Monografías del Semin. Matem. García de Galdeano. **26:** 483–490, (2003).

Some results obtained in the study of fault detection from scattered data.

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Abstract

The purpose of this paper is to establish a characterization of jump discontinuities for bivariate functions when they are not explicitly known and only the values at a limited number of points are available.

Keywords: Fault detection, Discontinuities, Explicit surfaces.

AMS Classification: 65D10, 65D15, 65D99

1 Introduction

The approximation of faulted explicit surfaces has received increasing attention in the last few years, due to its application in oil exploration, medical imaging, Geology, Geophysics and other sciences.

Let S be an explicit surface of the form $x_3 = f(x_1, x_2)$, where f is a real function defined over a bounded open subset $\Omega \subset I\!R^2$, and such that f is discontinuous on each point of an unknown subset $\mathcal{F}$ contained in the closure of Ω. The first step in the process of approximation of f is to localize the subset $\mathcal{F}$. For this step, in [3] and [4], an algorithm is given, which is derived from a characterization of jump discontinuities. This characterization uses a ‘‘continuous’’ functional which assumes that f is explicitly known. In practice, this functional needs to be discretized because only a finite subset of scattered data points is available. In this paper, we give a new characterization directly expressed in terms of the discretized functional.

This paper is organized as follows. We recall, in Section 2, some notations and preliminaries. Section 3 is devoted to the study of jump discontinuities. In Subsection 3.1 we present some hypotheses and previous results and, finally, in Subsection 3.2, we prove the characterization of the discontinuities.

2 Notations and preliminaries

Let A be a subset of $I\!R^2$. We denote by $\overline{A}$, ∂A and $\mu(A)$, respectively, the closure, the boundary and the (Lebesgue) measure of A. We write $P_1(A)$ for the space formed by the restrictions to A of all polynomial functions of degree ≤ 1, with respect to the set of variables, defined over $I\!R^2$.

Let ω be an open subset of $I\!R^2$. For any nonempty ordered finite set $T \subset \omega$ which contains at least three non–aligned points and for any function $f : T \to I\!R$, we denote by $\Pi_T f$ the discrete least–squares projection of f in $P_1(\omega)$, i.e. the unique element of $P_1(\omega)$ such that

$$\sum_{x\in T}(f(x) - \Pi_T f(x))^2 = \min_{p\in P_1(\omega)} \sum_{x\in T}(f(x) - p(x))^2.$$

For any two functions $u, v : T \to I\!R$, we write

$$\mu^T_{uv} = M \sum_{x\in T} u(x)v(x) - \sum_{x\in T} u(x) \sum_{x\in T} v(x),$$

M being the cardinal of T. Likewise, we write $\Delta^T_{uv} = \mu^T_{uu}\mu^T_{vv} - (\mu^T_{uv})^2$. It can be shown that T contains three non–aligned points if and only if $\Delta^T_{x_1x_2} > 0$, where we have indentified x_i, with the mapping $x = (x_1, x_2) \to x_i$, $i = 1, 2$. For any $f : T \to I\!R$ and any bounded, closed, rectangle $K \subset I\!R^2$ such that $T \subset K$, let J_T be the functional introduced by Arcangéli and Manzanilla [1], as follows:

$$J_T(f) = \frac{1}{\mu(K)} \int_K \|\nabla(\Pi_T f)\|^2 \, dx \,,$$

where $\| \,.\, \|$ denotes the Euclidean norm in $I\!R^2$. It is readily seen that $\Pi_T f$ can be expressed by $\Pi_T f(x_1, x_2) = \alpha^f_{0T} + \alpha^f_{1T} x_1 + \alpha^f_{2T} x_2$, and that

$$J_T(f) = \left(\alpha^f_{1T}\right)^2 + \left(\alpha^f_{2T}\right)^2 , \tag{1}$$

where

$$\alpha^f_{1T} = \frac{\mu^T_{x_2x_2}\mu^T_{x_1f} - \mu^T_{x_1x_2}\mu^T_{x_2f}}{\Delta^T_{x_1x_2}} , \quad \alpha^f_{2T} = \frac{\mu^T_{x_1x_1}\mu^T_{x_2f} - \mu^T_{x_1x_2}\mu^T_{x_1f}}{\Delta^T_{x_1x_2}} . \tag{2}$$

