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From Hamiltonian PDEs to Hamiltonian ODEs through normal forms

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Abstract

We present a method to reduce a polynomial Hamiltonian PDE $\mathcal{H}$ to a Hamiltonian ODE in n degrees of freedom under some conditions that the quadratic part of $\mathcal{H}$ must satisfy. The technique is based on the construction of adequate normal forms for PDEs and generalises the one given by A. Mielke. We apply this procedure to an example of nonlinear PDEs.
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1 Introduction

This work deals with the construction of formal changes of variables for special types of PDEs with the aim of transforming these PDEs into equivalent ODEs up to a certain order of approximation. The approach for the general case (including dissipative and symplectic situations) appears for the first time in [8] and rigorous results will be given in reference [9]. Here we focus on the Hamiltonian context, outlining some specific aspects of the reduction process for Hamilton PDEs.

Specifically, given an n–degree–of–freedom Hamiltonian function, one can introduce formal (and continuous) integrals (also called symmetries or constants of motion) by means of truncated changes of variables. This approach can be extended almost naturally to partial differential equations of Hamiltonian nature. Indeed, after expressing the partial differential equation as a system of infinite ordinary differential equations (Hamiltonian equations with infinite degrees of freedom), it is usually possible to build a normal form transformation such that the number of formal integrals introduced in the process is infinite. In these circumstances the partial differential equation gets reduced to a Hamiltonian differential equation having a finite number of degrees of freedom.

Our technique generalises two methods: on the one hand, the centre manifold reduction procedure for Hamiltonians, which allows to transform some PDEs to ODEs [6]; on the other hand, the Birkhoff normal form setting for Hamiltonian PDEs [10]. Note that these two methods can be readily derived from the formal symmetry approach we propose.

The present work contains theoretical results for Hamiltonian PDEs without justification. The proof of the main theorem we give here will appear elsewhere [9]. The note has three sections. In Section 2 we establish the method to reduce Hamiltonians in infinite dimensions, extending thereafter the procedure to take into account PDEs with dissipative terms. In Section 3 we illustrate our method by applying it to the nonlinear Schrödinger equation, showing how the introduction of infinite asymptotic integrals allows us to transform the equation into a nonlinear ODE.

2 Reduction of Infinite–Dimensional Hamiltonians to Finite–Dimensional Hamiltonians

As each type of Hamiltonian PDE is defined on a different infinite–dimensional phase space, we prefer to present the results in a generic context. Afterwards, we shall particularize for the example of Section 3.

Given two scalar functions $\mathcal{P}$ and $\mathcal{Q}$ depending on infinite coordinates $\mathbf{x} = (x_1, \ldots, x_n, \ldots)$ and their associate moments $\mathbf{X} = (X_1, \ldots, X_n, \ldots)$, we define their infinite Poisson bracket as the formal bilinear operator $\{\,\mathcal{P}\,,\,\mathcal{Q}\,\} = \sum_{i=1}^{\infty} \big(\frac{\partial \mathcal{P}}{\partial x_i}\frac{\partial \mathcal{Q}}{\partial X_i} - \frac{\partial \mathcal{P}}{\partial X_i}\frac{\partial \mathcal{Q}}{\partial x_i}\big)$. Now, we state the following result:

Theorem 2.1 *(Normal form Theorem for Hamiltonian PDEs) Let $\mathcal{H}(\mathbf{x}, \mathbf{X}) = \mathcal{H}_2 + \mathcal{H}_3 + \frac{1}{2}\mathcal{H}_4 + \ldots$ be an infinite–dimensional Hamiltonian in the coordinates $\mathbf{x} = (x_1, \ldots, x_n, \ldots)$ and conjugate moments $\mathbf{X} = (X_1, \ldots, X_n, \ldots)$ coming from a nonlinear PDE, such that, for $i \geq 2$, $\mathcal{H}_i$ is an infinite homogeneous polynomial in $\mathbf{x}$ and $\mathbf{X}$ of degree i with arbitrary real (or complex) coefficients. Let $\mathcal{T}_j(\mathbf{x}, \mathbf{X})$, with $j \geq m+1$ (for some $m \in \mathbf{N}$) be infinite linear–independent integrals of $\mathcal{H}_2$ (i.e., $\{\mathcal{H}_2, \mathcal{T}_j\} = 0$, $\forall j \geq m+1$). Let $\mathcal{K}_i$ be some Hamiltonians in $\mathbf{x}$ and $\mathbf{X}$ satisfying $\{\mathcal{K}_i, \mathcal{T}_j\} = 0$, $\forall j \geq m+1$, $\forall i \geq 2$ and such that the homology equation*

$$\{\mathcal{W}_{i+2}, \mathcal{H}_2\} = \widetilde{\mathcal{H}}_{i+2} - \mathcal{K}_{i+2}$$

has a solution $\mathcal{W}_{i+2}$ for $i \geq 1$. Note that $\widetilde{\mathcal{H}}_{i+2}$ represents terms known from previous orders.

