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Long–run dynamic equilibrium in a state–space model under symmetrical controlability

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Abstract

In several occasions the dynamic behaviour of a specific economic variable is influenced by the optimal behaviour of agents that participate in the Economy. This economic variable denominated state variable is observed by each agent in a different way and it offers different answers for each one of them in the optimal process relative to decision making. These answers represent the control parameters for state variables and they determine the future behaviour.

The answer of each agent depends on the perception they have about the state variables since their expectative about other agents interact in the system behaviour.

We offer a model with a only state variable and only control parameter and two agents with symmetrical behaviour and it is enough to analyse one of them and generalise the results for the other.

One of the most common examples that it is possible to analyse under this view is the two oligopolistics behaviour case. State variable will be joint volume offered by them and control variable will be the volume produced for each one of them.

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1 Introduction

We suppose a state equation in which x_{t+1} expresses the state of some situation at moment $t+1$: we will refer to x_{t+1} as a state variable. There are two agents, 1 and 2 that observe this state variable, of distorted form, by means of $z_{1,t}$ y de $z_{2,t}$ respectively:

$$\left\{ \begin{array}{lcll} z_{1,t} & = & h_1 x_t + v_{1,t} & v_{1,t} \sim N(0, \sigma_{v_1}^2) \\ z_{2,t} & = & h_2 x_t + v_{2,t} & v_{2,t} \sim N(0, \sigma_{v_2}^2), \end{array} \right.$$

where $v_{1,t}$ y $v_{2,t}$ are aleatory perturbations. The agents choose a strategy that they will follow accord with this reality: $u_{i,t}$ (control variable) that represents the strategy of the i-th agent at time t. Suppose that the state equation is

$$\boxed{x_{t+1} = \phi x_t + \gamma_1 u_{1,t} + \gamma_2 u_{2,t} + w_t,}$$

where w_t is a stochastic component (or *noise*) such that $w_t \sim N(0, \sigma_w^2)$ and ϕ, γ_1, γ_2 are constants. We also assume that each one of the agents has an *objective function* to be optimised

$$\sum_{t=0}^{T} F_{i,t} \frac{1}{(1+\beta)^t},$$

being β a discount factor and $F_{i,t}$ a function that depends on the perception of the reality that the i-th agent has at time t, $z_{i,t}$, on his strategy, $u_{i,t}$, and on his expectation with respect to the strategy that the other agent (the j-th) chooses and that we will represent by $E_i(u_{j,t}), i, j = 1, 2, i \neq j$. That is,

$$\begin{aligned} F_{1,t} &= F_{1,t}(z_{1,t}, u_{1,t}, E_1(u_{2,t})) \\ F_{2,t} &= F_{2,t}(z_{2,t}, u_{2,t}, E_2(u_{1,t})). \end{aligned}$$

Our problem consist of *defining conditions on the functions* $F_{i,t}$ such that these conditions ensure the resolution of a maximisation problem over the objective function in such a way that the solutions of this maximisation problem are a sequence of control variables $\{u_{i,t}\}, being \quad i = 1, 2.$

First, we wonder if we could express the objective function depending only the state variables (being z_i the distorted state variable) and the control variables respectively. We are supposing that there exists a reality $Q = (z_{i,t}, u_{i,t}, E_i(u_{j,t}))$ in which

1. $F_{i,t}(z_{i,t}, u_{i,t}, E_i(u_{j,t})) = 0,$
2. the partial derivatives $\dfrac{\partial F_{i,t}}{\partial z_{i,t}}(Q)$, $\dfrac{\partial F_{i,t}}{\partial u_{i,t}}(Q)$ both exist and are continuous, and
3. $\dfrac{\partial F_{i,t}}{\partial E_i(u_{j,t})}(Q) \neq 0.$