By an open subset with a Lipschitz–continuous boundary we shall understand a bounded, connected, nonempty, open subset of $I\!R^2$ with a Lipschitz-continuous boundary in the J. Nečas [2] sense. Finally, we recall that, if ω is bounded and f is a Lipschitz–continuous function on ω, there exists a unique continuous extension of f to $\overline{\omega}$, which we shall denote with the same letter f.

Remark 1 *The definition of $J_T(f)$ is analogous to the definition of $J_K(f)$ in [4] by using the continuous least-squares projection instead of the discrete least-squares projection.*

3 Jump discontinuities characterization

3.1 Hypotheses and previous results

Suppose we are given

- a nonempty, bounded, open rectangle $\Omega \subset I\!R^2$ and a point $c = (c_1, c_2) \in \Omega$,
- a function $f : \overline{\Omega} \to I\!R$,
- an ordered finite set $T = \{\, x_i^0 = (x_{1i}^0, x_{2i}^0)\,,\; i = 1, \ldots, M \,\} \subset \overline{\Omega}$ such that $\Delta_{x_1x_2}^T > 0$,
- for any $j = 1, 2$, a real sequence $(r_{jn})_{n\in I\!N}$, such that $r_{j0} = 1$ and

$$\forall n \in I\!N^*,\; r_{jn} > 1\,, \tag{3}$$

$$\lim_{n\to+\infty} P_{jn} = +\infty\,, \tag{4}$$

where, for any $n \in I\!N$, $P_{jn} = \prod_{i=0}^{n} r_{ji}$. Let $(T_n)_{n\in I\!N}$ be the sequence defined by

$$\left.\begin{array}{l} T_0 = T\,, \\ T_n = \{\, x_i^n = (x_{1i}^n, x_{2i}^n) \mid x_i^n = v_n(x_i^{n-1})\,,\; x_i^{n-1} \in T_{n-1} \,\}\,,\; n \geq 1, \end{array}\right| \tag{5}$$

where, for any $n \in I\!N$, v_n denotes the mapping from $\overline{\Omega}$ into $I\!R^2$, defined by

$$v_n(x_1, x_2) = \left(c_1 + \frac{x_1 - c_1}{r_{1n}}\,,\; c_2 + \frac{x_2 - c_2}{r_{2n}} \right). \tag{6}$$

From (3) it follows that, for any $n \in I\!N$, $T_n \subset \overline{\Omega}$.

Proposition 1 *Under hypothesis (3), we have, for any $n \in I\!N$, that $\Delta_{x_1x_2}^{T_n} > 0$.*

Proof. From (5) and (6), it follows that

$$\forall n \in I\!N,\, \forall i = 1, \ldots, M,\; x_i^n = \left(c_1 + \frac{x_{1i}^0 - c_1}{P_{1n}}\,,\; c_2 + \frac{x_{2i}^0 - c_2}{P_{2n}} \right), \tag{7}$$

then, for any $j, k = 1, 2$, $\mu_{x_jx_k}^{T_n} = \mu_{x_jx_k}^{T}/(P_{jn}P_{kn})$, and therefore, for any $n \in I\!N$, we have $\frac{1}{P_{1n}^2 P_{2n}^2}\,\Delta_{x_1x_2}^T > 0$. From which, we obtain the result. □

Proposition 1 implies that the coefficients $\alpha_{jT_n}^f$ of $\Pi_{T_n} f$, $j = 1, 2$, can be obtained as in (2). Then, we can write, for any $n \in I\!N$ and for $j = 1, 2$,

$$\alpha_{jT_n}^f = P_{jn}\, \alpha_{jT}^{g_n}\,, \tag{8}$$

where $g_n : \overline{\Omega} \to I\!R$ is the function defined, for any $(x_1, x_2) \in \overline{\Omega}$, by

$$g_n\,(x_1, x_2) = f\left(c_1 + \frac{x_1 - c_1}{P_{1n}}, c_2 + \frac{x_2 - c_2}{P_{2n}} \right). \tag{9}$$