In such circumstances there is a symplectic formal change of variables $(\mathbf{x}, \mathbf{X}) \to (\mathbf{y}, \mathbf{Y})$ which transforms $\mathcal{H}$ into Hamiltonian $\mathcal{K} = \mathcal{K}_2 + \mathcal{K}_3 + \frac{1}{2}\mathcal{K}_4 + \ldots$, where $\mathcal{K}$ defines a Hamiltonian system of m degrees of freedom and $\mathcal{K}_2 = \mathcal{H}_2$. Moreover, it is usually possible to construct explicitly $\mathcal{K}$ and the change of variables up to a certain degree $M \geq 3$.

Proof It will appear in reference [9]. □

We stress that indeed the reduction to a finite (symplectic) manifold is possible since, by virtue of the normal form transformation, we introduce an infinite number of formal integrals (the functions $\mathcal{T}_j$) in the Hamiltonian $\mathcal{K}$ or, in other words, we construct $\mathcal{K}$ in such a way that it has an infinite set of independent integrals. Furthermore, with the aid of the generating function $\mathcal{W} = \mathcal{W}_3 + \mathcal{W}_4 + \ldots$, we are allowed to calculate the formal integrals of the original Hamiltonian, following (at least formally) the same procedure that the one we use for Hamiltonian ODEs, see for instance [7].

We also remark that if A represents the infinite–dimensional matrix associated to the quadratic Hamiltonian $\mathcal{H}_2$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \in \mathbf{C}$ (that is, $A = \mathcal{J}\, B$ with $\mathcal{J}$ an infinite skew–symmetric matrix and B an infinite symmetric matrix with real entries) then, whether the number of eigenvalues with null real part is finite, say $2\,m$, the latter result gives the reduction of the original PDE to the Hamiltonian centre manifold of dimension $2\,m$, see details in [6]. Moreover, if the number of eigenvalues λ_i with positive (negative) real part is finite, Theorem 2.1 represents the reduction to the unstable (stable) manifold. In addition to this, if a finite number of eigenvalues satisfy adequate nonresonant conditions then, it is possible the reduction to other finite–dimensional manifolds, which are a combination of the local centre, stable and unstable manifolds.

If $\mathcal{H}_2$ is an infinite linear combination of oscillators and diffusors, our result represents a Birkhoff normalisation procedure for PDEs, which yields Hamiltonian ODEs. Two references dealing with Birkhoff normal forms for Hamiltonians in $\mathbf{R}^n$ are [2] and [5]. The extension to infinite dimensions is straightforward, at least from a pure formal standpoint. On this occasion, the homology equation of Theorem 2.1 is readily satisfied for polynomials $\mathcal{W}$ and $\mathcal{K}$, due to the specially easy form acquired by the Lie operator $\{\cdot, \mathcal{H}_2\}$.

Let us emphasize that Hamiltonian $\mathcal{K}$ is defined on a phase space whose dimension is $2\,m$. The construction of such phase space must be done extending the results already established for the finite case, see for example the paper by Walcher [11]. The original PDE is defined over an infinite–dimensional phase space, usually a Hilbert space that we call ℓ. Associated to the infinite integrals $\mathcal{T}_j$ there is an infinite–dimensional Abelian Lie subgroup G of $GL(\ell)$ (the Lie group of infinite–dimensional square invertible matrices with real entries). Now we can define a smooth mapping ϱ between ℓ and G (the so–called reduction map), and pass from ℓ to a finite–dimensional space, the orbit space, which is defined through ϱ as the quotient space ℓ/G. Note that since both ℓ and G are infinite–dimensional and we have an infinite number of constants of motion, the variables which are not constants in ℓ/G are related to the m degrees of freedom. Indeed, the orbit space ℓ/G has dimension $2\,m$ and is parameterized by the infinite first integrals constructed from the constants of motion $\mathcal{T}_j$, as those linearly–independent polynomials satisfying $\{\varphi_i, \mathcal{T}_j\} = 0$, $1 \leq i \leq k$ (for some finite k) and $j \leq 2\,m + 1$. The phase space ℓ/G can be a regular hypersurface or can have singularities, depending on the nature of the reduction map.

