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On a nonlinear hyperbolic-parabolic equation

G. Gagneux and E. Rouvre

Laboratoire de Mathématiques Appliquées
Université de Pau et des Pays de l'Adour
BP 1155, 64013 Pau Cedex, France.

Abstract

We are concerned with existence, uniqueness and stability of a strong entropy solution for a Cauchy-Dirichlet problem, associated to the following scalar conservation law $u_t - \Delta\varphi(u) - div(\nu(u)\mathbf{G}) = 0$ on a bounded domain of $\mathbb{R}^{n+1}$. The problem particularity is that φ is a nondecreasing function, constant on an interval $[0, \varphi_c]$. Therefore, we study a free boundary problem which degenerates, for some solution values, into a nonlinear hyperbolic problem with boundary conditions.

Keywords: entropy solutions, degenerate parabolic equations, free boundaries
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1 Introduction

In this paper, we are interested in studying nonlinear conservation laws which present the particularity that the diffusion phenomenon is inhibited under critical solution value. When the solution is less than a critical value noted φ_c, the diffusion term vanishes and the parabolic equation degenerates into a first order nonlinear hyperbolic equation in *a priori* unknown parts of the domain. This kind of problems with free boundaries (separating parabolic and hyperbolic areas) is expressed by the following formulation:

$$u_t - \Delta\varphi(u) - div(\nu(u)\mathbf{G}) = 0 \text{ in } Q = (0,T) \times \Omega \tag{1}$$

$$\left.\begin{array}{l} \varphi(u) = 0 \text{ on } \Sigma = (0,T) \times \partial\Omega \\ \text{linked with an appropriate boundary condition on } \Sigma^+ = (0,T) \\ \times \{x \in \partial\Omega,\ \mathbf{G}.\mathbf{n}(x) > 0\} \text{ if the need arises (first-order behaviour)} \end{array}\right\} \tag{2}$$

$$u(0,.) = u_0 \text{ in } \Omega, \tag{3}$$

where φ is a nondecreasing function such that $\varphi' \equiv 0$ on $[0, \varphi_c]$; that means that the points set where φ' vanishes has a Lebesgue measure different from 0.

Thus, it concerns strong degenerate parabolic equations such that, when the solution value is always less than the critical value φ_c (for example $\varphi_c = 1$), we find the classical results of C. Bardos, A.Y. Leroux and J.C. Nédélec [1] concerning first order nonlinear hyperbolic equations, as opposed to the weak formulation of F. Otto [5]. Due to the nonlinear hyperbolic character of the problem we could not find approximately continuous solutions (we will work in the space of functions of bounded variation). Moreover, with intent to select through all the weak solutions the most physically acceptable of a well-posed problem, we will have to define an entropy condition, suited to the diffusion second order term. The aim of this paper is to develop the results established in [6], in other words, to prove existence and uniqueness of a strong entropy solution in the multidimensional case.

2 Statement of the problem

Let Ω be a bounded open set of $\mathbb{R}^n$, whose boundary Γ belongs to $W^{2,\infty}$ (we will see later that this regularity is needed to give realistic meaning to the outflow boundary condition). We are concerned with the following problem:
find u *a priori* belonging to $L^\infty(Q)$, $0 \leq u \leq 1$, $\mathcal{L}^{n+1} - a.e.$ in Q and satisfying (1)-(3), where $\mathbf{G}$ denotes a constant field of $\mathbb{R}^n$, ν is a Lipschitz increasing function on $[0,1]$, and φ is a C^1-continuous function vanishing on a values range $[0, \varphi_c]$, ($\varphi_c \in (0,1)$) and is strictly increasing on $(\varphi_c, 1]$. At last, the initial datum u_0 is chosen in $L^\infty(\Omega) \cap BV(\Omega)$, and we assume the following assumptions hold:

$$(\mathcal{H}_1) \quad 0 \leq u_0 \leq 1 \;\; \mathcal{L}^n - a.e \text{ in } \Omega, \quad (\mathcal{H}_2) \quad \varphi(u_0) \in H_0^1(\Omega),\; \Delta\varphi(u_0) \in \mathcal{M}_b(\Omega),$$

$$(\mathcal{H}_3) \quad \begin{cases} \Omega_+ = \{x \in \Omega,\; \varphi(u_0(x)) > 0\} \text{ is an open set,} \\ \Omega_0 = \{x \in \Omega,\; \varphi(u_0(x)) = 0\} \text{ is nonempty, } \partial\Omega_0 \text{ belongs to } W^{2,\infty}, \end{cases}$$

where $\mathcal{M}_b(\Omega)$ denotes the space of Radon measures on Ω.