Then, by (7),

$$\forall n \in I\!N, \forall i = 1, \ldots, M,\; g_n\,(x_i^0) = f(x_i^n)\,. \tag{10}$$

Now, taking into account (2), (8) can be written, for $j = 1, 2$, by

$$\forall n \in I\!N,\ \alpha^f_{jT_n} \ = \ \frac{P_{jn}}{\Delta^T_{x_1x_2}} X_{jn}, \tag{11}$$

where

$$X_{jn} \ = \ \mu^T_{x_kx_k}\, \mu^T_{x_jg_n} \ - \ \mu^T_{x_jx_k}\, \mu^T_{x_kg_n}\,,\ k \in \{1, 2\} \setminus \{j\}\,. \tag{12}$$

Then, we obtain from (1) that

$$\forall n \in I\!N,\ J_{T_n}(f) = (\alpha^f_{1T_n})^2 \ + (\alpha^f_{2T_n})^2\,. \tag{13}$$

We suppose now that there exists a positive constant C_0, such that

$$\forall n \in I\!N,\ \frac{1}{C_0} < \frac{P_{1n}}{P_{2n}} < C_0\,. \tag{14}$$

Theorem 2 *Let $K \subset \overline{\Omega}$ be a bounded rectangle, with sides parallel to the coordinate axes and such that $T \cup \{c\} \subset K$. Suppose that $f|_K$ is a Lipschitz–continuous function with Lipschitz constant L and $(T_n)_{n\in I\!N}$ is a sequence as in (5). Suppose that hypotheses (3) and (14) hold. Then, there exists a constant $C > 0$ such that, for any $n \in I\!N$, $J_{T_n}(f) \le CL^2$.*

PROOF. From $T \cup \{c\} \subset K$, we derive, for all $n \in I\!N$, that $T_n \subset K$. For simplicity, we shall write f and g_n instead of $f|_K$ and $g_n|_K$, respectively, for any $n \in I\!N$. From (9) and the continuity of f, it follows that

$$\forall n \in I\!N,\ \exists d_n \in H_T,\ g_n(d_n) = \frac{1}{M} \sum_{i=1}^{M} g_n(x^0_i)\,, \tag{15}$$

where H_T is the convex hull of T, and obviously

$$\forall i = 1, \ldots, M,\ \left\|x^0_i - d_n\right\| \le \max_{x,y\in T} \|x - y\|\,. \tag{16}$$

On the other hand, for any $n \in I\!N$ and any $j = 1, 2$, we have

$$\mu^T_{x_jg_n} = M \sum_{i=1}^{M} x^0_{ji} g_n(x^0_i) - \sum_{i=1}^{M} x^0_{ji} \sum_{i=1}^{M} g_n(x^0_i)\,. \tag{17}$$

Now, using (15) and (12), we obtain, for any $n \in I\!N$, and any $j = 1, 2$, that

$$X_{jn} \ = \ M \sum_{i=1}^{M} \left(\mu^T_{x_kx_k} x^0_{ji} - \mu^T_{x_jx_k} x^0_{ki} \right) (g_n(x^0_i) - g_n(d_n)),\ k \in \{1, 2\} \setminus \{j\}. \tag{18}$$

Since f is Lipschitz–continuous with constant L, from (9) it follows that g_n is also Lipschitz–continuous on K, with constant $L_n = L/\min\{P_{1n}\,, P_{2n}\}$. Then, from (16) and (18), for any $n \in I\!N$ and any $j = 1, 2$, we have

$$|X_{jn}| \le \ \frac{ML}{P_n} \max_{x,y\in T} \|x - y\| \ \sum_{i=1}^{M} \left| \mu^T_{x_kx_k} x^0_{ji} - \mu^T_{x_jx_k} x^0_{ki} \right|,\ k \in \{1, 2\} \setminus \{j\}.$$