3 The Nonlinear Schrödinger Equation

Our next task is to apply the latter theory to the nonlinear Schrödinger equation, which represents an important class of complex nonlinear Hamiltonian systems arising in quantum mechanics [1]. The equation

$$\imath\, u_t \;=\; u_{x\,x} - r\,u - f(|u|^2)\,u,$$

is defined on the finite x–interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(t,0) = 0 = u(t,\pi), \qquad -\infty < t < \infty.$$

Symboil $\imath$ stands for the imaginary unity. The parameter r is real and f is real analytic in some neighbourhood of the origin in $\mathbf{C}$. We also assume that $f(0) = 0$ and require f to be nondegenerate, i.e. such that $f'(0) \neq 0$. Furthermore we suppose that the sign of $f'(0)$ is positive. After rescaling u it yields:

$$\imath\, u_t \;=\; u_{x\,x} - r\,u - |u|^2\,u + \mathcal{O}(u^5).$$

Next we follow the paper by Kuksin and Pöschel [4]. As the boundary conditions are of Dirichlet's type we choose the phase space as $W_0^1([0,\pi])$, i.e. the Sobolev space of all complex–valued L^2–functions on the interval $[0,\pi]$ with an L^2–derivative and vanishing boundary values. Next we take the inner product

$$\langle u, v\rangle = \mathrm{Re} \int_0^\pi u\,\bar{v}\,d\,x,$$

and the Hamiltonian

$$H = \tfrac{1}{2}\,\langle A\,u, u\rangle + \tfrac{1}{2}\int_0^\pi g(|u|^2)\,d\,x,$$

where $A = -d^2/d\,x^2 + r$ and g a primitive of f. Then, Equation (3) can be written as

$$\frac{d\,u}{d\,t} = \imath\,\nabla\,H(u),$$

where the gradient of H is taken with respect to $\langle\cdot,\cdot\rangle$.

We rewrite H as a Hamilton function in infinitely many coordinates by making the following ansatz based on standard Fourier series:

$$u = \sum_{j\geq 1} q_j \phi_j, \qquad \phi_j = \sqrt{\tfrac{2}{\pi}}\,\sin\,(j\,x), \quad j \geq 1.$$

The latter coordinates belong to the Hilbert space $\ell^{a,p}$ of all complex–valued sequences $\mathbf{q} = (q_1, \ldots, q_n, \ldots)$ with norm

$$\|\mathbf{q}\|_{a\,p}^2 = \sum_{j\geq 1} |q_j|^2 j^{2\,p} \exp\left(2\,j\,a\right) < \infty,$$

where $a > 0$ and $p > \frac{1}{2}$ will be fixed later. If λ_j, $j \geq 1$, represents the eigenvalues of A, we have that $\lambda_j = j^2 + r$, arriving finally at

$$H \;=\; \tfrac{1}{2} \sum_{j\geq 1} \lambda_j\, |q_j|^2 + \tfrac{1}{2} \int_0^{\pi} g(\sum_{j\geq 1} q_j \phi_j)\, d\,x,$$

on the phase space $\ell^{a,p}$. The equations of motion associated to H are:

$$\frac{dq_j}{dt} \;=\; 2\,\imath\, \frac{\partial H}{\partial \bar{q}_j}, \qquad j \geq 1.$$

The above corresponds to the equations of motion of a Hamiltonian in complex variables q_j which can be split into real and imaginary parts by taking into account that $q_j = x_j + \imath\, X_j$.

The first term of Eq. (3) corresponds to the linear part of H, whereas the second term refers to the nonlinearity. The relevant part of the nonlinear terms is related to $|u|^2\, u$ and can be written as

$$\tfrac{1}{2} \int_0^{\pi} g(\sum_{j\geq 1} q_j \phi_j)\, d\,x \;=\; \tfrac{1}{4} \sum_{j,k,l,m} G_{jklm} q_j\, q_k\, \bar{q}_l\, \bar{q}_m,$$

with

$$G_{jklm} = \int_0^{\pi} \phi_j\, \phi_k\, \phi_l\, \phi_m\, d\,x.$$

It is proven in [4] that $G_{jklm} = 0$ unless $j \pm k \pm l \pm m = 0$, e.g. only a codimension–one set of coefficients is actually different from 0. We can conclude that the above sum extends only over $j \pm k \pm l \pm m = 0$.

The validity of H as a vector field on $\ell^{a,p}$ is analysed in [4] and we do not detail here. Therefore, we can take H as the starting point to perform the reduction process and apply Theorem 2.1.

