

Stability of motion by similarity transformations with a fixed point

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Abstract

In this communication we analyze the nonlinear stability of pseudo-rigid motions constituted by similarity transformations which leave a point of the mechanical systems fixed. For this we apply the energy-momentum method to a class of relative equilibria formed by upright motions of a deformable Lagrangian top modelled by a Saint Venant-Kirchhoff material.

Keywords: nonlinear stability, energy-momentum method, pseudo-rigid body.

AMS Classification: 58F10,70E15.

1 Introduction

The dynamics of deformable three-dimensional bodies can be approached using the theory of pseudo-rigid bodies proposed by Cohen and Muncaster [1] and Sławianowski [2]. In this theory the configuration space of an elastic body, given by the diffeomorphism group, is replaced by the linear group so that the equations of balance are transformed into ordinary differential equations. The finite-dimensional dynamical system corresponding to a pseudo-rigid motion has the structure of a Hamiltonian system with symmetries and for a certain class of motions which leave the rotation axis fixed, the corresponding steady motions and its orbital stability have been studied by Cohen and Muncaster [1], in the case of linear stability, and by Lewis and Simo [3] the nonlinear stability using the energy-momentum method.

Let us consider the most general class of motions formed by affine motions such that a point remains fixed throughout the motion, and study, in particular, the evolution of a deformable asymmetric spinning top where one of the principal axes is vertical and the

angular velocity about that axis increases monotonically. This motion is interesting because a top in unstable motion can be gyroscopically stabilized when the velocity increases beyond certain critical value.

The aim of this communication is to determine a set of values of the angular velocity for which the motion is nonlinearly stable for motions given by similarity transformations with a fixed point. This communication is organized as follows: in the forthcoming section the mechanical system and the reduction of the problem to a submanifold of the phase space is examined. Then, in section 3, the relative equilibria and its nonlinear stability properties are derived by means of the reduced energy-momentum method.

2 Symmetries and reduction

Consider a continuum medium in \mathbb{R}^3 . Let \mathcal{D} be the closure of an open set in \mathbb{R}^3 that represents the reference configuration of the medium at the time $t = 0$. Denote by $X \in \mathcal{D}$ the material points in \mathcal{D} . The instantaneous configuration of \mathcal{D} at an arbitrary time t is a mapping $\varphi_t : \mathcal{D} \rightarrow \mathbb{R}^3$ which is smooth, orientation preserving, and invertible on its image. The spatial points of $\varphi_t(\mathcal{D}) \subset \mathbb{R}^3$ are denoted by $x(t)$. The motion of \mathcal{D} is given by the time dependent family of configurations

$$x(t) = \varphi_t(X). \quad (1)$$

Let us assume that point $O_t = \varphi_t(O)$, for a point $O \in \mathcal{D}$, remains fixed at the origin for all values of time t . Then the motion of \mathcal{D} is

$$x(t) = F(t)X \quad (\text{or simply, } x = FX). \quad (2)$$

The condition that the motion is carried out through similarity transformations means that the matrix representation F of the configuration may be factorized, by the polar decomposition, in the form

$$F = UR, \quad (3)$$

where U is a symmetric matrix proportional to the 3×3 unity matrix, $\mathbf{1}$, and R is an orthogonal matrix, so that the configuration space for similarity motions with a fixed point is the Lie group $Q := \mathbb{R}^+ SO(3)$

$$Q = \{F \mid F = UR, U = u\mathbf{1}, (u \in \mathbb{R}^+), R \in SO(3)\}. \quad (4)$$

The material distribution at the reference configuration is characterized by the matter density $\varrho(X)$ and the Euler tensor

$$\mathfrak{E} := \int_{\mathcal{D}} \varrho(X) X \otimes X dX, \quad (5)$$

which determines a Riemannian metric on the phase space $\mathcal{M} := T^*Q$, whose elements will be denoted by (F, P) , defined by the kinetic energy

$$K(P) := \frac{1}{2} \text{Tr}(P\mathfrak{E}^{-1}P^T).$$

We assume that the external forces derive from a potential function $U : Q \rightarrow \mathbb{R}$:

$$U(F) := V(F) + W(F),$$

where $V(F)$ and $W(F)$ are associated to the gravitational and elastic forces, respectively. The gravitational force when referred to an orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ on \mathbb{R}^3 is given by a vector $\mathbf{g} = g\mathbf{e}_3$, and acts only on the centre of mass, \mathbf{x} , of the continuum medium, the gravitational potential being

$$V(F) = \mathbf{e}_3 \cdot \mathbf{x} = \mathbf{e}_3 \cdot F\mathbf{X}. \quad (6)$$

On the other hand, in order to describe the elastic properties of the continuum medium we consider a Saint Venant-Kirchhoff hyperelastic model for which the potential function is given by

$$W(F) = \frac{1}{2} (\text{Tr}(C - \mathbf{1}))^2 + \mu \text{Tr}(C - \mathbf{1})^2, \quad (7)$$

where $C := F^T F$ is the Cauchy-Green strain tensor and λ, μ are the Lamé coefficients satisfying the conditions

$$\nu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0. \quad (8)$$

We choose this model because it is one of the simplest models that exhibit nonlinear response.