Now, by (11) and (14), it follows that

$$\left|\alpha^f_{jT_n}\right| \le \frac{MLC_0}{\Delta^T_{x_1x_2}} \max_{x,y\in T} \|x-y\| \sum_{i=1}^{M} \left|\mu^T_{x_kx_k} x^0_{ji} - \mu^T_{x_1x_2} x^0_{ki}\right|, \; k \in \{1,2\} \setminus \{j\} .$$

To complete the proof it is sufficient to take (13) into account. □

Let $\omega, \omega_1, \omega_2$ be three open subsets of Ω with a Lipschitz–continuous boundary such that

$$\left.\begin{array}{l} \omega_1 \cap \omega_2 = \emptyset , \\ \overline{\omega}_1 \cup \overline{\omega}_2 = \overline{\omega} , \\ c \in \partial\omega_1 \cap \partial\omega_2 . \end{array}\right| \tag{19}$$

We suppose now that $T \subset \overline{\omega}$ verifies

$$T_{i,0} = T \cap \overline{\omega}_i \neq \emptyset, \; i = 1, 2 , \tag{20}$$

$$T \cap \; \partial\omega_1 \cap \partial\omega_2 = \emptyset , \tag{21}$$

$$b_{T_{1,0}} \neq b_{T_{2,0}} , \tag{22}$$

where $b_{T_{i,0}}$ denotes the barycentre of $T_{i,0}$, for $i = 1, 2$, and that the sequence $(T_n)_{n\in \mathbb{N}}$ given as in (5) verifies

$$\forall n \in \mathbb{N}^*, \; i = 1, 2, \; T_{i,n} = v_n(T_{i,n-1}) \subset \overline{\omega}_i . \tag{23}$$

Let $\mathcal{F}$ be an open (with induced topology) connected nonempty subset of $\omega \cap \partial\omega_1$.

Theorem 3 *Let $c \in \mathcal{F}$ and let $(T_n)_{n\in \mathbb{N}}$ be the sequence defined by (5). Suppose that hypotheses (3), (4), (20)–(23) hold. Assume that f is such that $f_1 = f|_{\omega_1}$ and $f_2 = f|_{\omega_2}$ are Lipschitz–continuous, f is continuous on $\omega \setminus \overline{\mathcal{F}}$ and presents a jump discontinuity on every point of $\mathcal{F}$, so that, for any $x \in \mathcal{F}$, $f_1(x) \neq f_2(x)$. Then, the sequence $(J_{T_n}(f))_{n\in \mathbb{N}}$ is divergent.*

Proof. It can be assumed, without loss of generality, that T is such that $T_{1,0} = \{x^0_i \mid i = 1, \dots, N\}$ and $T_{2,0} = \{x^0_i \mid i = N+1, \dots, M\}$, with $N \in \mathbb{N} \cap (0, M)$. Then, from (7), (9), (4) and (23), and taking into account that, for $j = 1, 2$, f_j is Lipschitz–continuous on ω_j, which is an open set with a Lipschitz–continuous boundary, we derive that

$$\left.\begin{array}{ll} \forall i = 1, \dots, N, & \displaystyle\lim_{n\to+\infty} g_n\left(x^0_i\right) = f_1(c) , \\ \forall i = N+1 \dots M, & \displaystyle\lim_{n\to+\infty} g_n\left(x^0_i\right) = f_2(c) . \end{array}\right| \tag{24}$$

On the other hand, writing in (12) $j = 1$ and $k = 2$, we have

$$\begin{aligned} X_{1n} = \; & \mu^T_{x_2x_2} \left(M \sum_{i=1}^{M} x^0_{1i} g_n(x^0_i) - \left(\sum_{i=1}^{M} x^0_{1i}\right)\left(\sum_{i=1}^{M} g_n(x^0_i)\right)\right) \; - \\ & \mu^T_{x_1x_2} \left(M \sum_{i=1}^{M} x^0_{2i} g_n(x^0_i) - \left(\sum_{i=1}^{M} x^0_{2i}\right)\left(\sum_{i=1}^{M} g_n(x^0_i)\right)\right) . \end{aligned} \tag{25}$$

