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Saturation of k-convex operators

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Abstract

In this paper we enrich a result of the authors on the local saturation of k-convex operators, so that it can be applied to more general situations. We deal with different shape properties of the operators and with different asymptotic expressions for them. Some applications are shown to the well known approximation operators of Bleimann-Butzer-Hahn, Kantorovich and to a generalized version of the Szász-Mirakyan operators.

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1 Introduction

In 1964 Lorentz [8] solved the problem of the saturation of the classical Bernstein operators B_n defined on $C[0,1]$. He established that $|B_nf(x)-f(x)| \leq (1/2n)Mx(1-x)$, $0 < x < 1$ if and only if $f' \in Lip_M 1$. This result found further development in two papers of Mühlbach [10] and Lorentz and Schumaker [9]. They dealt with a general sequence of linear positive operators satisfying an asymptotic formula of Voronovskaya type.

Recently, the authors [4], given $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, a closed real interval I and a sequence of linear operators $L_n : C^k(I) \to C^k(I)$, have proved a new result on local saturation for D^kL_n (D^k is the k-th differential operator) under the following more general assumptions:

A) the operators are k-convex, that is to say, $D^kf \geq 0$ implies $D^kL_nf \geq 0$,

B) there exist a sequence λ_n of real positive numbers and a function $p \in C^k(I)$ strictly positive on $\mathrm{Int}(I)$ such that for all $g \in C^k(I)$, bounded on I and $k+2$-times differentiable in some neighborhood of $x \in \mathrm{Int}(I)$,

$$\lambda_n(D^kL_ng(x) - D^kg(x)) \stackrel{n\to\infty}{\longrightarrow} Tg(x), \tag{1}$$

T being the operator defined by $Tg := D^k(pD^2g)$.

Moreover, recent advances on the establishment of this kind of asymptotic expressions for the k-th derivatives of certain shape preserving operators allowed the authors to show as well some applications to outstanding sequences of operators.

In this paper we enrich this result so that it can be applied to sequences of operators that either they are not k-convex or they do not satisfy an asymptotic formula as the one above. Specifically, in the first case we shall obtain a result on the saturation of D^2L_n without the 2-convexity of the operators L_n and in the second case we shall replace the limit in (1) by generalizing slightly the definition of the operator T. The results will be applied to the well-known operators of Bleimann-Butzer-Hahn, Kantorovich and to a generalized version of the Szász-Mirakyan's.

2 A local saturation result revisited.

We begin assuming **A)** and **B)** and recovering the essential statements that the authors proved in [4].

Result 1 *a) The linear differential equation $Tg \equiv 0$, with the unknown function g, can be reduced to a second order one, $Lz \equiv 0$ say, by using the variable change $z = D^k g$.*
b) If $f \in C^k(I)$, bounded on I, is a solution of $Tg \equiv 0$ on some neighborhood of a point $x \in Int(I)$, then

$$D^k L_n f(x) - D^k f(x) = o(\lambda_n^{-1}).$$

c) Let $M \geq 0$ and let $a, b \in Int(I)$ with $a < b$. Assume that there exists a fundamental system of solutions of $Lz \equiv 0$ (see a)), say $\{z_0, z_1\}$, which form an Extended Complete Tchebycheff System on (a, b) (see [9] for details) and consider a function $w \in C^k(I)$, bounded on I, such that for all $t \in (a, b)$

$$D^k w(t) = z_0(t) \int_c^t \frac{W(z_0, z_1)}{z_0^2}(\alpha) d\alpha \int_c^\alpha d\beta,$$

being $W(z_0, z_1)$ the Wronskian of z_0, z_1 and c a fixed point $a < c < b$. Then for $f \in C^k(I)$, bounded on I,

$$\lambda_n \left| D^k L_n f(x) - D^k f(x) \right| \leq M D^k(pD^2 w)(x) + o(1), \; x \in (a, b),$$

if and only if

$$\frac{1}{W(z_0, z_1)} \left(z_0 D^{k+1} f - D^1 z_0 D^k f \right) \in Lip_M 1 \text{ on } (a, b).$$

In the following two propositions we modify separately the aforementioned general hypotheses **A)** and **B)** and prove that a result as the one above holds.