The mechanical system (\mathcal{M}, H) , with Hamiltonian function

$$H : \mathcal{M} \rightarrow \mathbb{R}, \quad H(F, P) = K(P) + U(F), \quad (9)$$

is left invariant under spatial rotations that preserve the gravity vector \mathbf{g} . Therefore, the symmetry group is the abelian group

$$G := \{\exp(s\hat{\mathbf{e}}_3) \mid s \in \mathbb{R}\} \simeq S^1, \quad (10)$$

$\hat{\mathbf{e}}_3$ being the skew-symmetric matrix associated to the vector \mathbf{e}_3 , and its Lie algebra is $\mathfrak{g} = \{s\hat{\mathbf{e}}_3 \mid s \in \mathbb{R}\} \simeq \mathbb{R}$, whose dual space, \mathfrak{g}^* , is also isomorphic to \mathbb{R} .

For every vector $\xi \in \mathfrak{g}$ the infinitesimal generator associated to ξ by means of the G -action on Q is

$$\xi_Q(F) := \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi \hat{\mathbf{e}}_3) \cdot F = \xi \hat{\mathbf{e}}_3 F \in T_F Q, \quad (11)$$

and for the corresponding momentum map,

$$J : \mathcal{M} \rightarrow \mathfrak{g}^*, \quad \langle J(F, P), \xi \rangle = \langle P, \xi_Q(F) \rangle, \quad (12)$$

one obtains

$$J(F, P) = \text{Tr}(FP^T \hat{\mathbf{e}}_3), \quad (13)$$

where the duality pairing is defined in the usual form $\langle A, B \rangle := \text{Tr}(A^T B)$. The constant momentum constrain given by Nöther's theorem states that a trajectory $(F(t), P(t))$ with momentum μ_e stays on the submanifold $J^{-1}(\mu_e) \subset \mathcal{M}$.

Furthermore, the submanifold $J^{-1}(\mu_e)$ can be identified with the set of zero momentum $J^{-1}(0) \subset \mathcal{M}$ through the locked inertia tensor $\mathcal{J}(F)$ defined by means of the following commutative diagram

$$\begin{array}{ccc} \nu \in \mathfrak{g} & \xrightarrow{\Psi_F} & T_q Q \ni \nu_Q(F) \\ \mathcal{J}(F) \downarrow & & \downarrow \mathbb{F}L_F \\ \mathfrak{g}^* & \xleftarrow{J(F, \cdot)} & T_F^* Q \end{array} \quad (14)$$

where Ψ_F is the map that assigns, to each $\xi \in \mathfrak{g}$, the infinitesimal generator $\xi_Q(F)$ on Q given in (11), $\mathbb{F}L$ denotes the Legendre transformation and $J(F, \cdot)$ is defined by (13). Therefore the locked inertia tensor takes the form

$$\mathcal{J}(F) \cdot \xi = \xi \text{Tr}(F \mathfrak{E} F^T \hat{\mathbf{e}}_3^T \hat{\mathbf{e}}_3) \in \mathfrak{g}. \quad (15)$$

Then, for a configuration $F \in Q$ the locked velocity field is defined by the map $\xi : \mathcal{M} \rightarrow \mathfrak{g}$ given by

$$(F, P) \mapsto \xi(F, P) = \mathcal{J}^{-1} J(F, P) = \frac{\text{Tr}(FP^T \hat{\mathbf{e}}_3)}{\text{Tr}(F \mathfrak{E} F^T \hat{\mathbf{e}}_3^T \hat{\mathbf{e}}_3)} \hat{\mathbf{e}}_3, \quad (16)$$

and the corresponding momentum is

$$P_J := \mathbb{F}L(\xi(F, P)_Q(F)) = \frac{\text{Tr}(FP^T \hat{\mathbf{e}}_3)}{\text{Tr}(F \mathfrak{E} F^T \hat{\mathbf{e}}_3^T \hat{\mathbf{e}}_3)} \hat{\mathbf{e}}_3 F \mathfrak{E}. \quad (17)$$

