Monografías del Semin. Matem. García de Galdeano. **27:** 193–200, (2003).

An integral method for exterior transmission problems with applications to scattering of thermal waves

Ricardo Celorrio[1], María–Luisa Rapún[2], Francisco–Javier Sayas[3]

Departamento Matemática Aplicada. Universidad de Zaragoza
[1]E.U.I.T.I.Z. E-mail: celorrio@posta.unizar.es
[2]Facultad de Ciencias. E-mail: mlrapun@posta.unizar.es
[3]Centro Politécnico Superior. E-mail: jsayas@posta.unizar.es

Abstract

Integral equation methods are often used to deal with exterior problems of wave propagation. This approach is used here for an exterior problem where a side of an homogeneous opaque heat–conducting material (drilled by a finite number of cylinders made of a different material) is illuminated by a laser beam at constant frequency. By an indirect method for the two–dimensional Helmholtz equation the problem is reduced to a system of integral equations. We propose a Petrov–Galerkin method with piecewise constant functions to approximate the unknowns on the boundaries (densities). The method is shown to be stable and convergent.
Keywords: Boundary integral methods, Galerkin methods, scattering, transmission problems
AMS Classification: 65R20, 65N38

1 Statement of the problem

Let us consider a finite number of simply connected bounded open sets $\Omega_1, \ldots, \Omega_d$ strictly contained in the half–plane $\mathbb{R}^2_- := \{(x_1, x_2) \mid x_2 < 0\}$ and such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \neq j$. Let also $\Pi := \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$. The boundary of each Ω_k, denoted Γ_k, is assumed to be a $\mathcal{C}^2$ curve (see figure 1).

Let u_{inc} be a solution of the Helmholtz equation in the half plane (incident wave)

$$\Delta u_{inc} + \lambda^2 u_{inc} = 0 \quad \text{in} \;\; \mathbb{R}^2_-.$$

We are looking for a solution of the problem

$$\begin{aligned} \Delta u + \lambda^2 u &= 0, && \text{in} \;\; \Omega := \mathbb{R}^2_- \setminus (\cup_{k=1}^d \overline{\Omega}_k), \\ \Delta u + \lambda_k^2 u &= 0, && \text{in} \;\; \Omega_k \quad k = 1, \ldots, d, \end{aligned}$$

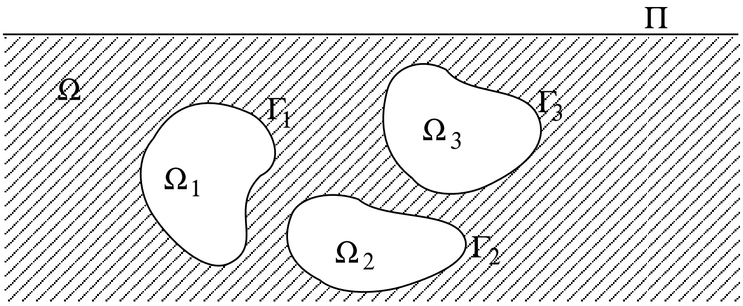

Figure 1: The geometry of the problem

where $\lambda, \lambda_k \in (1+i)\,\mathbb{R}^+ = \{z \in \mathbb{C} \,|\, \mathrm{re}\, z = \mathrm{im}\, z > 0\}$. We also demand that the solution satisfies the boundary condition

$$\partial_n u|_\Pi = \partial_n u_{inc}|_\Pi$$

and the transmission conditions on the inner boundaries

$$\begin{aligned} u|_{\Gamma_k}^{int} &= u|_{\Gamma_k}^{ext}, \\ \nu_k \partial_n u|_{\Gamma_k}^{int} &= \nu \partial_n u|_{\Gamma_k}^{ext}, \end{aligned}$$

with $\nu_k, \nu > 0$. Normals are directed towards the exterior of Ω_k for each k. The normal on Π is directed upwards (pointing towards the exterior of Ω).

Finally we demand that $u - u_{inc}$ is a radiating wave, i.e., that $u - u_{inc}$ satisfies the Sommerfeld condition at infinity [3], i.e.,

$$\lim_{r\to\infty} r^{1/2}(\partial_r(u - u_{inc}) - i\lambda(u - u_{inc})) = 0$$

uniformly in all available directions (∂_r denotes the radial derivative).

Physical motivation. This problem arises in quality control of composite materials consisting in a base with cilindrical incrustations to strengthen its structure. A particularly suitable means of inspecting this kind of materials is to use photothermal techniques as illuminating the upper surface by a defocused laser beam modulated at a given frequency ω.