At this point Kuksin and Pöschel apply Birkhoff normal form theory to H, arriving at an integrable normal form K, which contains terms of degree two and four in q_j. However, we propose another transformation valid for a constant parameter $r < 0$. This constraint implies that some eigenvalues λ_j are negative. More specifically, the number of nonpositive eigenvalues of A is $[\sqrt{-r}]$ (symbol $[\cdot]$ denotes the integer part). In fact, these eigenvalues are all negative excepting for the case $-r = n^2$ for some $n \in \mathbf{N}$. In this latter situation we have $[\sqrt{-r}] - 1$ negative eigenvalues plus one null eigenvalue.

Our goal is to apply Theorem 2.1 so that we reduce Hamiltonian H into a Hamiltonian K with a finite number of degrees of freedom. Concretely, taking $r \in (-m-1, -m)$ with $m \in \mathbf{N}$ we assure that K is a Hamiltonian of dimension $2\,[\sqrt{-r}]$ and we can reduce H to its stable manifold K. The computation will be carried out only to second order (quartic terms) although higher–order Hamiltonians can be also derived.

We define a complex symplectic change of coordinates in order to resolve the homology equation at every order. This is given through:

$$x_j = \tfrac{1}{\sqrt{2}}\,(v_j + \imath\, V_j), \qquad X_j = \tfrac{1}{\sqrt{2}}\,(\imath\, v_j + V_j), \qquad \forall\, j \geq 1.$$

Henceforth, the quadratic part of $H = H_2 + H_4$ reads as:

$$H_2(\mathbf{v}, \mathbf{V}) = \imath \sum_{j\geq 1} \lambda_j\, v_j\, V_j, \qquad H_4(\mathbf{v}, \mathbf{V}) = - \sum_{j\pm k\pm l\pm m=0} G_{jklm} V_j\, V_k\, v_l\, v_m,$$

where $\mathbf{v} = (v_1, \ldots, v_n, \ldots)$ and $\mathbf{V} = (V_1, \ldots, V_n, \ldots)$. In addition to this we have $H_3 = 0$ and $H_j = 0$ for all $j \geq 5$ as we only are interested in a second–order theory, e.g. in terms up to degree $M = 4$.

Note that the terms $T_j = v_j\, V_j$, $j \geq 1$ are the integrals associated to H_2. However, since we are interested in the Hamilton function K as a function of the first $[\sqrt{-r}]$ variables, the normal form we compute is such that T_j with $j \geq [\sqrt{-r}] + 1$ are the new integrals of K.

According to the form of H_2, the homology equation to be solved at each order $i \geq 1$ is:

$$\imath \sum_{j\geq 1} \lambda_j \left(v_j\, \frac{\partial\, W_{i+2}}{\partial\, v_j} - V_j\, \frac{\partial\, W_{i+2}}{\partial\, V_j} \right) = \widetilde{\mathcal{H}}_{i+2} - \mathcal{K}_{i+2}.$$

Now, we can establish the criterion to obtain $\mathcal{K}_{i+2}$ together with its associated generator $\mathcal{W}_{i+2}$ at each order $i \geq 1$. Indeed, given a monomial of degree $i+2$: $p_{i+2}(\mathbf{v}, \mathbf{V}) = \alpha_i\, v_1^{j_1}\, V_1^{k_1}\, v_2^{j_2}\, V_2^{k_2} \ldots v_n^{j_n}\, V_n^{k_n} \ldots$, i.e. $\sum_{n\geq 1}(j_n + k_n) = i+2$, then p_{i+2} belongs to $\ker\{\cdot, \mathcal{H}_2\}$, if and only if the finite sum $s_{i+2} = \sum_{n\geq 1} \lambda_n (j_n - k_n) \equiv 0$. According to the requirements of the normal form transformation we are building up, p_{i+2} is incorporated to $\mathcal{K}_{i+2}$ whenever $\sum_{n\geq [\sqrt{-r}]+1} \lambda_n (j_n - k_n) \equiv 0$, otherwise it goes to the generator as the term $\imath\, p_{i+2}/s_{i+2}$. Notice that due to the form of λ_i, the monomial s_{i+2} cannot vanish.