3 Entropy solution

3.1 Definition of an entropy solution

As opposed to the weak formulation of J. Carrillo [2], the elaboration of an entropy formulation associated to problem (1)-(3), requires here the normal derivative definition of $\varphi(u)$ along $\Sigma = (0,T) \times \Gamma$. So, the following definition will find a sense in the future entropy condition.

Definition 1 *Let f be a function belonging to $L^2(0,T; H^1(\Omega))$ such that $\Delta f \in \mathcal{M}_b(Q)$. The normal derivative of f along Σ is given by the integration by parts formula*

$$\forall v \in \mathcal{C}(\overline{Q}) \cap L^2(0,T;H^1(\Omega)),\ \left\langle\left\langle \frac{\partial f}{\partial n}, v \right\rangle\right\rangle = \int_Q \nabla f.\nabla v\, dx\, dt + \int_Q v d[\Delta f].$$

Inasmuch as the definition 1, we can now give the following definition, adapted from the S.N. Kruzhkov's formulation:

Definition 2 *A function u is called entropy solution of the problem (1)-(3), if*

$$0 \leq u \leq 1 \quad \mathcal{L}^{n+1} - a.e.\ in\ Q, \quad u \in L^\infty(Q) \cap BV(Q),$$

and if it satisfies the following integral inequalities:
$\forall k \in [0,1],\ \forall \Phi \in \mathcal{D}([0,T) \times \overline{\Omega}),\ \Phi \geq 0,$

$$\begin{aligned} &\int_Q |u-k| \frac{\partial \Phi}{\partial t}\, dx\, dt + \int_\Omega |u_0 - k|\, \Phi(0,x)\, dx \\ - &\int_Q |\nu(u) - \nu(k)|\, \mathbf{G}.\nabla\Phi\, dx\, dt - \int_Q \nabla\, |\varphi(u) - \varphi(k)|\, .\nabla\Phi\, dx\, dt \\ - &\ sign(k) \int_\Sigma (\nu(u) - \nu(k))\mathbf{G}.\mathbf{n}\Phi d\mathcal{H}^{n-1} dt - sign(k) \left\langle\left\langle \frac{\partial \varphi(u)}{\partial n}, \Phi \right\rangle\right\rangle \geq 0. \end{aligned}$$

3.2 Existence of an entropy solution

Theorem 1 *The problem (1)-(3) admits an entropy solution u, in the sense of the definition 2, which satisfies*

$$\begin{gathered} 0 \leq u \leq 1\ \mathcal{L}^{n+1} - a.e.\ in\ Q,\ u(0,.) = u_0\ in\ \Omega \\ u \in L^\infty(Q) \cap BV(Q) \cap Lip([0,T];L^1(\Omega)),\ \frac{\partial u}{\partial t} \in L^2(0,T;H^{-1}(\Omega)), \\ \varphi(u) \in H^1(Q) \cap L^\infty(0,T;H_0^1(\Omega)),\ \Delta\varphi(u) \in \mathcal{M}_b(Q). \end{gathered}$$

proof: We give here the general idea of the proof.
Firstly, we use the vanishing viscosity method, which consists in defining $\varphi_\varepsilon = \varphi + \varepsilon Id$, a bilipschitz continuous increasing function. Then, we show existence of the unique solution u_ε of the problem whose general equation is: $\dfrac{\partial u_\varepsilon}{\partial t} - \Delta\varphi_\varepsilon(u_\varepsilon) - div(\nu(u_\varepsilon)\mathbf{G}) = 0.$
Secondly, we establish the following *a priori* estimates: threre exists a constant C only depending on the boundary geometry such that:

$$\begin{gathered} f.a.a.\, t \in (0,T),\ \left| \frac{\partial u_\varepsilon}{\partial t}(t,.) \right|_{L^1(\Omega)} \leq |\Delta\varphi_\varepsilon(u_0^\varepsilon)|(\Omega) + |\mathrm{div}(\nu(u_0^\varepsilon)\mathbf{G})|(\Omega) \leq C, \\ \text{for each } t \in [0,T],\ \ |\nabla u_\varepsilon(t)|_{(L^1(\Omega))^n} \leq C, \\ \|\varphi(u_\varepsilon)\|_{L^\infty(0,T;H_0^1(\Omega))} + \left\| \frac{\partial \varphi(u_\varepsilon)}{\partial t} \right\|_{L^2(Q)} + \sqrt{\varepsilon}\, \|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C, \end{gathered}$$

where $|\mu|(\Omega)$ denotes the total variation on Ω of the Radon measure μ.
Then, by using Ascoli's theorem, we show existence of an element u in $BV(Q) \cap L^\infty(Q)$, limit in $\mathcal{C}([0,T], L^1(\Omega))$ of the sequence $(u_\varepsilon)_\varepsilon$, which satifies the entropy inequalities given in definition 2. ■