Then, the identification between $J^{-1}(\mu_e)$ and $J^{-1}(0)$ is derived taking into account that for every momentum P one can obtain a corresponding momentum (the shifted momentum) on $J^{-1}(0)$ as $\tilde{P} := P - P_J$, and in the new variables (F, \tilde{P}) on $J^{-1}(0)$ the functional energy-momentum defined as ([4])

$$H_{\mu_e} : \mathcal{M} \times \mathfrak{g} \rightarrow \mathbb{R}, \quad ((F, P), \xi_e) \mapsto H(F, P) - (J(F, P) - \mu_e) \cdot \xi, \quad (18)$$

is reduced to

$$h_{\mu_e} : J^{-1}(0) \rightarrow \mathbb{R}, \quad (F, \tilde{P}) \mapsto \frac{1}{2} \|\tilde{P}\|_{\mathfrak{E}}^2 + V_{\mu_e}, \quad (19)$$

where Smale's amended potential $V_{\mu_e}(F)$ defined on the configuration space takes here the form

$$V_{\mu_e} = U(F) + \frac{1}{2} \mu_e^2 [\text{Tr}(F \mathfrak{E} F^T \hat{\mathbf{e}}_3^T \hat{\mathbf{e}}_3)]^{-1}. \quad (20)$$

The Hamiltonian (9) is written as the sum of the kinetic and potential energy so that the stability analysis for relative equilibria can be restricted from the full phase space to a subset of the configuration space by means of the amended potential (20).

3 Nonlinear stability of relative equilibria

From the relative equilibrium theorem (see [6]) the relative equilibria of the mechanical systems with symmetry (\mathcal{M}, H, G) coincides with the critical points of the function h_{μ_e} given by

$$\tilde{P}_e = 0, \quad \frac{\delta V_{\mu_e}}{\delta F} = 0.$$

Thus, a point $(F_e, P_e) \in \mathcal{M}$ is a relative equilibrium if the conditions

$$P_e = \text{Tr}(F_e P_e^T \hat{\mathbf{e}}) [\text{Tr}(F_e \mathfrak{E} F_e^T \hat{\mathbf{e}}_3^T \hat{\mathbf{e}}_3)]^{-1} \hat{\mathbf{e}}_3 F_e \mathfrak{E}, \quad (21)$$

$$0 = \frac{\delta U}{\delta F} - \mu_e^2 [\text{Tr}(F \mathfrak{E} F^T \hat{\mathbf{e}}_3^T \hat{\mathbf{e}}_3)]^{-2} \hat{\mathbf{e}}_3^T \hat{\mathbf{e}}_3 F_e \mathfrak{E}, \quad (22)$$

are satisfied. From these we get that an equilibrium configuration F_e must satisfy the equation

$$0 = \mathbf{e}_3 \otimes \mathbf{x} + 2\lambda \text{Tr}(F_e^T F_e - \mathbf{1}) F_e + 4\mu F_e (F_e^T F_e - \mathbf{1}) + \xi \hat{\mathbf{e}}_3 \xi \hat{\mathbf{e}}_3 F_e \mathfrak{E}, \quad (23)$$

which is equivalent to the relations

$$\mathbf{e} \cdot \mathbf{x} + 6(3\lambda + 2\mu)(u^2 - 1)u^2 - \xi^2(\alpha_1 + \alpha_2)u^2 = 0, \quad (24)$$

$$(\mathbf{x} - \xi^2 \mathcal{E} \mathbf{e}_3) \times \mathbf{e}_3 = \mathbf{0}, \quad (25)$$

where $\alpha_1, \alpha_2, \alpha_3$ are the eigenvalues of the Euler tensor \mathfrak{E} . The condition (25) is analogous to the Staude condition for the relative equilibrium of rigid tops (see [5]).

For the analysis of the nonlinear stability of relative equilibria we apply the reduced energy-momentum due to Simo *et al.* [4], which synthesizes the classical techniques of Arnol'd and Smale [7]. In this method it is shown that if the quadratic form $D^2 V_{\mu_e}$ is positive definite on certain space \mathcal{V} —the admissible configuration variations—, then the quadratic form $D^2 h_{\mu_e}$ is also positive definite on \mathcal{V} . This leads to sufficient conditions for orbital nonlinear stability in the case of dynamical systems on configuration spaces of finite dimension.

To determine the admissible configuration space we first observe that for the symmetry group (10) the isotropy group for a prefixed values μ_e is given by

$$G_{\mu_e} = \{\varphi \in G \mid \exp(s \hat{\mathbf{e}}_3) \mu_e \exp(-s \hat{\mathbf{e}}_3) = \mu_e\}. \quad (26)$$

Secondly, that the tangent space $T_{F_e}Q$ has the vector basis $\mathcal{B} := \{\mathbf{e}_{iQ}(F_e)\}_{i=1}^4$ given by

$$\mathcal{B} = \{\hat{\mathbf{e}}_1 F_e, \hat{\mathbf{e}}_2 F_e, \hat{\mathbf{e}}_3 F_e, \mathbf{e}_4 F_e\}, \quad (27)$$

where \mathbf{e}_4 represents the unity matrix. And, finally, that the tangent space to the orbit $G_{\mu_e} \cdot F_e$ of the relative equilibrium F_e can be written as

$$\mathfrak{g}_{\mu_e} \cdot F_e \equiv T_{F_e}(G_{\mu_e} \cdot F_e) = \{s\hat{\mathbf{e}}_3 \mid \forall s \in \mathbb{R}\}. \quad (28)$$