We start by identifying K_2 with H_2 and making $K_3 = W_3 = 0$. So, for $i = 2$, we need to determine K_4 and W_4. As H_4 is formed by linear combinations of fourth–degree monomials in the moments V_i, thus K_4 has to be:

$$K_4 = - \sum_{\substack{j\pm k\pm l\pm m=0 \\ 1\leq j,k,l,m\leq [\sqrt{-r}]}} G_{jklm} V_j\, V_k\, v_l\, v_m - \sum_{\substack{j\pm k\pm l\pm m=0 \\ j,k,l,m\geq [\sqrt{-r}]+1,\ \{j,k\}=\{l,m\}}} G_{jklm} V_j\, V_k\, v_l\, v_m.$$

Hence, taking into consideration that the antiimage of every monomial of degree four $p_4 = G_{jklm} V_j\, V_k\, v_l\, v_m$ is $\imath\, p_4/s_4$ with $s_4 = -\lambda_j - \lambda_k + \lambda_l + \lambda_m$, we conclude that W_4 yields:

$$W_4 = \imath \sum_{\substack{j\pm k\pm l\pm m=0 \\ j,k,l,m\geq [\sqrt{-r}]+1,\ \{j,k\}\neq\{l,m\}}} \frac{G_{jklm}}{\lambda_j + \lambda_k - \lambda_l - \lambda_m} V_j\, V_k\, v_l\, v_m.$$

We stress that $\lambda_j + \lambda_k - \lambda_l - \lambda_m = j^2 + k^2 - l^2 - m^2 \neq 0$ provided that $\{j, k\} \neq \{l, m\}$ (see reference [4]), therefore small denominators do not occur. For $i = 3$ we have $K_5 = W_5 = 0$ and straightforward computations will determine K_6 and W_6 and the process can be continued to degree six. However we do not pursue this as details will appear in [9]. The convergence of our approach is not discussed, but it follows similar steps to those used in [4].

Our normal form differs from that of [4], as the Hamiltonian we calculate has $[\sqrt{-r}]$ degrees of freedom (corresponding to the terms of K_4, with indexes j, k, l and m not bigger than $[\sqrt{-r}]$) whereas the one determined by Kuksin and Pöschel is of zero degrees of freedom. Hence, the subsequent analysis of $K = K_2 + K_4$ would yield new results which will help to the better understanding of the dynamics of the nonlinear Schrödinger equation. For example, from K one can calculate explicitly up to degree M the k–invariant tori with $1 \leq k \leq [\sqrt{-r}]$ and may use KAM techniques. It is also possible to recover the original variables by using the direct change of coordinates, see [3].

One can also perform a numerical integration of the Hamilton equations associated to $\mathcal{K}$ with a well–suited method for finite–dimensional Hamiltonian systems (perhaps with continuous output), and make use of this to obtain precise expressions of the coordinates of the initial Schrödinger equation in a neighbourhood of the origin. By doing so one has a semianalytical integration of the original PDE.

There are many Hamiltonian PDEs where these results apply: water waves, deformations of beams, inviscid channel flows, diffusion in pipes, etc, see references [1, 6]. The theorems we propose will be demonstrated in [9] using norms in appropriate Hilbert spaces.

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References

[1] R. Abraham and J. E. Marsden. *Foundations of Mechanics.* Addison–Wesley Publishing Company, Redwood City. (1985).

[2] R. C. Churchill, M. Kummer and D. L. Rod. On averaging, reduction, and symmetry in Hamiltonian systems. *Journal of Differential Equations* **vol.**(49), 359–414. (1983).

[3] A. Deprit. Canonical transformations depending on a small parameter. *Celestial Mechanics* **vol.**(1), 12–30. (1969).

[4] S. Kuksin and J. Pöschel. Invariant Cantor manifolds of quasi–periodic oscillations for a nonlinear Schrödinger equation. *Annals of Mathematics* **vol.**(143), 149–179. (1996).

[5] K. R. Meyer and G. R. Hall. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*. Applied Mathematical Sciences **90.** Springer–Verlag, Berlin–New York. (1992).

[6] A. Mielke. *Hamiltonian and Lagrangian Flows on Center Manifolds*. Lecture Notes in Mathematics **1489.** Springer–Verlag, Berlin–Heidelberg. (1991).

[7] J. Palacián and P. Yanguas. Reduction of polynomial Hamiltonians by the construction of formal integrals. *Nonlinearity.* **vol.**(13), 1021–1054. (2000).

[8] J. Palacián and P. Yanguas. Formal reduction of partial differential equations to ordinary differential equations. *Preprint.* (2001).

[9] J. Palacián and P. Yanguas. On the transformation of PDEs into ODEs using formal symmetries. *In preparation.* (2002).

[10] J. Pöschel. Nonlinear partial differential equations, Birkhoff normal forms, and KAM theory. *Progress in Mathematics* **vol.**(169), 167–186. (1998).

[11] S. Walcher. On differential equations in normal form. *Mathematische Annalen* **vol.**(291), 293–314. (1991).