Proposition 1 *We can show that* $|\varphi(u)-\varphi(k)| \in H^1(Q)$ *and* $\Delta|\varphi(u)-\varphi(k)| \in \mathcal{M}_b(Q)$*; thus, it is possible to define its normal derivative in the sense of the definition 1.*
Moreover, the function u *satisfies* <u>*formally*</u> *the entropy boundary condition,* $\forall k \in [0,1]$,

$$(sign(u-k)+sign(u))(\nu(u)-\nu(k))\mathbf{G}.\mathbf{n}+\left[sign(k)\frac{\partial\varphi(u)}{\partial n}+\frac{\partial|\varphi(u)-\varphi(k)|}{\partial n}\right]\leq 0. \quad (4)$$

Remark 1 *To obtain the uniqueness result, we want to take advantage of the strong framework by exploiting pieces of information of the boundary condition. Consequently, we set out to define inequality (4) in a rigorous way.*

4 Strong entropy solutions

In this section, we show that any solution of (1)-(3) satisfies some remarkable properties concerning the total variation of the function $u(t)$ and the measure $\Delta\varphi(u(t))$ ($f.a.a.\ t \in (0,T)$), which allow us to define, in a rigorous way, the normal derivative of $\varphi(u)$ along Σ.

Proposition 2 *For all* t *of* $[0,T]$, $u(t)$ *belongs to* $BV(\Omega)\cap L^\infty(\Omega)$.
Moreover, there exists a constant $C>0$ *only depending on the functions* φ *and* ν*, on the initial datum* u_0 *and on the boundary* Γ *geometry, such that*

$$\forall t \in [0,T], \quad TV_\Omega(u(t)) \leq C.$$

proof: We know that u is the limit in $\mathcal{C}([0,T],L^1(\Omega))$ of the sequence $(u_\varepsilon)_\varepsilon$, *i.e.* $\forall t \in [0,T]$, $(u_\varepsilon(t))_\varepsilon$ converges to $u(t)$ with respect to the L^1-norm. It follows that $\forall t \in [0,T]$, $u(t)$ belongs to $BV(\Omega)$. Furthermore, by the classical result of lower semi-continuity of variation measure, we have for all t of $[0,T]$

$$TV_\Omega(u(t)) \leq \liminf TV_\Omega(u_\varepsilon(t)).$$

Using *a priori* estimates on u_ε, we show that there exists a constant C, independent of ε such that $\forall t \in [0,T]$: $TV_\Omega(u_\varepsilon(t)) \leq C+O(\varepsilon)$. Then, when ε tends to 0, we obtain the result. ■

Lemma 1 *Let* u *be an entropy solution of problem (1)-(3); then*

$$f.a.a.\ t \in (0,T), \quad \Delta\varphi(u(t)) \in \mathcal{M}_b(\Omega).$$

proof: The application $\varphi(u)$ defined on $[0,T]$ into $H_0^1(\Omega)$ is continuous, $H_0^1(\Omega)$ being provided with the weak topology and the operator $(-\Delta)$ is weakly-weakly continuous on $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Thus, for each t of $[0,T]$, $-\Delta\varphi(u(t))$ has a sense and defines an

element of $H^{-1}(\Omega)$. In the same way, since operator $\dfrac{d}{dx_i} : L^2(\Omega) \to H^{-1}(\Omega)$ is continuous and since $\forall t \in [0,T],\ \nu(u(t)) \in L^2(\Omega)$, we deduce that $div(\nu(u(t))\mathbf{G}) \in H^{-1}(\Omega)$.
The fact that u is a weak solution of $(1)-(3)$ yields

$$f.a.a.\ t \in (0,T),\quad \frac{\partial u}{\partial t} = \Delta\varphi(u(t)) + div(\nu(u(t))\mathbf{G})) \text{ in } H^{-1}(\Omega). \tag{5}$$