Separating the terms in which g_n is evaluated in points of T_1 from those in which g_n is evaluated in T_2, taking limits and using (24), it follows that

$$\lim_{n\to+\infty} X_{1n} =$$

$$\mu^T_{x_2x_2}\Bigg(Mf_1(c)\sum_{i=1}^{N} x^0_{1i} + Mf_2(c)\sum_{i=N+1}^{M} x^0_{1i} - Nf_1(c)\sum_{i=1}^{M} x^0_{1i} - (M-N)f_2(c)\sum_{i=1}^{M} x^0_{1i}\Bigg) -$$

$$\mu^T_{x_1x_2}\Bigg(Mf_1(c)\sum_{i=1}^{N} x^0_{2i} + Mf_2(c)\sum_{i=N+1}^{M} x^0_{2i} - Nf_1(c)\sum_{i=1}^{M} x^0_{2i} - (M-N)f_2(c)\sum_{i=1}^{M} x^0_{2i}\Bigg).$$

Then,

$$\lim_{n\to+\infty} X_{1n} = (M-N)\,N\,\Big(f_1(c)-f_2(c)\Big)\Big(\mu^T_{x_2x_2}\,\xi_1 - \mu^T_{x_1x_2}\,\xi_2\Big)\,, \tag{26}$$

where, for $j=1,2\,$,

$$\xi_j = \frac{1}{N}\sum_{i=1}^{N} x^0_{ji} - \frac{1}{M-N}\sum_{i=N+1}^{M} x^0_{ji}\,. \tag{27}$$

Analogously, we obtain

$$\lim_{n\to+\infty} X_{2n} = (M-N)\,N\,\Big(f_1(c)-f_2(c)\Big)\Big(\mu^T_{x_1x_1}\,\xi_2 - \mu^T_{x_1x_2}\,\xi_1\Big)\,. \tag{28}$$

Let us prove, arguing by contradiction, that $\lim_{n\to+\infty}(X_{1n}\,,\,X_{2n}) \neq (0,0)\,$. So, suppose that

$$\lim_{n\to+\infty} X_{1n} = \lim_{n\to+\infty} X_{2n} = 0\,. \tag{29}$$

Obviously $(M-N)\,N \neq 0$ and, taking into account that $c\in\mathcal{F}$, we derive that $f_1(c) - f_2(c) \neq 0\,$. Then, by (26), (28) and (29), we deduce that (ξ_1,ξ_2) is the solution of the homogeneous system

$$\begin{pmatrix} \mu^T_{x_2x_2} & -\mu^T_{x_1x_2} \\ -\mu^T_{x_1x_2} & \mu^T_{x_1x_1} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,,$$

whose coefficient matrix is regular and, in consequence, $\xi_1 = \xi_2 = 0\,$. Then, by (27), it follows that

$$\left(\frac{1}{N}\sum_{i=1}^{N} x^0_{1i}\,,\,\frac{1}{N}\sum_{i=1}^{N} x^0_{2i}\right) = \left(\frac{1}{M-N}\sum_{i=N+1}^{M} x^0_{1i}\,,\,\frac{1}{M-N}\sum_{i=N+1}^{M} x^0_{2i}\right)\,,$$

in contradiction with (22). Therefore, (29) is not true and then, taking into account (8), (11) and (13), the theorem follows. □

Theorem 4 *Suppose that hypothesis (14) and the conditions in Theorem 3 with* $c\in\partial\omega_1\setminus\mathcal{F}$ *hold. Then, the sequence* $(J_{T_n}(f))_{n\in I\!N}$ *is bounded.*