Hence, by the Implicit Function Theorem, there exists a continuous and differentiable function $f_{i,t}$ -in some neighbourhood- such that

$$E_i(u_{j,t}) = f_{i,t}(z_{i,t}, u_{i,t}).$$

That means that the i-th agent has the same strategy as the j-th at each moment t. In order to simplify the initial problem, we are interested in expressing a variable as a function of the other two. For this, we are assuming that

$$E_1(u_{2,t}) = f_{1,t}(z_{1,t}, u_{1,t}) \tag{1}$$

$$E_2(u_{1,t}) = f_{2,t}(z_{2,t}, u_{2,t}) \tag{2}$$

Note that this problem in which there are two agents trying to optimise their objective functions could be simplified as a unique problem, since there exists a total symmetry in the study of the one and the other case. So that, in the future, we will only analyse the problem from the point of view of the first agent. Thus, our initial problem viewed by the first agent is

$$\begin{array}{|lll|}\hline \text{Maximise} & \displaystyle\sum_{t=0}^{T} F_{1,t}(z_{1,t}, u_{1,t}, E_1(u_{2,t}))\frac{1}{(1+\beta)^t} & \beta > 0 \\ \{u_{1,0}\dots u_{1,T-1}\} & & \\ \text{subject to} & x_{t+1} = \phi x_t + \gamma_1 u_{1,t} + \gamma_2 u_{2,t} + w_{1,t}, & \text{with } w_{1,t} \sim N(0, \sigma_{w_1}^2) \\ \hline \end{array}$$

We know that the perception of the reality that the first agent has is

$$z_{1,t} = h_1 x_t + v_{1,t},$$

so that implies that the equation (1) can be rewritten as

$$\begin{aligned} F_{1,t} &= F_{1,t}(h_1 x_t + v_{1,t}, u_{1,t}, f_{1,t}(z_{1,t}, u_{1,t})) \\ &= F_{1,t}(h_1 x_t + v_{1,t}, u_{1,t}, f_{1,t}(h_1 x_t + v_{1,t}, u_{1,t})) \\ &= G_{1,t}(x_t, u_{1,t}) \end{aligned}$$

that is, we have expressed $F_{1,t}$ by mean of the function $G_{1,t}$, which only depends on the state variable x_t and the control variable $u_{1,t}$. Hence, the *new* optimisation program is

$$\begin{array}{|l|}\hline (P) \left\{ \begin{array}{lll} \text{Maximise} & \displaystyle\sum_{t=0}^{T} G_{1,t}(x_t, u_{1,t})\frac{1}{(1+\beta)^t}, & \beta > 0 \\ \{u_{1,0}\dots u_{1,T-1}\} & & \\ \text{subject to} & x_{t+1} = \phi x_t + \gamma_1 u_{1,t} + \gamma_2 u_{2,t} + w_{1,t}, & \text{with } w_{1,t} \sim N(0, \sigma_{w_1}^2) \end{array} \right. \\ \hline \end{array}$$

So, the main result of this section is the following:

Theorem *Let $G_{1,t}$ be a concave function and assume that there exists other function $f_{1,t}$, derivable with continuity, in such a way that $E_1(u_{2,t}) = f_{1,t}(z_{1,t}, u_{1,t})$. Then the problem (P) has a solution.*

The concavity condition that is supposed to function $G_{1,t}$ is equivalent to the verification of the following conditions at moment t:

1. Both $G_{i,t} \in \mathcal{C}^2$, to $i = 1, 2$.
2. $\dfrac{\partial^2 G_{i,t}}{(\partial x_t)^2} < 0$.
3. $\dfrac{\partial^2 G_{i,t}}{(\partial x_t)^2}\dfrac{\partial^2 G_{i,t}}{(\partial u_{i,t})^2} > \left(\dfrac{\partial^2 G_{i,t}}{\partial x_t \partial u_{i,t}}\right)^2$ that is, $\dfrac{\partial^2 G_{i,t}}{(\partial u_{i,t})^2} < \dfrac{\left(\dfrac{\partial^2 G_{i,t}}{\partial x_t \partial u_{i,t}}\right)^2}{\dfrac{\partial^2 G_{i,t}}{(\partial x_t)^2}} < 0$.