Consequently, the space of admissible configuration variations $\mathcal{V} = T_{F_e}(Q/G_{\mu_e})$ defined in the energy-momentum method (see [4]) is the linear manifold

$$\mathcal{V} := \text{span}\{\hat{\mathbf{e}}_1 F_e, \hat{\mathbf{e}}_2 F_e, \mathbf{e}_4 F_e\}. \quad (29)$$

On the other hand, for the second variation of the amended potential (20) at the relative equilibrium we obtain the following bilinear form on $\mathcal{V} \times \mathcal{V}$:

$$\begin{aligned} D^2V_{\mu_e}(\delta F, \Delta F) &= \mathbf{e}_3 \cdot \delta F(F^{-1}\Delta F F^{-1})\mathbf{x} \\ &+ \xi_e^2 [4\mathcal{J}(F)^{-1} \text{Tr}(\mathfrak{E}F^T A \Delta F) \text{Tr}(\mathfrak{E}F^T A \delta F) + \text{Tr}(\mathfrak{E}\Delta F^T A \delta F)] \\ &+ \lambda [4 \text{Tr}(\Delta F^T F) \text{Tr}(\delta F^T F) + 2 \text{Tr}(C - \mathbf{1}) \text{Tr}(\delta F^T \Delta F^T) \\ &\quad + 2 \text{Tr}(C - \mathbf{1}) \text{Tr}(F^T \delta F F^{-1} \Delta F)] \\ &+ 2\mu [\text{Tr}((\Delta F^T F)(\delta F^T F)) + \text{Tr}((C - \mathbf{1})(\delta F^T \Delta F^T))] \\ &+ 2 \text{Tr}[(C - \mathbf{1})F^T \delta F F^{-1} \Delta F], \end{aligned} \quad (30)$$

where $\delta F, \Delta F \in \mathcal{V}$, $\xi_e := \mu_e/\mathcal{J}(F_e)$ is the velocity at the equilibrium and A is the orthogonal projector on the subspace orthogonal to \mathbf{e}_3 .

Now we use (30) to study the orbital stability of the relative equilibria family constituted by configurations for which the vertical vector \mathbf{e}_3 is an eigenvector of the spatial Euler tensor $\mathcal{E} := F_e \mathfrak{E} F_e^T$. In this case, from the equilibrium condition (25) one obtains that the centroid vector is

$$\mathbf{x} = \chi \mathbf{e}_3, \quad (31)$$

where the constant χ is given by the product mgl , m being the mass of the top and l the distance from the origin to the centre of mass.

Let us assume that the initial body frame is constituted by the eigenvectors of \mathcal{E} at $t = 0$, and choose the reference configuration F_e so that this reference frame coincides with the spatial reference $\{\mathbf{e}_i\}_{i=1}^3$. Then the polar decomposition in (4) is reduced to $F_e = U_e$. For this class of relative equilibria the matrix expression of the bilinear form $D^2V_{\mu_e}(\delta F, \Delta F)$ with respect to the basis (27) is

$$D^2V_{\mu_e}(F_e) = \text{diag}(A_1, A_2, A_3). \quad (32)$$

with

$$A_1 := -16\mu u^4 + \xi^2(\alpha_2 - \alpha_3)u^2 - \chi \quad (33)$$

$$A_2 := -16\mu u^4 + \xi^2(\alpha_1 - \alpha_3)u^2 - \chi \quad (34)$$

$$A_3 := 24(3\lambda + 2\mu)u^4 - (\xi^2(\alpha_1 + \alpha_2) + 12(3\lambda + 2\mu))u^2 + \chi. \quad (35)$$

3.1 Conclusion

The vertical relative equilibria (31) of a heavy top moving by similarity transformations with a fixed point and modelled by a Saint Venant-Kirchhoff material are nonlinearly stable (modulo rotations around the axis \mathbf{e}_3) if the rotation is about the axis for which $\alpha_3 < \alpha_1$ (assuming $\alpha_1 < \alpha_2$), and ξ satisfies the conditions $A_2 > 0$, $A_3 > 0$, where $u = u(\xi)$ is given by the implicit relation

$$6(3\lambda + 2\mu)(u^2 - 1)u^2 - \xi^2(\alpha_1 + \alpha_2)u^2 + \chi u = 0. \quad (36)$$

Acknowledgements This work has been supported by project VA014/02 from the Junta de Castilla y León.

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