Moreover, $\forall t \in [0,T]$, $u(t)$ is a function of bounded variation on Ω. Thus, $\forall t \in [0,T]$, $div(\nu(u(t))\mathbf{G})$ is a Radon measure on Ω and we show that for almost all t of $(0,T)$, $\dfrac{\partial u(t)}{\partial t} \in \mathcal{M}_b(\Omega)$. Indeed, u belongs to $Lip([0,T]; L^1(\Omega))$ with a Lipschitz constant L. Thus, t being fixed in $(0,T)$, for each h sufficiently small, the solution u satisfies

$$\left| \frac{u(t+h)-u(t)}{h} \right|_{L^1(\Omega)} \leq L.$$

There exists $\mu_t \in \mathcal{M}_b(\Omega)$ such that, up to a subsequence, $\left(\dfrac{u(t+h'_t)-u(t)}{h'_t}\right)_{h'_t>0}$ converges to μ_t in the sense of measures on Ω.
As $\dfrac{\partial u}{\partial t}$ belongs to $L^2(0,T;H^{-1}(\Omega))$, we deduce that $\forall t \in [0,T] \setminus Z$ (where $\mathcal{L}^1(Z)=0$),

$$\forall w \in H_0^1(\Omega),\quad \lim_{h'_t \to 0} \left\langle \frac{u(t+h'_t)-u(t)}{h'_t}, w \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \left\langle \frac{\partial u(t)}{\partial t}, w \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

It follows that, for almost all t of $(0,T)$, $\dfrac{\partial u(t)}{\partial t} = \mu_t$ which is a Radon measure on Ω.
From inequality (5), we obtain: $f.a.a.\ t \in (0,T),\quad \Delta\varphi(u(t)) \in \mathcal{M}_b(\Omega)$. ■

Proposition 3 *Since Ω is a bounded open set of $\mathbb{R}^n$ whose boundary belongs to $W^{2,\infty}$, and since $f.a.a.\ t \in (0,T)$, $\Delta\varphi(u(t)) \in \mathcal{M}_b(\Omega)$, we can show that the normal derivative $\dfrac{\partial\varphi(u)}{\partial n}$ is well-defined on Σ in the following sense:*

$$\frac{\partial\varphi(u)}{\partial n} \in L^\infty(0,T;L^1(\Gamma)).$$

Moreover, we have the following formula of Gauss-Green type, $f.a.a.\ t \in (0,T)$:

$$\forall v \in \mathcal{C}^1(\overline{\Omega}),\quad \int_\Gamma v \frac{\partial\varphi(u(t))}{\partial n}\, d\mathcal{H}^{n-1} = \int_Q vd\left[\Delta\varphi(u(t))\right] + \int_Q \nabla v.\nabla\varphi(u(t))\, dx. \tag{6}$$

proof: A result given by Ph. Bénilan in [3] (proposition 9, p.580) which uses the lemma 1 result allows us to show immediately $f.a.a.\ t \in (0,T),\quad \dfrac{\partial\varphi(u)(t)}{\partial n} \in L^1(\Gamma)$.
Indeed, if Ω is an open set whose boundary belongs to $W^{2,\infty}$, then for each U of $W_0^{1,1}(\Omega)$ such that $\Delta U = \mu$ (where μ is Radon measure on Ω), we have $\dfrac{\partial U}{\partial n} \in L^1(\Gamma)$. ■

Remark 2 *Now the entropy boundary condition given in (4) is well-defined. Such solutions are called strong entropy solutions of which we recall the essential properties:*

$$\begin{cases} \forall t \in [0,T],\ u(t) \in BV(\Omega) \cap L^\infty(\Omega),\ \dfrac{\partial u}{\partial t} \in L^2(0,T;H^{-1}(\Omega)), \\ \varphi(u) \in H^1(Q) \cap L^2(0,T;H_0^1(\Omega)),\ \Delta\varphi(u) \in L^\infty(0,T;\mathcal{M}_b(\Omega)),\ \dfrac{\partial\varphi(u)}{\partial n} \in L^\infty(0,T;L^1(\Gamma)). \end{cases}$$

The uniqueness result essentially relies on a comparison theorem which is a Carrillo's extension to the second-order equation of the usual hyperbolic method of doubling the variables (*cf.* J. Carrillo [2]); it is a comparison result of Kato's inequality type.

