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Decomposition of Mueller matrices in pure optical media

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Abstract

The mathematical model of Mueller matrices is able to represent polarimetric properties of every material samples. In this paper an algebraic operation is performed to decompose a Mueller matrix M into the corresponding matrices of the pure optical media embedded into the complex material sample modelized by M.
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1 Introduction

In optical polarimetry, the state of polarization of a light beam is represented by the "Stokes vector" with four real elements arranged in a column vector. When a light beam interacts with a material medium, the Stokes vector $\mathbf{s}$ that characterizes the incident light is transformed by the 4×4 real Mueller matrix M that corresponds to the medium. The emerging light beam is characterized by another Stokes vector $\mathbf{s}'$ given by the product $\mathbf{s}' = M\mathbf{s}$.

The mathematical structure of M depends on the complexity of the optical medium, so that we can distinguish between "pure Mueller matrices" and "non-pure Mueller matrices". Some previous papers [1-5] deal with this property and its mathematical formulation.

The structure of Mueller matrices is studied by means of a transformation of M into a "coherency matrix" H given by [6]

$$H=\frac{1}{2}\left[\begin{array}{cccc} m_{00}+m_{01}+ & m_{02}+m_{12}+ & m_{20}+m_{21}- & m_{22}+m_{33}+ \\ m_{10}+m_{11} & i\left(m_{03}+m_{13}\right) & i\left(m_{30}+m_{31}\right) & i\left(m_{23}-m_{32}\right) \\ m_{02}+m_{12}- & m_{00}-m_{01}+ & m_{22}-m_{33}- & m_{20}-m_{21}- \\ i\left(m_{03}+m_{13}\right) & m_{10}-m_{11} & i\left(m_{23}+m_{32}\right) & i\left(m_{30}-m_{31}\right) \\ m_{20}+m_{21}+ & m_{22}-m_{33}+ & m_{00}+m_{01}- & m_{02}-m_{12}+ \\ i\left(m_{31}+m_{31}\right) & i\left(m_{23}+m_{32}\right) & m_{10}-m_{11} & i\left(m_{03}-m_{13}\right) \\ m_{22}+m_{33}- & m_{20}-m_{21}+ & m_{02}-m_{12}- & m_{00}-m_{01} \\ i\left(m_{23}-m_{32}\right) & i\left(m_{30}-m_{31}\right) & i\left(m_{03}-m_{13}\right) & -m_{10}+m_{11} \end{array}\right]. \quad (1)$$

This expression indicates that there exists a simple linear relation between M and H, so that we can analyze the problem in terms of coherency matrices, which have a simpler mathematical characterization.

The degree and indices of purity of the material sample are given through the eigenvalues of H, so that H can be decomposed into a sum of one to four pure coherency matrices (i.e. the optical media can be represented by a combination of one to four pure material elements). Each pure coherency matrix contains a unique non-null eigenvalue, whereas non-pure coherency matrices contain 2, 3 or 4 non-null eigenvalues.

In this paper we deal with the mathematical resolution of a physical problem that appears frequently in polarimetry: Once obtained a measurement of the coherency matrix corresponding to the whole complex media, we want to "subtract" the action of a pure component that we know (or suspect) is present into the complex medium under measurement. We have developed a mathematical procedure to perform a proper subtraction and obtain the coherency matrix corresponding to the complex system regardless the effect of the known component.

2 Algebraic decomposition procedure

Given two positive semidefinite hermitic matrices H and A with dimension n such that rang$A$=1, rang$H$=r, $0 < r \leq n$, the following problem is stated:

"Is there exist a positive real number α such that rang$(H - \alpha A) = r - 1$?"

This question can be completely answered in two steps: regular H case, and general H case.

Regular H case

If rang$H = n$,then $\alpha \neq 0$ is a necessary condition for $\det(H - \alpha A) = 0$. Therefore the problem stated before is equivalent to the problem of eigenvalues

$$\det(\frac{1}{\alpha}I - H^{-1}A) = 0. \quad (2)$$

It is easy to prove that only one eigenvalue $1/\alpha > 0$ exists, whereas 0 is the other eigenvalue with multiplicity degree $n-1$.

The solution to this problem also could be solved using the classical procedure for simultaneous diagonalization of two cuadratic forms [7], extending it to the case of two hermitic forms with coordenate matrices H(positive definite) and A(positive semidefinite) in a base $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. The procedure consists of building an orthonormal base $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ by means of the Gramm-Schmidt method applied to the hermitic product

$$P_H(\mathbf{z}, \mathbf{w}) = \mathbf{z}^T H \overline{\mathbf{w}}, \qquad \mathbf{z},\, \mathbf{w} \in \mathcal{C}^n. \tag{3}$$

