequence $(u_j) \nearrow u_0$, weakly in $W^{2,p}(K)$, strongly in $C^1(K)$ for any compact set $K \subset \Omega$. Hence

$$-\Delta u_0 = 1 \text{ in } \Omega.$$

Moreover, on the boundary, u_0 can be extended to a continuous function at every point y of $\partial\Omega$ admitting a strong barrier by setting

$$u_0(y) = 0.$$

The boundary-function u_0 defined above is independent of the choice of subsets H_j.

Definition 2. We say that a sequence of points $(x_k)_{k\in I\!N^*}$ in Ω *"tends to the boundary* $\partial\Omega$ $w.r.t.\, u_0$*"*, and we write $x_k \overset{u_0}{\to} \partial\Omega$ if $u_0(x_k) \to 0$.

Definition 3. ("The Refined Dirichlet Boundary Condition") We say that u "vanishes as" u_0 on the boundary $\partial\Omega$ and we write $u \overset{u_0}{=} 0$ on $\partial\Omega$ if, for any sequence $x_k \overset{u_0}{\to} \partial\Omega$, we have $u(x_j) \to 0$.

Example: If B is the unit ball in $I\!R^3$ and if $\Omega = B\backslash\{0\}$, then $u_0(x) = \dfrac{1}{6}(1-|x|^2)$ .Then, in this case, $u \overset{u_0}{=} 0\ on\ \partial\Omega$ if and only if $u = 0\ on\ \partial B$.

3.2. Eigenvalue Problem

Now, we consider the eigenvalue problem

$$(REVP) \qquad \begin{cases} -\Delta u = \lambda m(x) u \ \ on\ \Omega \\ u \overset{u_0}{=} 0 \ \ on\ \partial\Omega \end{cases}$$

For $m \equiv 1$ or $m > 0$, the problem is studied in ([BNV]) and ([Bi]).
We assume here that $m \in C(\overline{\Omega})$, and there is $x \in \Omega$ such that $m(x) > 0$.

Theorem 4. *: There exists a positive function ϕ_1 in $W^{2,p}_{loc}(\Omega)$, $\forall p < \infty$, called a "principal eigenfunction" , and a positive real λ_1, called "principal eigenvalue" satisfying*

$$-\Delta\phi_1 = \lambda_1 m(x)\phi_1,\ \ \phi_1 \overset{u_0}{=} 0\ \ on\ \partial\Omega.$$

The proof of this theorem can be found in [Le].
A similar result is shown in ([FlHeTh]), under the assumptions $m \in L^\infty(\Omega)$, and $\Omega^+ = \{x \in \Omega | m(x) > 0\}$, $\Omega^- = \{x \in \Omega | m(x) < 0\}$ such that their Lebesgue measure $|\Omega^+| > 0$ and $|\Omega^-| > 0$

Consequence: if m changes of sign, there are two principal eigenvalues.

Remark: We know nothing about existence of other eigenvalues.

3.3. The case of systems

The existence of a principal eigenfunction can be extended to the case of some systems; let us consider :

$$(RCSP) \qquad \begin{cases} \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \ \ in\ \Omega \\ u \overset{u_0}{=} 0 \ \ and\ v \overset{u_0}{=} 0 \ \ on\ \partial\Omega \end{cases}$$

The coefficients $a,\ b,\ c,\ d$ are in $L^\infty(\Omega)$. We assume the existence of two positive numbers β and γ such that $b(x) \geq \beta$ and $c(x) \geq \gamma$.

Theorem 5. *: There exists a positive vector* $\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}$ *in* $W^{2,p}_{loc}(\Omega)^2$, $\forall p < \infty$, *called a "principal eigenvector" , and a positive* λ_1, *called "principal eigenvalue" satisfying*

$$\begin{cases} \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} \ \ in \ \Omega \\ \phi_1 \overset{u_0}{=} 0 \ \ and \ \psi_1 \overset{u_0}{=} 0 \ \ on \ \partial\Omega \end{cases}$$

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