Proposition 1 *Result 1 holds true if we keep on assuming* **A)** *and in* **B)** *we redefine the operator T to be $Tg := D^k(qD^1g + pD^2g)$ with p in the same conditions and $q \in C^k(I)$.*

Proof. It suffices to recover the proof of Result 1 in [4] and recall that the specific form of the asymptotic expression entered the picture just when we considered a function $\tilde{w}$ in order that $T\tilde{w}(x) = D^k(pD^2\tilde{w})(x)$ was a positive constant. Here we can consider $\tilde{w}$ such that for all $t \in (a, b)$,

$$D^1\tilde{w}(t) = e^{-\int_c^t \frac{q(z)}{p(z)}dz} \int_c^t \frac{e_k(x)}{p(x)} e^{\int_c^x \frac{q(z)}{p(z)}dz} dx.$$

Merely for the sake of completeness and thinking about applications, observe that the differential equation $Lz \equiv 0$ which appears in a) has the following form:

$$Lz = pD^2z + (q + kD^1p)D^1z + \left(\frac{k(k-1)D^2p}{2} + kDq\right) z \equiv 0.$$

Indeed, for $k \in \mathbb{N}$ the k-th convexity of the operators implies that for $i = 0, \ldots, k-1$ $D^kL_ne_i \equiv 0$ so from **B)** one has that $Te_i(t) = D^k(qD^1e_i + pD^2e_i)(t) = 0$ for all $t \in \mathrm{Int}(I)$. Then it suffices to consider the change $z = D^kg$ in the equation $Tg \equiv 0$. □

Proposition 2 *Result 1 holds true in the particular case $k = 2$ and $I = [I_1, +\infty)$ with $I_1 \in \mathbb{R}$, if we keep on assuming* **B)** *and, instead of the 2-convexity of the operators (see* **A)***), we assume that $D^2L_nf \geq 0$ whenever simultaneously $D^2f \geq 0$ and $D^1f \leq 0$.*

Proof. Notice that for $i = 0, 1$, $Te_i \equiv 0$, so a) is clear. Moreover, b) is a very direct consequence of **B)**. On the other hand, the proof of c) resembles closely the corresponding one in [4] but some steps concerning the use of the shape preserving properties of the operators must be modified, so we write it completely. We first state two lemmas which are proved at the end of the section. □

Lemma 1 *If $f \in C^2(I)$, bounded on I, satisfies that $D^1f \leq 0$ and $D^2f \geq 0$ on some neighborhood N_x of a point $x \in Int(I)$, then $D^2L_nf(x) + o(\lambda_n^{-1}) \geq 0$*

Lemma 2 *Let $g \in C^2(I)$, bounded on I. D^2g is convex on (a, b) with respect to z_0 and z_1 (see [9] for details) if and only if*

$$D^2L_ng(x) \geq D^2g(x) + o(\lambda_n^{-1}), \ x \in (a, b).$$

Now, we first observe that from **B)**

$$\lim_{n\to+\infty} \lambda_n \left(D^kL_nw(x) - D^kw(x)\right) = D^k \left(pD^2w\right)(x), \ x \in (a, b),$$

so applying Lemma 2 to the functions $Mw \pm f$, it is derived that

$$\lambda_n \left|D^2L_nf(x) - D^2f(x)\right| \leq MD^2(pD^2w)(x) + o(1), \ x \in (a, b),$$

if and only if $MD^2w \pm D^2f$ are convex on (a,b) with respect to z_0 and z_1.

On the other hand, if we consider a constant $\alpha \in \mathbb{R}$ such that $\tilde{z}_1 := z_1 + \alpha z_0$ verifies $\tilde{z}_1(c) = 0$, then $u_0 := z_0$, $u_1 := \tilde{z}_1$ and $u_2 := D^k w$ constitute an ECT-system formed from the functions $w_0 := z_0$, $w_1 := W(z_0, \tilde{z}_1)/z_0^2$ and $w_2 := e_0$ in the way that it is shown in Chap.XI of [5].