PROOF. We shall prove that the sequence $(\alpha^f_{1T_n})_{n\in I\!N}$ is bounded. For any $n \in I\!N$, taking into account that $f_1(c) = f_2(c)$ and (10), it follows that (25) can be written

$$\begin{aligned} X_{1n} = \ & M \sum_{i=1}^{N} \left(\mu^T_{x_2x_2}(x^0_{1i} - \overline{x}^0_1) - \mu^T_{x_1x_2}(x^0_{2i} - \overline{x}^0_2) \right)(f_1(x^n_i) - f_1(c)) \\ + \ & M \sum_{i=N+1}^{M} \left(\mu^T_{x_2x_2}(x^0_{1i} - \overline{x}^0_1) - \mu^T_{x_1x_2}(x^0_{2i} - \overline{x}^0_2) \right)(f_2(x^n_i) - f_2(c)), \end{aligned}$$

where, for $j = 1, 2$, $\overline{x}^0_j = \sum_{i=1}^{M} x^0_{ji}/M$. Then, since f_j is Lipschitz–continuous on ω_j with constant L_j, for $j = 1, 2$, we have

$$\begin{aligned} |X_{1n}| \leq \ & ML_1 \sum_{i=1}^{N} \left|\mu^T_{x_2x_2}(x^0_{1i} - \overline{x}^0_1) - \mu^T_{x_1x_2}(x^0_{2i} - \overline{x}^0_2)\right| \, \|x^n_i - c\| \\ + \ & ML_2 \sum_{i=N+1}^{M} \left|\mu^T_{x_2x_2}(x^0_{1i} - \overline{x}^0_1) - \mu^T_{x_1x_2}(x^0_{2i} - \overline{x}^0_2)\right| \, \|x^n_i - c\| \end{aligned}$$

and, using (7), we deduce that

$$|X_{1n}| \leq \frac{LM}{P_n} \sum_{i=1}^{M} \left|\mu^T_{x_2x_2}(x^0_{1i} - \overline{x}^0_1) - \mu^T_{x_1x_2}(x^0_{2i} - \overline{x}^0_2)\right| \, \left\|x^0_i - c\right\|,$$

where $P_n = \min\{P_{1n}, P_{2n}\}$ and $L = \max\{L_1, L_2\}$. Now, from (11), using (14), it follows

$$\exists C > 0, \, \forall n \in I\!N, \ \left|\alpha^f_{1T_n}\right| \leq CL\,.$$

Analogously, $(\alpha^f_{2T_n})_{n\in I\!N}$ is bounded. The result is then a consequence of (13). □

3.2 The main result

Suppose, for simplicity, that $\Omega = (a_1, b_1) \times (a_2, b_2) \subset I\!R^2$ with $\mu(\Omega) > 0$.

For $i = 1, 2$, let $k_i \in I\!N$, such that $k_1 + k_2 \neq 0$, for $j = 1, \dots, k_i$, let $R^i_j \subset (a_i, b_i)$ be a nonempty open interval, and let $g^i_j : R^i_j \to I\!R$ be a piecewise–monotone, Lipschitz–continuous function such that $\mathcal{F}^i_j \subset \Omega$, where $\mathcal{F}^i_j = \{(x_1, g^i_j(x_1)) \mid x_1 \in R^i_j\}$, if $i = 1$, and $\mathcal{F}^i_j = \{(g^i_j(x_2), x_2) \mid x_2 \in R^i_j\}$, for $i = 2$. Assume that, for any $(i_1, j_1) \neq (i_2, j_2)$, with $(i_l, j_l) \in \{1, 2\} \times \{1, \dots, k_l\}$, for $l = 1, 2$, $\overline{\mathcal{F}^{i_1}_{j_1}} \cap \overline{\mathcal{F}^{i_2}_{j_2}} = \emptyset$ is verified. Then we shall write $\mathcal{F} = \bigcup_{\substack{j=1,\dots,k_i \\ i=1,2}} \mathcal{F}^i_j$ and $\Omega' = \Omega \setminus \overline{\mathcal{F}}$.