After a sufficiently long time the temperature distribution becomes time-harmonic. The periodic term of the temperature has the form $T(\mathbf{x}, t) = \mathrm{Re}(u(\mathbf{x}) \exp(i\omega t))$. Our unknown is $u(\mathbf{x})$, the complex amplitude of the thermal wave.

The boundary condition models an adiabatic situation whereas the transmission conditions model the continuity of temperature and heat flux [8]. □

2 Uniqueness

If we take as unknown

$$v := \left| \begin{array}{ll} u - u_{inc}, & \text{in } \Omega, \\ u, & \text{in } \Omega_k \quad k = 1, \ldots, d, \end{array} \right.$$

($u - u_{inc}$ is called scattered wave), the transmission problem becomes

$$\left| \begin{array}{ll} \Delta v + \lambda^2 v = 0, & \text{in } \Omega, \\ \Delta v + \lambda_k^2 v = 0, & \text{in } \Omega_k, \quad k = 1, \ldots, d, \\ v|_{\Gamma_k}^{int} - v|_{\Gamma_k}^{ext} = g_0^k, & k = 1, \ldots, d, \\ \nu_k \partial_n v|_{\Gamma_k}^{int} - \nu \partial_n v|_{\Gamma_k}^{ext} = g_1^k, & k = 1, \ldots, d, \\ \partial_n v|_{\Pi} = 0, & \\ \lim_{r\to\infty} r^{1/2}(\partial_r v - i\lambda v) = 0 & \text{(Sommerfeld condition)} \end{array} \right. \tag{1}$$

where $g_0^k := -u_{inc}|_{\Gamma_k}$ and $g_1^k := -\nu \partial_n u_{inc}|_{\Gamma_k}$.

By construction we have that for all k

$$g_0^k \in H^{1/2}(\Gamma_k), \qquad g_1^k \in H^{-1/2}(\Gamma_k),$$

(see [3] for definitions of these usual Sobolev spaces in the boundary). The solution is assumed to be such that $v|_{\Omega_k} \in H^1(\Omega_k)$ for all k and

$$v|_{\Omega} \in H^1_{loc}(\overline{\Omega}) := \{v \in \mathcal{D}'(\Omega) \,|\, v\,\phi \in H^1(\Omega), \forall \phi \in \mathcal{D}(\overline{\Omega})\}.$$

It can be seen that any solution to this problem is smooth, up to Π, and therefore the boundary condition $\partial_n v|_{\Pi} = 0$ is satisfied in a classical way, as happens with the Sommerfeld radiation condition.

Moreover, we can prove the following result (see [2]).

Proposition 1 *For arbitrary $g_0^k \in H^{1/2}(\Gamma_k)$, $g_1^k \in H^{-1/2}(\Gamma_k)$, there exists a unique solution to* (1) *in the sense specified above.*

3 Boundary integral formulation

In order to simplify the discussion we consider the simplest transmission problem in $\mathbb{R}^2_-$ with a single obstacle. Superscripts "+" and "–" correspond to the exterior (unbounded) and the interior (bounded) domain respectively and Γ is the common boundary.

We propose an indirect formulation with unknown densities $\psi^+, \psi^- : \Gamma \to \mathbb{C}$

$$v = \left| \begin{array}{ll} \mathcal{S}^+\psi^+, & \text{in } \Omega^+, \\ \mathcal{S}^-\psi^-, & \text{in } \Omega^-, \end{array} \right.$$

where

$$\begin{aligned}\mathcal{S}^+\psi &:= -\int_\Gamma (\Phi(\lambda^+|\cdot-\mathbf{y}|)+\Phi(\lambda^+|\cdot-\tilde{\mathbf{y}}|))\,\psi(\mathbf{y})\,d\gamma(\mathbf{y}) : \Omega^+ \longrightarrow \mathbb{C},\\ \mathcal{S}^-\psi &:= -\int_\Gamma \Phi(\lambda^-|\cdot-\mathbf{y}|)\,\psi(\mathbf{y})\,d\gamma(\mathbf{y}) : \Omega^- \longrightarrow \mathbb{C},\end{aligned}$$

being $\tilde{\mathbf{y}} = (y_1, -y_2)$ the reflected point of $\mathbf{y} = (y_1, y_2)$ and $\Phi(x) = -\frac{i}{4}H_0^{(1)}(x)$ the Hankel function of first kind and order zero (cf [1]), that can be decomposed as $\Phi(x) = a(x^2)\log(x) + b(x^2)$ with a, b entire functions.

By definition, v satisfies the corresponding Helmholtz equations in the exterior and in the interior domains, the boundary condition on Π and the Sommerfeld radiation condition at infinity (see [3] Chapter 7).