The hermitic form associated with the matrix H expressed in the base E is

$$P_H(\mathbf{z}, \mathbf{z}) = \mathbf{z}^T \mathbf{z}, \tag{4}$$

and the hermitic form given by the matrix A is

$$P_A(\mathbf{z}, \mathbf{z}) = \mathbf{z}^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \mathbf{z}, \tag{5}$$

where $\lambda_i,\ i = 1, \ldots, n$ are the eigenvalues of the hermitic matrix $\overline{C}^T \overline{A} C$ and C is the matrix of the basis change from the base B to the base E. The equation $\text{rang}(H - \alpha A) = n - 1$ formerly stated is solved computing α such that $1 - \alpha\lambda_i = 0$ for some $i = 1, \ldots, n$, with $\lambda_i \neq 0$. Taking into account that $\overline{C}^T \overline{A} C$ is a hermitic matrix, we have that λ_i are positive, and then $\alpha > 0$. The uniqueness of the solution α is justified by the hypothesis $\text{rang}A = 1$.

General H case

From the simultaneous diagonalization procedure above described, we can approach the general case of $\text{rang}H = r$, where $0 < r < n$. If $\text{rang}H = r < n$, is not always possible to make the subtraction $H - \alpha A$ with $\text{rang}(H - \alpha A) = r - 1$. On the light of this consideration, we see that now the goal is to characterize the existence of solution α by means of the most efficient test.

To obtain the test proposed below, we need some suitable notation.
The matrix A, with $\text{rang}A = 1$, is denoted by $A = (a_1\mathbf{t}, a_2\mathbf{t}, \ldots, a_n\mathbf{t})^T$ where $\mathbf{t}$ is a nonzero row vector.
The vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$ is named *the proportionality vector of the matrix A.*

If $L = [l_{kj}]$ is a order n regular matrix such that $LH = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, 0, \ldots, 0)^T$, with $\mathbf{v}_i$ row vectors, then:

$$L(H - \alpha A) = (\mathbf{v}_1 - \alpha\lambda_1\mathbf{t}, \ldots, \mathbf{v}_r - \alpha\lambda_r\mathbf{t}, -\alpha\lambda_{r+1}\mathbf{t}, \ldots, -\alpha\lambda_n\mathbf{t})^T, \tag{6}$$

where $\lambda_k = \sum_{j=1}^{n} l_{kj} a_j$, $k = 1, \ldots, n$.

Denoting $V_l = span\{\text{columns of } A\}$, $V_r = span\{\text{rows of } H\}$ it is shown that $\lambda_i = 0$ $(i = r+1, \ldots, n)$, is a necessary and sufficient condition to $V_l \subset V_r$, and there exists α such that $\text{rang}(H - \alpha A) = r - 1$.

On the other hand, if $\lambda_i \neq 0$ for some $i = 1, \ldots, n$, then $\mathcal{C}^{r+1} = V_l \oplus V_r$, and we have that the subtraction $H - \alpha A$ is not possible.

A similar test can be stated regarding the product $H\overline{L}^T = (\mathbf{s}_1, \ldots, \mathbf{s}_r, 0, \ldots, 0)$ where $\mathbf{s}_j$ $(j = 1, \ldots, n)$, are column vectors. Taking the matrix A made by columns, both tests are equivalents.