Finally, from Lemma 3.1 of [9], $MD^2w \pm D^2f$ are convex on (a,b) with respect to z_0 and z_1 if and only if D^2f belongs to the class $Lip_M 1$ with respect to u_0, u_1, u_2 (see [9] again for a detailed definition), or equivalently if and only if $\frac{1}{w_1} D^1 \left(\frac{1}{w_0} D^2 f \right) \in Lip_M 1$ (now in the classical sense), what ends the proof of c), Result 1 just observing that $W(z_0, z_1) = W(z_0, \tilde{z}_1)$. □

Proof of Lemma 1. First of all we observe that there exists another neighborhood of x, $\theta_x \subset N_x$, and a function $\tilde{f} \in C^2(I)$, bounded, with $D^1\tilde{f} \le 0$ and $D^2\tilde{f} \ge 0$ such that for all $t \in \theta_x$, $D^2f(t) = D^2\tilde{f}(t)$.

Indeed, it suffices to take any $x_1, x_2 \in N_x$, with $x_1 < x < x_2$, the choose any $x_0 \in N_x$, $x_2 < x_0$, satisfying $Df(x_2) \le \frac{x_2 - x_0}{2} D^2 f(x_2)$ and finally let $\theta_x = (x_1, x_2)$ and let $\tilde{f}$ be any function in $C^2(I)$, bounded on I such that

$$D\tilde{f}(t) = \begin{cases} Df(x_1) + D^2f(x_1)(t - x_1) & si\ 0 \le t \le x_1 \\ Df(t) & si\ x_1 < t \le x_2 \\ u(t) & si\ \ x_2 < t \le x_0 \\ v(t) & si\ \ x_0 < t \end{cases}$$

where

$$u(t) = \left(Df(x_2) - \frac{x_2^2 - 2x_0x_2}{2(x_2 - x_0)} D^2 f(x_2) \right) - \frac{x_0 D^2 f(x_2)}{x_2 - x_0} t + \frac{D^2 f(x_2)}{2(x_2 - x_0)} t^2$$

$$v(t) = \frac{(3x_0^2 Du(x_0))t - 2x_0^3 Du(x_0)}{t^3}$$

Finally we apply the shape preserving property to the function $\tilde{f}$, the part b) of the Result 1 to the function $f - \tilde{f}$ and the proof follows easily. □

Proof of Lemma 2. Let $x \in (a,b)$. If D^2g is convex on (a,b) with respect to z_0 and z_1, then there exist a solution of $Lz \equiv 0$, $z = z(t)$ say, such that $z(t) \le D^2g(t)$ for all $t \in (a,b)$ and $z(x) = D^2g(x)$. Now we take $y \in C^k(I)$, bounded on I, solution of $Tg \equiv 0$ such that $D^2y(t) = z(t)$ for all $t \in (a,b)$ and $D^1y(b) = D^1g(b)$. From Lemma 1

$$D^2 L_n y(x) \le D^2 L_n g(x) + o(\lambda_n^{-1}),$$

or equivalently

$$D^2 L_n y(x) - D^2 y(x) \le D^2 L_n g(x) - D^2 g(x) + o(\lambda_n^{-1}),$$

from which the result follows just applying b), Result1 to the function y.

For the converse, we denote by $S\left(h,t_1,t_2\right)$ the unique solution of $Lz\equiv 0$ which interpolates the function h at the points t_1 and t_2. Now, if we assume that D^2g is not convex on (a,b) with respect to z_0 and z_1, then there exist $a<t_1<x<t_2<b$ such that

$$S\left(D^2g,t_1,t_2\right)(x)<D^2g(x).$$

Now, given any $\tilde{w}\in C^2(I)$, bounded on I, we can find a positive constant $\epsilon>0$ such that

$$S\left(\epsilon D^2\tilde{w}+D^2g,t_1,t_2\right)(x)<\left(\epsilon D^2\tilde{w}+D^2g\right)(x).$$

Indeed, if $D^2\tilde{w}(x)-S\left(D^2\tilde{w},t_1,t_2\right)(x)\geq 0$ we can take any $\epsilon>0$, and otherwise we can take

$$0<\epsilon<\frac{D^2g(x)-S\left(D^2g,t_1,t_2\right)(x)}{S\left(D^2\tilde{w},t_1,t_2\right)(x)-D^2\tilde{w}(x)}.$$

Now, if we call $\tilde{z}$ a solution of $Lz\equiv 0$ which is strictly positive on (a,b) (its existence if guaranteed), then the function

$$\frac{\epsilon D^2\tilde{w}+D^2g-S\left(\epsilon D^2\tilde{w}+D^2g,t_1,t_2\right)}{\tilde{z}}$$

is continuous in $[t_1,t_2]$, it vanishes at the end points of this interval and it is strictly positive at the point x, so it reaches its maximun, say m, at a point $\tilde{x}\in(t_1,t_2)$.