The traces of the single layer potentials are given by the operators

$$\begin{aligned}V^+\psi := \mathcal{S}^+\psi|_\Gamma &= -\int_\Gamma (\Phi(\lambda^+|\cdot-\mathbf{y}|)+\Phi(\lambda^+|\cdot-\tilde{\mathbf{y}}|))\,\psi(\mathbf{y})\,d\gamma(\mathbf{y}) : \Gamma \longrightarrow \mathbb{C},\\ V^-\psi := \mathcal{S}^-\psi|_\Gamma &= -\int_\Gamma \Phi(\lambda^-|\cdot-\mathbf{y}|)\,\psi(\mathbf{y})\,d\gamma(\mathbf{y}) : \Gamma \longrightarrow \mathbb{C}.\end{aligned}$$

On the other hand, the traces of the normal derivatives of the single layer potentials satisfy

$$\begin{aligned}\partial_n\mathcal{S}^+\psi|_\Gamma^+ &= -\tfrac{1}{2}\psi - J^+\psi,\\ \partial_n\mathcal{S}^-\psi|_\Gamma^- &= \tfrac{1}{2}\psi - J^-\psi,\end{aligned}$$

being

$$\begin{aligned}J^+\psi &:= \int_\Gamma \partial_{n(\,\cdot\,)}(\Phi(\lambda^+|\cdot-\mathbf{y}|)+\Phi(\lambda^+|\cdot-\tilde{\mathbf{y}}|))\,\psi(\mathbf{y})\,d\gamma(\mathbf{y}) : \Gamma \longrightarrow \mathbb{C},\\ J^-\psi &:= \int_\Gamma \partial_{n(\,\cdot\,)}\Phi(\lambda^-|\cdot-\mathbf{y}|)\,\psi(\mathbf{y})\,d\gamma(\mathbf{y}) : \Gamma \longrightarrow \mathbb{C}.\end{aligned}$$

Remark. The kernels of the operators $J^\pm$ are continuous whereas the operators $V^\pm$ have kernels with logarithmic singularities. □

Now we can express the transmission conditions in matrix form

$$\mathcal{L}\begin{bmatrix}\psi^-\\ \psi^+\end{bmatrix} := \begin{bmatrix}V^- & -V^+\\ \nu^-(\frac{1}{2}I - J^-) & \nu^+(\frac{1}{2}I + J^+)\end{bmatrix}\begin{bmatrix}\psi^-\\ \psi^+\end{bmatrix} = \begin{bmatrix}g_0\\ g_1\end{bmatrix}. \qquad (2)$$

The first equation is the continuity of temperature and the second one is the continuity of heat flux. Then we have the following result (see [2])

Theorem 2 *$\mathcal{L} : H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ is an isomorphism. Moreover, there exists an elliptic operator $V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ (i.e., $\xi\langle V_0\psi,\psi\rangle \geq \|\psi\|^2_{-1/2,\Gamma}$, $\xi \in \mathbb{C}$) independent of $\lambda^\pm$ such that*

$$\mathcal{L} - \begin{bmatrix}V_0 & -V_0\\ \frac{\nu^-}{2}I & \frac{\nu^+}{2}I\end{bmatrix}$$

is compact.

4 A Petrov–Galerkin method

Let $\mathbf{x} : [0,1] \to \Gamma$ be a regular parameterization of the boundary Γ which henceforth is assumed to be smooth. We consider new unknowns,

$$\psi^{\pm} := \psi^{\pm}(\mathbf{x}(\cdot))\,|\mathbf{x}'(\cdot)| : [0,1] \longrightarrow \mathbb{C},$$

data

$$\begin{aligned} g_0 &:= g_0(\mathbf{x}(\cdot)), \\ g_1 &:= |\mathbf{x}'(\cdot)| g_1(\mathbf{x}(\cdot)), \end{aligned}$$