In the previous maximisation problem we are focused on determining conditions on the objective function that implies the existence of solution for the problem. As this is a optimisation program of a two variable-function subject to equality restrictions, this can be solved by means of the Lagrange dynamical multipliers method. For this, firstly, we define the Lagrangian function

$$\begin{aligned}\underline{\mathcal{L}(x_t, u_{1,t}, \lambda(t))} = & \sum_{t=0}^{T} [G_{1,t}(x_t, u_{1,t})\frac{1}{(1+\beta)^t}] \\ + & \frac{\lambda(t)}{(1+\beta)^{t+1}} \left(x_{t+1} - \phi x_t - \gamma_1 u_{1,t} - \gamma_2 u_{2,t} - w_{1,t}\right), \quad t = 0, \ldots, T.\end{aligned}$$

Second, we calculate the points which cancel simultaneously all partial derivatives of the Lagrangian function, where this function is considered as a function of the variables $x_t, u_{1,t}$ and $\lambda(t)$. These conditioned critical points are $p^* = (x_t^*, u_{1,t}^*)$ such that $\left(x_t^*, u_{1,t}^*, \lambda(t)\right)$ is a solution of the equations system, obtained by conditions of the first order, at each moment $t = 0, 1, \ldots, T$. We note that from the first two equations it could be obtained a common value for $\lambda(t)$. We are assuming that constants ϕ, γ_1, γ_2 are nonzero, and so

$$\begin{cases} \dfrac{\partial G_{1,t}}{\partial x_t} & = & \dfrac{1}{(1+\beta)}\lambda(t)\phi \\ \dfrac{\partial G_{1,t}}{\partial u_{1,t}} & = & \dfrac{1}{(1+\beta)}\lambda(t)\gamma_1 \\ x_{t+1} - \phi x_t - \gamma_1 u_{1,t} - \gamma_2 u_{2,t} - w_{1,t} & = & 0 \end{cases}$$

Hence,

$$\frac{\partial G_{1,t}}{\partial u_{1,t}} = \frac{\gamma_1}{\phi}\frac{\partial G_{1,t}}{\partial x_t} \text{ that is,} \qquad \gamma_1 = \phi\frac{\dfrac{\partial G_{1,t}}{\partial u_{1,t}}}{\dfrac{\partial G_{1,t}}{\partial x_t}}. \tag{3}$$

With the same argument, for the second agent, we have the following expression:

$$\gamma_2 = \phi\frac{\dfrac{\partial G_{2,t}}{\partial u_{2,t}}}{\dfrac{\partial G_{2,t}}{\partial x_t}}. \tag{4}$$

As one of the second-order conditions for calculating conditionated extremal points, we analyse the sign of quadratic form associated to the hessian matrix, restricted to certain subspace. For this, we determine the second-order partial derivatives of the Lagrangian function in order to form the hessian matrix. Thus, for each conditionated critical point p^*, the matrix is H_{p^*}. As the Schwarz Theorem guarantees, under certain conditions, the equality of crossed partial derivatives, the hessian matrix H_{p^*} is symmetrical and so, it represents a quadratic form, the following one:

$$Q(y_1, y_2) = \frac{1}{(1+\beta)^t}\left(\frac{\partial^2 G_{1,t}(p^*)}{(\partial x_t)^2}y_1^2 + \frac{\partial^2 G_{1,t}(p^*)}{(\partial u_{1,t})^2}y_2^2 + 2\frac{\partial^2 G_{1,t}(p^*)}{\partial x_t \partial u_{1,t}}y_1 y_2\right)$$

This quadratic form must be restricted to these points (x,u) that verify the following condition, where x represents the state variable and u the control variable:

$$x\frac{\partial g}{\partial x_t}(p^*)+u\frac{\partial g}{\partial u_{1,t}}(p^*)=0$$

being (p^*) a critical conditionated point. With a simple calculation, we obtain the following expressions

$$\begin{aligned}\frac{\partial g}{\partial x_t}(p^*) &= -\phi\\ \frac{\partial g}{\partial u_{1,t}}(p^*) &= -\gamma_1\end{aligned}$$

that allows us to reduce the initial optimisation program to the determination of the sign of quadratic form represented by matrix H_{p^*} restricted to subspace given by

$$u=-\frac{\phi}{\gamma_1}x, \tag{5}$$

Then, the quadratic form with restriction (5)

