On a critical Hamiltonian system on \( \mathbb{R}^N \).

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Abstract

In this paper, we elucidate how abstract concentration compactness established in [7], can be used in solving variational systems, by giving an application to a problem treated in [6] by concentration compactness of P. L. Lions.

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1 Introduction and main result

Via an abstract concentration compactness approach, we prove the existence of a weak solution of the problem:

\[
(P) : \begin{cases}
-\Delta u = q|v|^{q-2}v \text{ in } \mathbb{R}^N \\
-\Delta v = p|u|^{p-2}u \text{ in } \mathbb{R}^N \\
\lim_{|x|\to\infty} u(x) = 0 \\
\lim_{|x|\to\infty} v(x) = 0
\end{cases}
\]

where \( N \geq 3 \), and \( p, q \) are two real numbers satisfying \( \frac{1}{p} + \frac{1}{q} = \frac{N-2}{N} = \frac{2}{2^*} \).

P.L. Lions has proved in [6], that the corresponding scalar equation \( -\Delta((-\Delta u)^{1/q}) = u^p \), has radial ground states for all values of \( p \) and \( q \), on the critical hyperbola.

In [3], authors have proved uniqueness of solution and its asymptotic behavior.

In this paper, we prove again, the existence result, by combining linking and abstract concentration compactness method.

Note that if \( p = q \), then \( p = q = 2^* \), \( v = u \), and
\[ (p) \iff \begin{cases} -\Delta u = 2^*|u|^{2^*-1} \text{ in } \mathbb{R}^N \\ \lim_{|x| \to \infty} u(x) = 0 \end{cases} \]

A weak solution of the problem \((P)\), is a critical point of the functional \(J\), defined by:

\[ J(u, v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} [|u|^p + |v|^q] \, dx \]

We note by \(\mathcal{H}\), the Banach space \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) equipped with the norm

\[ \|(u, v)\|_{\mathcal{H}} = \|u\|_{H^1(\mathbb{R}^N)} + \|v\|_{H^1(\mathbb{R}^N)}. \]

**Theorem 1.1** \((P)\) has a weak solution in \(\mathcal{H}\), for all positive real numbers \(p\) and \(q\), satisfying

\[ \frac{1}{p} + \frac{1}{q} = \frac{N - 2}{2N} = \frac{2}{2^*}. \]

### 2 Minimax theorem and Plais-Smale sequence

In this section, we establish the linking geometry of \(J\), to give a Palais-Smale sequence by the minimax principle used in [9] and [1].

**Definition 2.1** Let \(S\) be a closed subset of a Banach space \(X\), and \(Q\) a submanifold of \(X\) with relative boundary \(\partial Q\).

We say that \(S\) and \(\partial Q\) link if:

1. \(S \cap \partial Q = \emptyset\).
2. \(\forall h \in C^0(X, X)\) such that \(h_{|\partial Q} = \text{id}\), there holds \(h(Q) \cap S \neq \emptyset\).

**Theorem 2.2** Let \(J : X \to \mathbb{R}\) be a \(C^1\) functional. Consider a closed subset \(S \subset X\), and a submanifold \(Q \subset X\) with relative boundary \(\partial Q\). Suppose:

1. \(S\) and \(\partial Q\) link.
2. \(\exists \delta > 0\) such that

\[ J(z) \geq \delta \quad \forall z \in S, \]
\[ J(z) \leq 0 \quad \forall z \in \partial Q. \]

Let

\[ \Gamma := \{ h \in C^0(X, X) / h_{|\partial Q} = \text{id}\}, \]

and

\[ c := \inf_{h \in \Gamma} \sup_{z \in Q} J(h(z)) \geq \delta. \]
Then there exists a sequence \((z_k)_{k \in \mathbb{N}} \subset X\), such that
\[
J(z_k) \xrightarrow{k \to \infty} c,
\]
\[
J'(z_k) \xrightarrow{k \to \infty} 0.
\]
We choose numbers \(\mu > 1, \nu > 1\), such that \(\frac{1}{p} < \frac{\mu}{\mu + \nu}\), and \(\frac{1}{q} < \frac{\nu}{\mu + \nu}\).