Consequently $\left(\epsilon D^2\tilde{w}+D^2g\right)(\tilde{x})=\left(S\left(\epsilon D^2\tilde{w}+D^2g,t_1,t_2\right)+m\tilde{z}\right)(\tilde{x})$, and for all $t\in(t_1,t_2)$, $\left(\epsilon D^2\tilde{w}+D^2g\right)(t)\leq\left(S\left(\epsilon D^2\tilde{w}+D^2g,t_1,t_2\right)+m\tilde{z}\right)(t)$. Now we take $\tilde{y},s\in C^2(I)$, bounded on I, and a linear function r, solutions of $Tg\equiv 0$ on (t_1,t_2), such that on this interval $D^2s=S\left(\epsilon D^2\tilde{w}+D^2g,t_1,t_2\right)$, $D^2\tilde{y}=\tilde{z}$, and $s+m\tilde{y}-(\epsilon\tilde{w}+g)+r$ is a non increasing function. Then we apply Lemma 1 to obtain

$$\begin{aligned}&\epsilon D^2L_n\tilde{w}(\tilde{x})+D^2L_ng(\tilde{x})-\left(\epsilon D^2\tilde{w}(\tilde{x})+D^2g(\tilde{x})\right)\\ &\leq D^2L_ns(\tilde{x})+mD^2L_n\tilde{y}(\tilde{x})-\left(D^2s(\tilde{x})+mD^2\tilde{y}(\tilde{x})\right)+o(\lambda_n^{-1}),\end{aligned}$$

from which if we apply apply b), Result 1 to the functions $\tilde{y}$ and s we obtain

$$D^2L_ng(\tilde{x})-D^2g(\tilde{x})\leq-\epsilon\left(D^2L_n\tilde{w}(\tilde{x})-D^2\tilde{w}(\tilde{x})\right)+o(\lambda_n^{-1}).$$

Finally we get a contradiction if we choose for instance $\tilde{w}$ such that on (a,b) $D^2\tilde{w}=e_2/p$, apply **B)** to it, and recall the strict positivity of ϵ. □

3 Applications

In this section we shall apply the previous results to three well known sequences of operators. As we have already said we shall work with the Kantorovich operators, K_n, defined on the classical space $L_1[0,1]$ by

$$K_nf(t)=(n+1)\sum_{p=0}^{n}\binom{n}{p}t^p(1-t)^{n-p}\int_{\frac{p}{n+1}}^{\frac{p+1}{n+1}}f(z)dz,$$

and also with the generalized Szász-Mirakyan operators, $S_{v,n}$, $v \in \mathbb{N}_0$, and the Bleimann, Butzer and Hahn operators, H_n, defined both on $C[0,\infty)$ respectively as

$$S_{v,n}f(t) = e^{-(n+v)t} \sum_{p=0}^{\infty} f\left(\frac{p}{n}\right) \frac{(n+v)^p t^p}{p!},$$

$$H_n f(t) = (1+t)^{-n} \sum_{p=0}^{n} f\left(\frac{p}{n-p+1}\right) \binom{n}{p} t^p.$$

As far as the conservative properties that these operators possess, it is only too well-known that K_n and $S_{v,n}$ are k-convex for all $k \in \mathbb{N}_0$. When $v = 0$, $S_{v,n}$ are the already classical Szász-Mirakyan operators. In the sequel we do not consider this case as it was studied in [4]. H_n, as well as be 0-convex and 1-convex, they satisfy that $D^2 f \geq 0$ provided that simultaneously $D^2 f \geq 0$ and $D^1 f \leq 0$ (see [6]).