$$u_{1,t}=-\frac{\phi}{\gamma_1}x_t. \tag{6}$$

is

$$Q\left(x,-\frac{\phi}{\gamma_1}x\right)=Q_t\left(x\right)=\frac{1}{\left(1+\beta\right)^t}\left(\frac{\partial^2 G_{1,t}}{\left(\partial x_t\right)^2}x^2-2\frac{\partial^2 G_{1,t}}{\partial x_t\partial u_{1,t}}\frac{\phi}{\gamma_1}x^2+\frac{\partial^2 G_{1,t}}{\left(\partial u_{1,t}\right)^2}\left(-\frac{\phi}{\gamma_1}x\right)^2\right),$$

that can be expressed as the following matrix product:

$$\begin{aligned}Q_t\left(x\right) &= \frac{1}{\left(1+\beta\right)^t}x^2\begin{pmatrix}1 & -\frac{\phi}{\gamma_1}\end{pmatrix}\begin{pmatrix}\frac{\partial^2 G_{1,t}}{\left(\partial x_t\right)^2} & \frac{\partial^2 G_{1,t}}{\partial x_t\partial u_{1,t}}\\ \frac{\partial^2 G_{1,t}}{\partial x_t\partial u_{1,t}} & \frac{\partial^2 G_{1,t}}{\left(\partial u_{1,t}\right)^2}\end{pmatrix}\begin{pmatrix}1\\ -\frac{\phi}{\gamma_1}\end{pmatrix}\\ &= \frac{1}{\left(1+\beta\right)^t}x^2 Q_{1,t}^*\left(1,-\frac{\phi}{\gamma_1}\right).\end{aligned}$$

For $Q_t\left(x\right)<0$, it is neccesary and sufficient that $Q_{1,t}^*\left(1,-\frac{\phi}{\gamma_1}\right)$, is negative:

$$Q_{1,t}^*\left(1,-\frac{\phi}{\gamma_1}\right)=\begin{pmatrix}1 & -\frac{\phi}{\gamma_1}\end{pmatrix}\begin{pmatrix}\frac{\partial^2 G_{1,t}}{\left(\partial x_t\right)^2} & \frac{\partial^2 G_{1,t}}{\partial x_t\partial u_{1,t}}\\ \frac{\partial^2 G_{1,t}}{\partial x_t\partial u_{1,t}} & \frac{\partial^2 G_{1,t}}{\left(\partial u_{1,t}\right)^2}\end{pmatrix}\begin{pmatrix}1\\ -\frac{\phi}{\gamma_1}\end{pmatrix}<0.$$

Note that this matrix product is a quadratic form evaluated at points $\left(1,-\frac{\phi}{\gamma_1}\right)$ and thus, it negative definite, so the initial problem has solution under the conditions given before:

1. $\dfrac{\partial^2 G_{i,t}}{(\partial x_t)^2} < 0.$

2. $\dfrac{\partial^2 G_{i,t}}{(\partial x_t)^2}\dfrac{\partial^2 G_{i,t}}{(\partial u_{i,t})^2} > \left(\dfrac{\partial^2 G_{i,t}}{\partial x_t \partial u_{i,t}}\right)^2$ it implies, $\dfrac{\partial^2 G_{i,t}}{(\partial u_{i,t})^2} < \dfrac{\left(\dfrac{\partial^2 G_{i,t}}{\partial x_t \partial u_{i,t}}\right)^2}{\dfrac{\partial^2 G_{i,t}}{(\partial x_t)^2}} < 0.$

   Hence, the optimal strategy at moment t for each agent is

$$(x_t^*, u_{i,t}^*) = (x_t^*, \frac{-\phi}{\gamma_i} x_t^*), \qquad \text{being} \qquad i = 1, 2.$$

2 Conclusions

In order to obtain conclusions, we summarise now the contents of previous sections:

1. First, the state equation

$$\boxed{x_{t+1} = \phi x_t + \gamma_1 u_{1,t} + \gamma_2 u_{2,t} + w_t,}$$

   is a first-order difference equation which solution is obtained in a recursive form and can be expressed as

$$\boxed{x_{t+1} = \phi^{t+1} x_0 + \sum_{i=0}^{t} \phi^{t-i} [\gamma_1 u_{1,i} + \gamma_2 u_{2,i} + w_i].} \tag{7}$$