The following propositions give the linking geometry of \(J\). Its proofs are similar to those in [1], and will be omitted.

**Proposition 2.3** There exist \(\rho > 0, \delta > 0\), such that if we define
\[
S := \{(\rho^{\mu - 1} u, \rho^{\nu - 1} u) / \|(u, u)\| = \rho, \ u \in D(\nabla)\}
\]
than \(J(z) \geq \delta \forall z \in S\).

**Proposition 2.4** There exist \(\sigma > 0, M > 0\), such that if we define
\[
Q = \{\tau (\sigma^{\mu - 1} u, \sigma^{\nu - 1} u) + (\sigma^{\mu - 1} v, -\sigma^{\nu - 1} v) / 0 \leq \tau \leq \sigma, \ 0 \leq \|(v, -v)\|_H \leq M, \text{ and } u, v \in D(\nabla)\},
\]
then \(J(z) \leq 0 \forall z \in \partial Q\), where \(\partial Q\) is the boundary of \(Q\) relative to the subspace
\[
\{\tau (\sigma^{\mu - 1} u, \sigma^{\nu - 1} u) + (\sigma^{\mu - 1} v, -\sigma^{\nu - 1} v) / \tau \in \mathbb{R}, \ v \in D(\nabla)\}
\]

### 3 Abstract concentration compactness

In this section, we recall the abstract concentration compactness due to I. Schindler, and K. Tintarev [7], and we give a version adapted to our problem.

Let \(E\) be a separable reflexive Banach space, and let \(G\) be an infinite multiplicative group of bounded linear operator on \(E\).

Let \(u, u_k \in E\). We say that \(u_k\) converges to \(u\) weakly with concentration, and we note \(u_k \xrightarrow{cw} u\), if \(\forall \phi \in E^*\)
\[
\lim_{k \to \infty} \sup_{g \in G} (g(u_k - u), \phi) = 0.
\]

If \(G\) is a compact group, concentrated weak convergence is equivalent to weak convergence.

**Definition 3.1** Let \(\{\phi_k\}\) be a normalized basis for \(E^*\). Then we define the norm
\[
\|u\|_G := \sup_{g \in G} \left(\sum_{k=1}^{\infty} \frac{|(gu, \phi_k)|^2}{2^k}\right)^{\frac{1}{2}}.
\]

We suppose that \(G\) satisfies:

**P1)** \(\sup_{g \in G} \|g\| < \infty\), where \(\|g\| := \sup_{\|u\| = 1} \|gu\|\).
P2) If \((g_k)_{k \in \mathbb{N}} \subset G\), and for all \(u \in \mathcal{E}\) \(g_k u \xrightarrow[k \to \infty]{} g_0 u\), then \(g_0 \in G\).

P3) If \(g_k u \rightharpoonup w \neq 0\) for a \(u \in \mathcal{E}\), then \(g_k\) has a strongly convergent subsequence.

P4) Let \((u_k)_{k \in \mathbb{N}} \subset \mathcal{E}\) be a bounded sequence, and let \(\{w^{(n)}\} \subset \mathcal{E}\) and \(\{g_k^{(n)}\} \subset G\), \(k, n \in \mathbb{N}\), be such that the sequence
\[
(g_k^{(n)} g_k^{(m)^{-1}})_{k \in \mathbb{N}} \text{ dissipates for } m \neq n
\]

and
\[
g_k^{(n)} u_k \rightharpoonup w^{(n)}, \ n \in \mathbb{N},
\]
then \(||w^{(n)}||_G \xrightarrow[n \to \infty]{} 0||\).