and parameterized versions of the operators for which we keep the same notation,

$$\begin{aligned} V^+\eta &:= -\int_0^1 (\Phi(\lambda^+|\mathbf{x}(\cdot)-\mathbf{x}(t)|) + \Phi(\lambda^+|\mathbf{x}(\cdot)-\tilde{\mathbf{x}}(t)|))\,\eta(t)\,dt : [0,1] \longrightarrow \mathbb{C}, \\ V^-\eta &:= -\int_0^1 \Phi(\lambda^-|\mathbf{x}(\cdot)-\mathbf{x}(t)|)\,\eta(t)\,dt : [0,1] \longrightarrow \mathbb{C}, \\ J^+\eta &:= \int_0^1 |\mathbf{x}'(\cdot)|\partial_{n(\cdot)}(\Phi(\lambda^+|\mathbf{x}(\cdot)-\mathbf{x}(t)|) + \Phi(\lambda^+|\mathbf{x}(\cdot)-\tilde{\mathbf{x}}(t)|))\,\eta(t)\,dt : [0,1] \longrightarrow \mathbb{C}, \\ J^-\eta &:= \int_0^1 |\mathbf{x}'(\cdot)|\partial_{n(\cdot)}\Phi(\lambda^-|\mathbf{x}(\cdot)-\mathbf{x}(t)|)\,\eta(t)\,dt : [0,1] \longrightarrow \mathbb{C}. \end{aligned}$$

We consider the Sobolev spaces (see [4] Chapter 8, [7]),

$$H^r := \{\phi \in \mathcal{D}' \mid |\widehat{\phi}(0)|^2 + \sum_{0\neq k\in\mathbb{Z}} |k|^{2r}|\widehat{\phi}(k)|^2 < \infty\},$$

where $\mathcal{D}'$ is the space of 1–periodic distributions at the real line and

$$\widehat{\phi}(k) := \langle \phi, \exp(-2k\pi i\,\cdot)\rangle_{\mathcal{D}'\times\mathcal{D}}$$

are the Fourier coefficients of ϕ. For all $r \in \mathbb{R}$, H^r is a Hilbert space with inner product

$$(\phi,\psi)_r := \widehat{\phi}(0)\overline{\widehat{\psi}(0)} + \sum_{0\neq k\in\mathbb{Z}} |k|^{2r}\widehat{\phi}(k)\overline{\widehat{\psi}(k)}.$$

Proposition 3 *In the new notations,*

$$\mathcal{L} := \begin{bmatrix} V^- & -V^+ \\ \nu^-(\frac{1}{2}I - J^-) & \nu^+(\frac{1}{2}I + J^+) \end{bmatrix} : H^{-1/2}\times H^{-1/2} \to H^{1/2}\times H^{-1/2}$$

is an isomorphism and there exists an elliptic operator $V_0 : H^{-1/2} \to H^{1/2}$ independent of $\lambda^{\pm}$ such that

$$\mathcal{L} - \begin{bmatrix} V_0 & -V_0 \\ \frac{\nu^-}{2}I & \frac{\nu^+}{2}I \end{bmatrix}$$

is compact.

We propose a Petrov–Galerkin method for the parameterized versions of the equations above. Trial and test spaces are defined as follows: we construct a uniform mesh in $[0,1]$ with nodes $s_i = ih$ and take a space of piecewise constant functions,

$$S_h^0 := \{\varphi : [0,1] \to \mathbb{C} \mid \varphi|_{[s_{i-1},s_i]} \in \mathbb{P}_0\},$$

as trial space for both unknowns and as test space for the first equation. The space

$$S_h^1 := \{\varphi \in \mathcal{C}([0,1]) \mid \varphi(0) = \varphi(1), \quad \varphi|_{[s_i - \frac{h}{2}, s_i + \frac{h}{2}]} \in \mathbb{P}_1\}$$

is taken as test space for the second equation. S_h^1 is formed by periodic first degree polynomials between consecutive midpoints of the mesh nodes.

Remark. We base our choice of displaced polygonal functions on stability questions. The first equation takes place in $H^{1/2}$ whereas the second one occurs in $H^{-1/2}$; as they have different character, we take different test spaces to achieve the same convergence order. □

Given $f, g \in H^0$, we denote $(f,g) := \int_0^1 f(t)g(t)\,dt$. The numerical method is then:

$$\left|\begin{array}{ll} \text{Find } \psi_h^+, \psi_h^- \in S_h^0, \text{ such that} & \\ (V^-\psi_h^- - V^+\psi_h^+, u_h) = (g_0, u_h), & \forall u_h \in S_h^0, \\ (\nu^-(\tfrac{1}{2}I - J^-)\psi_h^- + \nu^+(\tfrac{1}{2}I + J^+)\psi_h^+, v_h) = (g_1, v_h), & \forall v_h \in S_h^1. \end{array}\right. \tag{3}$$

5 Analysis of convergence

Theorem 4 *The equations* (3) *are uniquely solvable for h small enough. There exist constants $C_1, C_2 > 0$ independent of $\psi^\pm$ and h such that*

$$\|\psi^+ - \psi_h^+\|_{-1/2} + \|\psi^- - \psi_h^-\|_{-1/2} \leq C_1\, h^{3/2}(\|\psi^+\|_1 + \|\psi^-\|_1), \tag{4}$$

$$\|\psi^+ - \psi_h^+\|_{-2} + \|\psi^- - \psi_h^-\|_{-2} \leq C_2\, h^3(\|\psi^+\|_1 + \|\psi^-\|_1). \tag{5}$$