2. In second place, at each moment t we obtained the following conditions:

$$\boxed{\gamma_1 = \phi \frac{\dfrac{\partial G_{1,t}}{\partial u_{1,t}}}{\dfrac{\partial G_{1,t}}{\partial x_t}} \qquad \text{and,} \qquad \gamma_2 = \phi \frac{\dfrac{\partial G_{2,t}}{\partial u_{2,t}}}{\dfrac{\partial G_{2,t}}{\partial x_t}}.} \tag{8}$$

   Since $u_{i,t} = -\dfrac{\phi}{\gamma_i} x_t$, then the optimal value for the control variables of both agents are

$$\boxed{u_{1,t}^* = -\frac{\phi}{\gamma_1} x_t = -\frac{\dfrac{\partial G_{1,t}}{\partial x_t}}{\dfrac{\partial G_{1,t}}{\partial u_{1,t}}} x_t, \qquad u_{2,t}^* = -\frac{\phi}{\gamma_2} x_t = -\frac{\dfrac{\partial G_{2,t}}{\partial x_t}}{\dfrac{\partial G_{2,t}}{\partial u_{2,t}}} x_t.} \tag{9}$$

   and so,

$$u_{1,t}^* = \frac{\gamma_2}{\gamma_1} u_{2,t}^*$$

   which can be reexpressed, by (8), as

$$u_{1,t}^* = \frac{\dfrac{\partial G_{2,t}}{\partial u_{2,t}}}{\dfrac{\partial G_{1,t}}{\partial u_{1,t}}} u_{2,t}^*. \tag{10}$$

The expression (9) collects the **reaction functions** of both agents. Theses reaction functions, evaluated at the optimal points, are connected by the relationship (10).

3. Finally, the following expression is obtained by inserting the previous optimal values in the state equation:

$$x^*_{t+1} = \phi x^*_t - \phi \frac{\dfrac{\partial G_{1,t}}{\partial x_t}}{\dfrac{\partial G_{1,t}}{\partial u_{1,t}}} x^*_t - \phi \frac{\dfrac{\partial G_{2,t}}{\partial x_t}}{\dfrac{\partial G_{2,t}}{\partial u_{2,t}}} x^*_t + w_t = \phi x^*_t \left[1 - \left(\frac{\dfrac{\partial G_{1,t}}{\partial x_t}}{\dfrac{\partial G_{1,t}}{\partial u_{1,t}}} + \frac{\dfrac{\partial G_{2,t}}{\partial x_t}}{\dfrac{\partial G_{2,t}}{\partial u_{2,t}}} \right) \right] + w_t.$$

From the economical point of view, we suppose that values of state variables are not changing notably, and that $\phi > 0$. From this expression, it is clear that the optimal solution of the *game* depends on the action of each agent. By (7) y (9), the above expression can be rewritten as

$$\boxed{x^*_{t+1} = \phi^{t+1} x^*_0 + \sum_{i=0}^{t} \phi^{t-i} [-2\phi x^*_i + w_i].} \tag{11}$$

Note that, in order to analyse the behaviour of this *best solution*, we consider it as solution of a first-order difference equation with constant coefficients, that is, as a *sequence* which convergence is particularly interesting in the following case:

$$0 \;<\; \phi \left| 1 - \left(\frac{\dfrac{\partial G_{1,t}}{\partial x_t}}{\dfrac{\partial G_{1,t}}{\partial u_{1,t}}} + \frac{\dfrac{\partial G_{2,t}}{\partial x_t}}{\dfrac{\partial G_{2,t}}{\partial u_{2,t}}} \right) \right| \;<\; 1$$