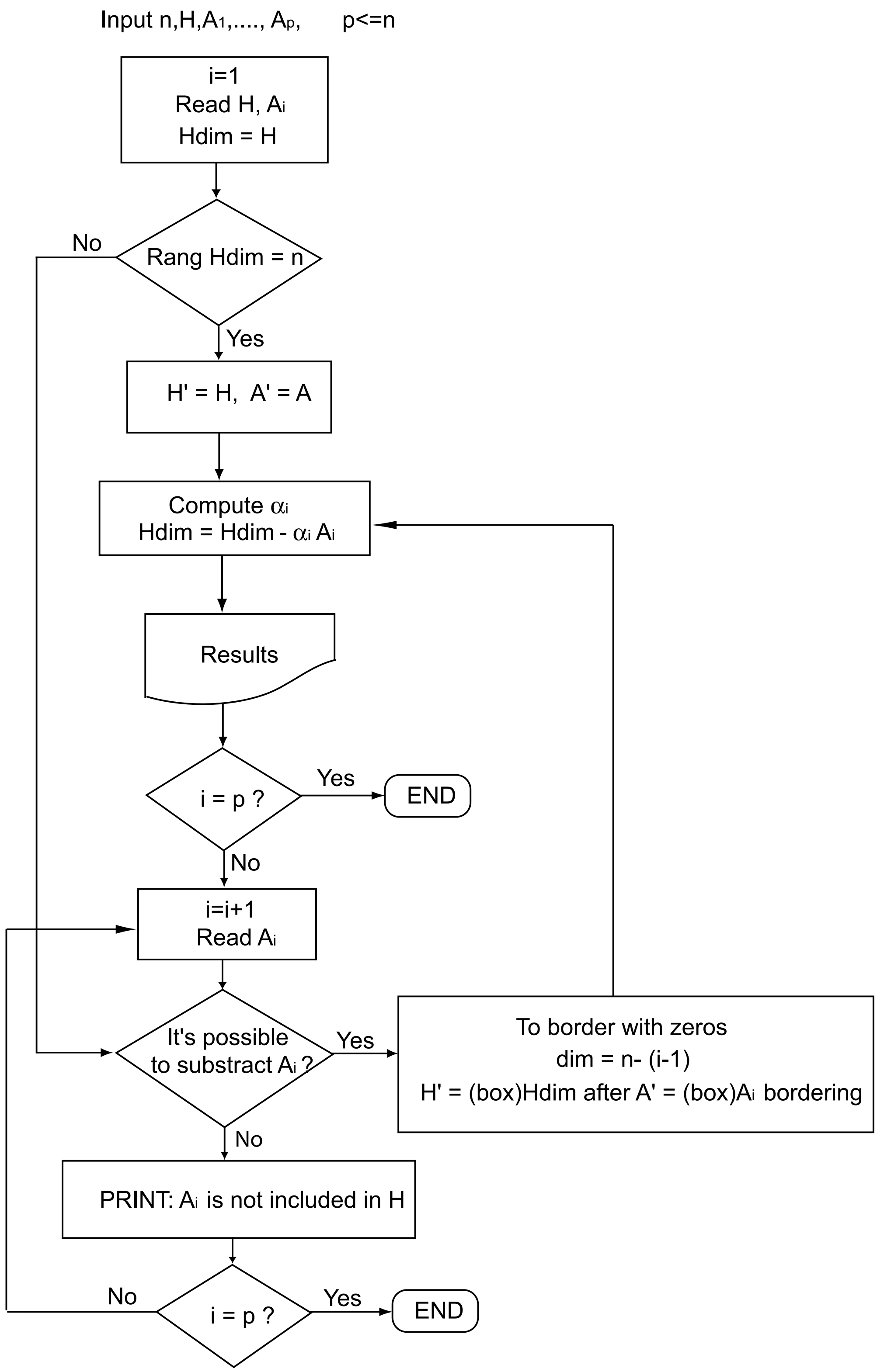
If the test performed with H and A has been successful, then the transformation $L(H - \alpha A)\overline{L}^T$ leads, for every α, to a matrix bordered with zeros in the last $n - r$ rows and columns:

$$LH\overline{L}^T = \begin{bmatrix} \begin{bmatrix} & & \\ & H' & \\ & & \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}, \quad LA\overline{L}^T = \begin{bmatrix} \begin{bmatrix} & & \\ & A' & \\ & & \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}. \tag{7}$$

Now, considering the previously solved regular case, and taking $H = H'$, $A = A'$ and $n = r$, the general problem is solved and we can finally state:

"Given $n \times n$ positive semidefinite hermitic complex matrices H and A such that $\text{rang}A = 1$, $\text{rang}H = r$, $0 < r \leq n$, there exists a unique real $\alpha > 0$ such that $\text{rang}(H - \alpha A) = r - 1$".

3 Iterative scheme



4 An application example

In order to clarify the method, we present here an example of application of the procedure described in the previous sections.

We consider the matrix

$$H = \frac{1}{320} \begin{bmatrix} 53 & 15 & 15 & 61 \\ 15 & 53 & 5-48i & 15 \\ 15 & 5+48i & 53 & 15 \\ 61 & 15 & 15 & 53 \end{bmatrix}, \tag{8}$$

that corresponds to a non-pure material, with rang$H = 3$, and the matrix

$$P = \frac{1}{32} \begin{bmatrix} 9 & 3 & 3 & 9 \\ 3 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 9 & 3 & 3 & 9 \end{bmatrix}, \tag{9}$$

that corresponds to a pure polarizer [8]. In this case, the test is successful an the subtraction is possible with $\alpha_1 = 1/2$.

This value of α_1 represents the proportion of the material represented by P into the complex material represented by H. Then, the rest is

$$H - \frac{1}{2}P = \frac{1}{40} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 6 & -6i & 0 \\ 0 & 6i & 6 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}, \tag{10}$$

and it can be tested if the pure material represented by

$$P_1 = \frac{1}{8} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}, \tag{11}$$

is a component of the rest (10). Now, we can verify the test again and we obtain $\alpha_2 = 1/5$. This value means the concentration of polarizer P_1 [8] into the material H.

Following the iterative procedure, the difference

$$H - \frac{1}{2}P - \frac{1}{5}P_1 = \frac{3}{20} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

is a new matrix, which range is 1, that constitutes a retarder [8] and represents the remaining rest material:

$$R = \frac{3}{20}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

It is easily tested that others optically pure materials are not present into the complex material represented by matrix H. For example, if the test is made for a retarder given by the matrix [8]

$$R_l = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

the result of the test is negative and the subtraction is not possible.

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