Theorem 2 *Let u_1 and u_2 be two entropy solutions of (1)-(3) respectively associated to initial data u_0^1 and u_0^2 satisfying assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ et $(\mathcal{H}_3)$.*
For all nonnegative element ξ of $\mathcal{D}(Q)$ (and $\forall \xi \in H_0^1(Q)$, $\xi \geq 0$ by density), we have

$$\int_Q \nabla \left|\varphi(u_1) - \varphi(u_2)\right| . \nabla \xi \, dx \, dt + \int_Q \left|\nu(u_1) - \nu(u_2)\right| \mathbf{G} . \nabla \xi \, dx \, dt - \int_Q |u_1 - u_2| \, \xi_t \, dx \, dt \leq 0.$$

proof: This result is adapted from a Carrillo's method exposed in [2]. One may find the proof details in [4]. ∎
Let us now give a consequence of this theorem which is preliminary result to obtain uniqueness.

Lemma 2 *For each nonnegative element Φ in $\mathcal{D}(0,T)$, we have*

$$\int_Q |u_1(t,x) - u_2(t,x)| \, \Phi'(t) \, dx \, dt + \int_\Sigma \frac{\partial \left|\varphi(u_1) - \varphi(u_2)\right|(t,x)}{\partial n} \Phi(t) d\mathcal{H}^{n-1} dt$$
$$+ \int_\Sigma \left|\nu(u_1) - \nu(u_2)\right|(t,x) \mathbf{G}.\mathbf{n} \; d\mathcal{H}^{n-1} dt \geq 0.$$

proof: For almost all t of $(0,T)$, by extending $\varphi(u_1)(t) - \varphi(u_2)(t)$ by 0 outside Ω, we can show that $\Delta|\varphi(u_1)(t) - \varphi(u_2)(t)|$ is a Radon measure on Ω. This property allows us to define the normal derivative of $|U(t)| = |\varphi(u_1)(t) - \varphi(u_2)(t)|$ along Γ; actually we have (according to the proposition 3)

$$f.a.a. \; t \in (0,T), \qquad \frac{\partial \left|\varphi(u_1)(t) - \varphi(u_2)(t)\right|}{\partial n} \in L^1(\Gamma).$$

In the comparison theorem 2, we consider the test-function given by $f.a.a.$ $(t,x) \in Q$, $\xi_j(t,x) = \Phi(t)(1 - \rho_j(x))$, which belongs to $H_0^1(\Omega)$ for almost all t of $(0,T)$, as $(\rho_j)_j \subset \mathcal{C}^1(\overline{\Omega})$ is such that

$$\lim_{j \to \infty} \rho_j = 0 \;\text{ everywhere in } \Omega, \;\; \rho_{j_{|\Gamma}} = 1, \;\; 0 \leq \rho_j \leq 1.$$

We obtain the following inequality:

$$\int_Q |u_1(t,x) - u_2(t,x)| \, \Phi'(t)(1 - \rho_j(x)) \, dx \, dt + \int_Q \nabla \left|\varphi(u_1) - \varphi(u_2)\right| . \nabla \rho_j(x) \Phi(t) \, dx \, dt$$
$$+ \int_Q \left|\nu(u_1) - \nu(u_2)\right| \mathbf{G}.\nabla \rho_j(x) \Phi(t) \, dx \, dt \geq 0.$$

Since the sequence $(\rho_j)_j$ converges everywhere to 0, it converges almost everywhere to 0 with respect to Radon measures $\Delta|\varphi(u_1(t)) - \varphi(u_2(t))|$ and $div(\nu(u_1(t)) - \nu(u_2(t))\mathbf{G}$, and the sequence $(1-\rho_j)_j$ converges almost everywhere to 1 with respect to the Lebesgue measure $\mathcal{L}^n$. Thus, as j tends to ∞, by using Green type formulas, we obtain the required result. ∎

5 Uniqueness and stability

In the following part, u_1 and u_2 denote strong entropy solutions respectively associated to initial data u_0^1 and u_0^2. We first expose a preliminary result given in

Lemma 3 *Any entropy solution of problem (1)-(3) satisfies conditions:*

$$\begin{cases} 0 \leq \gamma u \leq \varphi_c \quad \mathcal{H}^n - a.e. \text{ on } \Sigma \\ f.a.a.\ t \in (0,T),\ \nu(\gamma u(t))\mathbf{G}.\mathbf{n} + \dfrac{\partial \varphi(u(t))}{\partial n} \leq 0 \quad \mathcal{H}^{n-1} - a.e. \text{ on } \Gamma^+, \end{cases}$$

where $\Gamma^+ = \{x \in \Gamma,\ \mathbf{G}.\mathbf{n} > 0\}$.