Theorem 3.2 Let \((u_k)_{k \in \mathbb{N}} \subset \mathcal{E}\) be a bounded sequence. Then there exist \((w^{(n)})_{n \in \mathbb{N}} \subset \mathcal{E}\), \((g_k^{(n)})_{n \in \mathbb{N}} \subset G\), \(k \in \mathbb{N}\), such that for a renamed subsequence,
\[
g_k^{(n)} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m,
\]
\[
u_k \rightharpoonup \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)} \rightharpoonup 0.
\]

3.1 Concretisation of the abstract concentration compactness on \(\mathcal{H} := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\)

Let \(G\) be the infinite multiplicative group of bounded linear operators defined on \(\mathcal{H}\) by
\[
g_{t, \alpha}(u, v) = \left( t^{-\frac{N}{p}} u \left( \cdot + \frac{\alpha}{t} \right), t^{-\frac{N}{q}} v \left( \cdot + \frac{\alpha}{t} \right) \right) = (g_{1, t, \alpha} u, g_{2, t, \alpha} v)
\]
where \(t \in \mathbb{R}\) and \(\alpha \in \mathbb{R}^N\).

\(G\) satisfies the properties \(P1)\)-\(P4)\). See [7] for a proof.

We have the invariance \(J(g_{t, \alpha}(u, v)) = J(u, v)\).

The following theorem is a corollary of the Theorem 1.7 of [7].

Theorem 3.3 Let \((z_k = (u_k, v_k))_k\) be a bounded sequence in \(\mathcal{H}\). Then there exist \(w^{(1)}, w^{(2)}, \ldots \in \mathcal{H}\), and \((\alpha_k^{(1)}, \tau_k^{(1)}), (\alpha_k^{(2)}, \tau_k^{(2)}), \ldots \in \mathbb{R}^N \times \mathbb{R}^+\), such that for \(r \neq m\), either \(\tau_k^{(r)} / \tau_k^{(m)} \to \infty\), or \(\tau_k^{(r)} / \tau_k^{(m)} \to 0\), or \(|\alpha_k^{(r)} - \alpha_k^{(m)}| \to \infty\); and \(\forall r : \tau_k^{(r)} \to \infty\), or \(\tau_k^{(r)} \to 0\); where
\[
w^{(n)} = w - \lim_{k \to \infty} g_{\tau_k^{(m)}, -\alpha_k^{(m)}} z_k
\]
The series \(\sum_n g_{\tau_k^{(m)}, \alpha_k^{(m)}} w^{(n)}\) converges absolutely in \(\mathcal{H}\), and on a renamed subsequence:
\[
z_k - \sum_n g_{\tau_k^{(m)}, \alpha_k^{(m)}} w^{(n)} \rightharpoonup 0.
\]
Lemma 3.4 Let \((z_k)_k\) be a bounded sequence in \(\mathcal{H}\) such that \(z_k \overset{cw}{\rightarrow} 0\).

Then modulo a subsequence, \(\lim_{k \rightarrow +\infty} \|z_k\|_{L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)} = 0\), for all positive real numbers \(p\) and \(q\), satisfying \(\frac{1}{p} + \frac{1}{q} = \frac{N}{N-2} = 2^*\).

Proof of Lemma 3.4: Let \(z_k := (u_k, v_k) \in \mathcal{H}\).

\[\frac{1}{p} + \frac{1}{q} = 2^* \implies \left(\frac{N}{N-2} < p \leq 2^* \text{ and } q \geq 2^*\right) \text{ or } \left(p \geq 2^* \text{ and } \frac{N}{N-2} < q \leq 2^*\right)\]

Suppose that \(\frac{N}{N-2} < p < 2^* \text{ and } q \geq 2^*\).

Note that \(z_k \overset{cw}{\rightarrow} 0 \implies \forall g \in G: gz_k \rightarrow 0\).