Sketch of the proof:
The numerical method for the principal part of the operator $\mathcal{L}$ is:

$$\left|\begin{array}{ll} \text{Find } \varphi_h^+, \varphi_h^- \in S_h^0, \text{ such that} & \\ (V_0\varphi_h^- - V_0\varphi_h^+, u_h) = (g_0, u_h), & \forall u_h \in S_h^0, \\ (\frac{\nu^-}{2}\varphi_h^- + \frac{\nu^+}{2}\varphi_h^+, v_h) = (g_1, v_h), & \forall v_h \in S_h^1. \end{array}\right. \tag{6}$$

The principal part of the operator $\mathcal{L}$ can be decomposed as

$$\begin{bmatrix} V_0 & -V_0 \\ \frac{\nu^-}{2}I & \frac{\nu^+}{2}I \end{bmatrix} = \underbrace{\begin{bmatrix} V_0 & 0 \\ 0 & I \end{bmatrix}}_{=:\,\mathcal{P}} \underbrace{\begin{bmatrix} I & -I \\ \frac{\nu^-}{2}I & \frac{\nu^+}{2}I \end{bmatrix}}_{=:\,\mathcal{Q}}.$$

The operator $\mathcal{Q}$ allows to do a change of variables and the equations (6) for the new unknowns are uncoupled since $\mathcal{P}$ is diagonal. The numerical method in terms of the new unknowns is

$$\left|\begin{array}{ll}\text{Find } \eta_h^+, \eta_h^- \in S_h^0, \text{ such that} & \\ (V_0\eta_h^-, u_h) = (g_0, u_h), & \forall u_h \in S_h^0, \\ (\eta_h^+, v_h) = (g_1, v_h), & \forall v_h \in S_h^1.\end{array}\right. \tag{7}$$

The first equation in (7) is a S_h^0–Galerkin method for the operator V_0. Ellipticity of $V_0 : H^{-1/2} \to H^{1/2}$ implies that the S_h^0–Galerkin method for V_0 is $H^{-1/2}$–stable (see [4] Chapter 13).

The second equation is a Petrov–Galerkin method for the identity operator with S_h^0 and S_h^1 as trial and test spaces respectively. We prove in first term H^0–stability for the identity operator and then, by means of inverse inequalities in the space S_h^1 (see [5] for approximation and in verse properties of splines), we show $H^{-1/2}$–stability. This follows the same ideas given in [6].

Undoing the change of variables we obtain $H^{-1/2}$–stability for the principal part:

$$\|\varphi_h^+\|_{-1/2} + \|\varphi_h^-\|_{-1/2} \le C\,(\|\varphi^+\|_{-1/2} + \|\varphi^+\|_{-1/2}).$$

Hence we have Céa's estimate and from it and the approximation property of S_h^0 in $H^{-1/2}$ we obtain convergence of the method (6) in $H^{-1/2} \times H^{-1/2}$.

Standard compactness arguments yield stability and convergence in $H^{-1/2} \times H^{-1/2}$ for the global operator $\mathcal{L}$.

Finally, we obtain the convergence bounds: Céa's lemma leads to (4) and Aubin–Nitsche duality argument yields (5) (see [5] Chapter 1 for this kind of argumentation).

□

Acknowledgment The authors are partially supported by MCYT Project BFM2001-2521 and MAT2002-04153.

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical functions*, Dover, 1972.

[2] R. Celorrio, M.–L. Rapún, F.–J. Sayas, in preparation.

[3] G. Chen, J. Zhou, *Boundary element methods*, Academic Press, 1992.

[4] R. Kress, *Linear integral equations.* Second edition, Springer–Verlag, 1999.

[5] S. Prössdorf, B. Silbermann, *Numerical analysis for integral and related operator equations*, Akademie Verlag, 1991

[6] K. Ruotsalainen, J. Saranen, "On the convergence of the Galerkin method for nonsmooth solutions of integral equations", *Numer. Math.* **54** (1988), 295–302.

[7] Saranen, J.; Vainikko, G. *Periodic Integral and Pseudodifferential Equations with Numerical Approximation*, Springer–Verlag, 2002.

[8] J.M. Terrón, A. Sánchez–Lavega, A. Salazar, "Multiple scattering effects of thermal waves by two subsurface cylinders", *J. Appl. Phys.,* **87** (2000), 2600–2607.