Obviously, for any $x \in \mathcal{F}$, there exists a unique pair (i, j) with $x \in \mathcal{F}^i_j$. Then, for any $x \in \mathcal{F}$, there exists a nonempty, open ball $B^x \subset \Omega$, of centre x, such that the curve $\mathcal{F}^i_j$ passes through x and separates B^x into two connected regions, B^x_+ y B^x_-, which have not any point of $\mathcal{F}$ in their interior.

Let $f : \Omega \to I\!R$. We shall write $f \in L_{\mathcal{F}}(\Omega)$ if, for any open subset $\omega \subset \Omega'$ with a Lipschitz–continuous boundary, $f|_\omega$ is Lipschitz–continuous and, for any $x \in \mathcal{F}$, $\lim\limits_{\substack{z\to x \\ z\in B^x_+}} f(z) \neq \lim\limits_{\substack{z\to x \\ z\in B^x_-}} f(z)$, where B^x_- y B^x_+ are the regions associated with the ball B^x.

From now on, for any $c \in \Omega$, we suppose that an open ball $B^c \subset \Omega$, of centre c, is fixed such that there exists a curve $\gamma = \gamma_c$ which passes through c and separates B^c into two open sets with Lipschitz–continuous boundary, B^c_+ y B^c_-, and also that $\overline{\mathcal{F}} \cap B^c \subset \gamma$.

Theorem 5 *Suppose that $f \in L_{\mathcal{F}}(\Omega)$. Let $c \in \Omega$ and $(T_n)_{n\in I\!\!N}$ be a sequence associated with c as in Theorem 3, writing in (20) and (23), B^c, B^c_+ and B^c_- instead of ω, ω_1 and ω_2, respectively. Suppose that hypothesis (14) holds. Then, the sequence $(J_{T_n}(f))_{n\in I\!\!N}$ is divergent if and only if $c \in \mathcal{F}$.*

PROOF. (i) For any $c \in \Omega$ it follows that, writing B^c, B^c_+ and B^c_- instead of ω, ω_1 and ω_2, respectively, (19) is verified, where now $\partial\omega_1 \cap \partial\omega_2 = \overline{\gamma} \cap \overline{B^c}$. On the other hand, since $f \in L_{\mathcal{F}}(\Omega)$, it follows that the functions $f|_{B^c_+}$ and $f|_{B^c_-}$ are Lipschitz–continuous and, if $\mathcal{F} \cap B^c = \emptyset$, then $f|_{B^c}$ is also Lipschitz–continuous.

(ii) If $c \in \mathcal{F}$, from point (i), we can apply Theorem 3 which implies that the sequence $(J_{T_n}(f))_{n\in I\!\!N}$ is divergent. Conversely, suppose that $(J_{T_n}(f))_{n\in I\!\!N}$ is divergent. We shall prove, arguing by contradiction, that $c \in \mathcal{F}$. In fact, if $c \notin \mathcal{F}$, obviously $c \in \gamma \setminus \mathcal{F}$. Then, if $\mathcal{F} \cap B^c \neq \emptyset$, Theorem 4 proves that $(J_{T_n}(f))_{n\in I\!\!N}$ is bounded and this leads to the contradiction of the divergence of this sequence. Hence, $\mathcal{F} \cap B^c = \emptyset$ and $f|_{B^c}$ is Lipschitz–continuous. In this case, it is readily seen that there exists a closed rectangle, $K \subset B^c$, which contains the point c and, from (7), it is deduced that there exists $n_0 \in I\!\!N$ such that, for any $n \in I\!\!N$, $n \geq n_0$, $T_n \subset K$.

Now, taking into account that $f|_K$ is Lipschitz–continuous and applying Theorem 2, we deduce that $(J_{T_n}(f))_{n\in I\!\!N}$ is bounded in contradiction with the divergence of this sequence. As a consequence, we deduce that $c \in \mathcal{F}$, ending the proof. □

Acknowledgments This work has been partially supported by DGICYT (Project I+D, BFM2000-1058).

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