Let \(g = g_{t_k,0}\), where \(t_k \overset{k \rightarrow +\infty}{\rightarrow} +\infty\) is chosen such that \(\int |v_k|^{q} \overset{k \rightarrow +\infty}{\rightarrow} 0\);

and let \(w_k = g_{t_k,0}^2 v_k = t_k^{-\frac{N}{q}} v_k(t_k^{-1})\), i.e \(v_k(x) = t_k^{-\frac{N}{q}} w_k(t_k x)\).

\[\int_{|v_k| > t_k^\frac{N}{q}} |v_k|^q \overset{k \rightarrow +\infty}{\rightarrow} 0;\]

Hence \(\|v_k\|_{L^q(\mathbb{R}^N)} \overset{k \rightarrow +\infty}{\rightarrow} 0\).

Let now \(\{B(y,1), \ y \in Z \subset \mathbb{R}^N\}\) be a cover of \(\mathbb{R}^N\), and \(g = g_{1,-y}\).

By Rellich-Kondrachov inequality, we obtain:

\[\|u_k\|_{L^p(B(y,1))} \leq C\|u_k\|_{H^1_0(B(y,1))} \|u_k\|_{L^p(B(y,1))}^{p-2}\]  \(\text{(3.1)}\)

By summing inequalities (3.1) over \(y \in Z\), we obtain:

\[\|u_k\|_{L^p(\mathbb{R}^N)} \leq C\|u_k\|_{H^1(\mathbb{R}^N)} \sup_{y \in Z} \|g_{1,-y} u_k\|_{L^p(B(0,1))}^{p-2}\]

By the compactness of the imbedding of \(H^1_0(B(0,1))\) into \(L^p(B(0,1))\), it follows that modulo a subsequence, \(g_{1,-y} u_k \overset{k \rightarrow +\infty}{\rightarrow} 0\) in \(L^p(B(0,1))\).

Hence, \(\|u_k\|_{L^p(\mathbb{R}^N)} \overset{k \rightarrow +\infty}{\rightarrow} 0\).
4 Proof of the main result

Lemma 4.1 Let \((z_k = (u_k, v_k))_{k \in \mathbb{N}}\) be a sequence of \(\mathcal{H}\) such that
\[
J(z_k) \xrightarrow{k \to \infty} c \quad \text{and} \quad J'(z_k) \xrightarrow{k \to \infty} 0.
\] (4.1)
Then \((z_k)_{k \in \mathbb{N}}\) is bounded.

The proof is similar to that of Proposition 2.1 in [1].

Proof of Theorem 1.1. Let \((z_k)\) be a sequence satisfying (4.1).
According to Theorem 3.3, there exist \(w^{(1)}, w^{(2)}, \cdots \in \mathcal{H}\), and \((\alpha_k^{(1)}, t_k^{(1)}),
(\alpha_k^{(2)}, t_k^{(2)}), \cdots \in \mathbb{R}^N \times \mathbb{R}^+_\ast\), such that
\[
z_k - \sum_n g_k^{(n)} w^{(n)} \xrightarrow{w} 0,
\]
where \(g_k^{(n)} = g_{t_k^{(n)}}^{(n)} \alpha_k^{(n)}\).
\((z_k)\) does not converge weakly with concentration to 0. In fact, if we suppose that \(z_k \xrightarrow{w} 0\)
we will have by Lemma 3.4 \(\lim_{k \to \infty} \|z_k\|_{L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)} = 0\) (modulo a subsequence), which
shows that \(J(z_k) \xrightarrow{k \to \infty} 0\). Contradiction. Then there exists a \(w^{(n_0)} \neq 0\).
On the other hand, for some \(g_k \in G\), we have
\[
g_k z_k \xrightarrow{w} w^{(n_0)}.
\]
Then
\[
J'(g_k z_k) \xrightarrow{w} J'(w^{(n_0)})
\]
However, \(J'(z_k) \xrightarrow{k \to \infty} 0 \implies J'(g_k z_k) \xrightarrow{k \to \infty} 0\). Then, \(J'(w^{(n_0)}) = 0\).

\[\square\]

References