proof: 1) As φ is Lipschitz continuous on $[0,1]$, we have $\gamma(\varphi(u)) = \varphi(\gamma u)$. Moreover, $\varphi(u)$ vanishing along Σ, we deduce that $0 \leq \gamma u \leq \varphi_c \ \ \mathcal{H}^n - a.e.$ on Σ.
2) In the boundary condition (4), henceforth well-defined according to remark 2, we fix $k = 0$. Thus $f.a.a.\ t \in (0,T)$, one has

$$\nu(u(t))\mathbf{G}.\mathbf{n} + \frac{\partial \varphi(u)(t)}{\partial n} \leq 0 \quad \mathcal{H}^{n-1} - a.e. \text{ on } \Gamma.$$

On $\Gamma \setminus \Gamma^+$, this relation is obviously satisfied. Indeed, along this boundary part, the condition is translated into $\nu(u(t))\mathbf{G}.\mathbf{n} \leq 0$, what is always true by construction. The boundary condition forces on the solution to satisfy the required relation. ■
Consequence: Any strong entropy solution, subjected to the previous constraint, satisfies necessarily along Γ^+ one of the two following cases: $f.a.a.\ t \in (0,T)$

either $\begin{cases} 0 \leq (\gamma u)(t) < \varphi_c,\ \nu(u(t)) = 0 \\ \dfrac{\partial \varphi(u(t))}{\partial n} = 0, \end{cases}$ or $\begin{cases} (\gamma u)(t) = \varphi_c,\ \nu(u(t)) > 0 \\ \dfrac{\partial \varphi(u(t))}{\partial n} \leq -\nu(\varphi_c)\ \mathbf{G}.\mathbf{n} < 0. \end{cases}$

Lemma 4 *We have, for almost all t of* $(0,T)$

$$\frac{\partial |\varphi(u_1)(t) - \varphi(u_2)(t)|}{\partial n} + |\nu(u_1(t)) - \nu(u_2(t))|\mathbf{G}.\mathbf{n} \leq 0 \quad \mathcal{H}^{n-1} - a.e. \text{ on } \Gamma.$$

proof: 1) On $\Gamma\setminus\Gamma^+$: for almost all t of $(0,T)$, $|\varphi(u_1)(t) - \varphi(u_2)(t)|$ is nonnegative in Ω and $|\varphi(u_1)(t) - \varphi(u_2)(t)|$ vanishes along Γ; we deduce that its normal derivative is nonpositive. Furthermore, by construction, $|\nu(u_1)(t) - \nu(u_2)(t)|\ \mathbf{G}.\mathbf{n} \leq 0$ along $\Gamma\setminus\Gamma^+$ what implies the result on $\Gamma\setminus\Gamma^+$.

2) On Γ^+: using the result of the previous consequence, two cases arise.
1^{st} case : $\nu(\gamma u_1(t)) = \nu(\gamma u_2(t))$, the wanted result is then direct.

2^{nd} case : $\nu(\gamma u_2) = 0$ and $\gamma u_1 = \varphi_c$, (the case $\nu(\gamma u_1) = 0$ and $\gamma u_2 = \varphi_c$ being similar). We have

$$\frac{\partial\,|\varphi(u_1)(t) - \varphi(u_2)(t)|}{\partial n} = \frac{\partial \varphi(u_1)(t)}{\partial n}.$$

The wanted inequality is then written:

$$\frac{\partial \varphi(u_1)(t)}{\partial n} + \nu(\gamma u_1(t))\mathbf{G}.\mathbf{n} \leq 0 \quad \mathcal{H}^{n-1} - a.e. \text{ on } \Gamma^+,$$

property always satisfied by entropy solution, according to lemma 3. ∎

Theorem 3 *If u_1 and u_2 are both strong entropy solutions of problem (1)-(3) with respectively initial data u_0^1 and u_0^2, then*

$$\forall t \in (0,T], \quad |u_1(t) - u_2(t)|_{L^1(\Omega)} \leq \left|u_0^1 - u_0^2\right|_{L^1(\Omega)}.$$

proof: Immediately, from lemmas 2 and 4, we have, for any function $\Phi \geq 0$ belonging to $\mathcal{D}(0,T)$,

$$\int_0^T \Psi(t)\Phi'(t)dt \geq 0,$$

where $\forall t \in [0,T]$, $\Psi(t) = |u_1(t) - u_2(t)|_{L^1(\Omega)}$, $\Psi(0) = 0$. Thus, $\dfrac{d\Psi}{dt}$ is a nonpositive distribution and according to a result of L. Schwartz [7], Ψ is a nonincreasing function on $[0,T]$, what ensures the uniqueness result. ∎

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