3 Particular Cases.

3.1 Exponential objective function.

We assume the function $G_{1,t}(x_t, u_{1,t}) = -e^{x_t^2 + u_{1,t}^2}$, by this, now our problem is:

$$\boxed{(P) \begin{cases} \begin{array}{ll} \text{Maximise} & \displaystyle\sum_{t=0}^{T} \left(-e^{x_t^2 + u_{1,t}^2} \right) \frac{1}{(1+\beta)^t} \\ \{u_{1,0} \dots u_{1,T-1}\} & \\ \text{subject to} & x_{t+1} = \phi x_t + \gamma_1 u_{1,t} + \gamma_2 u_{2,t} + w_{1,t}, \quad \text{with } w_{1,t} \sim N(0, \sigma^2_{w_1}) \end{array} \end{cases}}$$

being β a discount factor. In this case, the Lagrange dynamical multiplier obtained is:

$$\lambda(t) = -\frac{1+\beta}{\phi} 2 x_t e^{x_t^2 + u_{1,t}^2} = -\frac{1+\beta}{\gamma_1} 2 u_{1,t} e^{x_t^2 + u_{1,t}^2}.$$

Which we deduce that $u_{1,t} = \dfrac{\gamma_1}{\phi} x_t$, this is, control variable that is proportional to the state variable, for each moment t.

The Hessian matrix that evaluated in points $p^* = (x_t, \frac{\gamma_1}{\phi}x_t)$, is the matrix

$$H_{p^*} = \left(\frac{1}{1+\beta}\right)^t \begin{pmatrix} -2e^{x_t^2(1+(\frac{\gamma_1}{\phi})^2)}(1+2(\frac{\gamma_1}{\phi})^2) & -4\frac{\gamma_1}{\phi}x_t^2 e^{x_t^2(1+(\frac{\gamma_1}{\phi})^2)} \\ \\ -4\frac{\gamma_1}{\phi}x_t^2 u_{1,t} e^{x_t^2(1+(\frac{\gamma_1}{\phi})^2)} & -2e^{x_t^2(1+(\frac{\gamma_1}{\phi})^2)}(1+2(\frac{\gamma_1}{\phi})^2) \end{pmatrix},$$

associated to a quadratic form that is negative definite, and we will restrict to points (x,u) that verify this equation

$$x\frac{\partial g(p^*)}{\partial x_t} + u\frac{\partial g(p^*)}{\partial u_{1,t}} = 0 \quad \text{this is,} \quad -\phi x - \gamma_1 u = 0.$$

3.2 Quadratic objective function.

In this example, we consider the concave function $G_{1,t}(x_t, u_{1,t})$, a quadratic form negative definite, which is,

$$G_{1,t} = a_{1,1}x_t^2 + a_{2,2}u_{1,t}^2 + 2a_{1,2}x_t u_{1,t}$$

being, $a_{1,1} < 0, a_{2,2} < 0 \quad \text{y} \quad a_{1,1}a_{2,2} > a_{1,2}^2$. Here the optimisation problem is:

$$\boxed{\begin{array}{ll} \text{Maximise} & \displaystyle\sum_{t=0}^{T} \frac{1}{(1+\beta)^t}\left(a_{1,1}x_t^2 + a_{2,2}u_{1,t}^2 + 2a_{1,2}x_t u_{1,t}\right) \quad \text{being}\ \beta > 0 \\ \{u_{1,0}\dots u_{1,T-1}\} & \\ \text{subject to} & x_{t+1} = \phi x_t + \gamma_1 u_{1,t} + \gamma_2 u_{2,t} + w_{1,t}, \quad \text{with}\ w_{1,t} \sim N(0, \sigma_{w_1}^2). \end{array}}$$

By conditions of the first order applied to the Lagrangian function, we obtain that $\lambda(t)$ (Lagrange dynamical multiplier) is:

$$\lambda(t) \quad = \quad \frac{1+\beta}{\phi}\left(2a_{1,1}x_t + 2a_{1,2}u_{1,t}\right) \quad = \quad -\frac{1+\beta}{\gamma_1}\left(2a_{2,2}u_{1,t} + 2a_{1,2}x_t\right),$$

simplifying, the following expression,

$$u_{1,t} = \frac{a_{1,1}\gamma_1 - a_{1,2}\phi}{a_{2,2}\phi - a_{1,2}\gamma_1}x_t,$$

from which we observe that the control variable is proportional to the state variable.

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