We shall apply Proposition 2, ii) to H_n and Proposition 1 to K_n and $S_{v,n}$, after showing of course that they posses an asymptotic expression of the required type, that in the more general form we reproduce here for the sake of clarity: 'there exist a sequence of real positive numbers λ_n and two functions, $p \in C^k(I)$ strictly positive on $\mathrm{Int}(I)$, and $q \in C^k(I)$' such that for all $g \in C^k(I)$, bounded on I and $k+2$-times differentiable in some neighborhood of $x \in \mathrm{Int}(I)$

$$\lambda_n(D^k L_n g(x) - D^k g(x)) \overset{n\to\infty}{\longrightarrow} D^k(qD^1 g + pD^2 g)(x). \tag{2}$$

In the following table one can read the interval I and the values of k, λ_n, $p = p(t)$ and $q = q(t)$ that make (2) be satisfied for each case. We also write the reference papers where the formulae were proved:

L_n	I	k	λ_n	$q(t)$	$p(t)$	References
H_n	$[0,\infty)$	$k=2$	$2n$	0	$t(1+t)^2$	[7]
K_n	$[0,1]$	$k=0,1,\dots$	$2(n+1)$	$1-2t$	$t(1-t)$	[2],[1],[7]
$S_{v,n}$	$[0,\infty)$	$k=0,1,\dots$	n	vt	$\frac{t}{2}$	[7]

Hence we are in a position to apply our results and obtain the so called saturation classes. We shall only take care of the application of claim c) since a) and b) can be seen as tools. With that purpose we present another table that summarizes the required information. It contains for each case a fundamental system of solutions of the reduced differential equation $Lz \equiv 0$, z_0 and z_1, that constitute an Extended Complete Tchebycheff System, and the values of $W(z_0, z_1)$ and $D^k(qD^1 w + pD^2 w)$ calculated maybe with the aid of a computer program. In the table it appears the function $\Gamma(t) = e^{-2vt} \int_{-2tv}^{+\infty} \frac{e^{-x}}{x} dx + \log t$.

L_n	$z_0(t)$	$z_1(t)$	$W(z_0,z_1)(t)$	$D^k(qD^1w+pD^2w)(t)$
H_n				
$k=2$	$\frac{1}{t(1+t)^2}$	$\frac{1}{(1+t)^2}$	$\frac{1}{t^2(1+t)^4}$	1
$S_{v,n}$				
$k=0$	1	$\frac{-e^{-2tv}}{2v}$	e^{-2tv}	$\frac{te^{-2tv}}{2}$
$k>0$	e^{-2tv}	$\frac{(-1)^{k+1}}{2(k-1)!v}D^k\Gamma(t)$	$\frac{e^{-2tv}}{t^k}$	$\frac{1}{2t^{k-1}}$
K_n				
$k=0$	1	$\log\frac{t}{1-t}$	$\frac{1}{t(1-t)}$	1
$k>0$	$\frac{1}{t^k}$	$\frac{(-1)^{k+1}}{k(1-t)^k}$	$\frac{1}{t^{k+1}(1-t)^{k+1}}$	$\frac{1}{(1-t)^k}$

Below we use the notation $e_i(t)=t^i$ and $\gamma(t)=e^{-2tv}$.

Corollary 1 *(H_n) Let $0<a<b$, then for $f\in C^2[0,\infty)$, bounded on $[0,\infty)$,*

$$2n\left|D^2H_nf(x)-D^2f(x)\right|\le M+o(1),\ x\in(a,b),$$

if and only if

$$e_1(e_0+e_1)^2D^3f+(e_0+4e_1+3e_2)D^2f\in Lip_M1\ \textit{on}\ (a,b).$$

Corollary 2 *($S_{v,n}$) Let $0<a<b$, then for $f\in C[0,\infty)$, bounded on $[0,\infty)$,*

$$n\left|S_{v,n}f(x)-f(x)\right|\le M\frac{xe^{-2vx}}{2}+o(1),\ x\in(a,b),$$

if and only if

$$\frac{1}{\gamma}D^1f\in Lip_M1\ \textit{on}\ (a,b),$$

and for $f\in C^k[0,\infty)$, $k\in\mathbb{N}$, bounded on $[0,\infty)$,

$$n\left|D^kS_{v,n}f(x)-D^kf(x)\right|\le M\frac{1}{2x^{k-1}}+o(1),\ x\in(a,b),$$

if and only if

$$e_kD^{k+1}f+2ve_kD^kf\in Lip_M1\ \textit{on}\ (a,b).$$

Corollary 3 *(K_n) Let $0<a<b<1$, then for $k\in\mathbb{N}_0$ and $f\in C^k[0,1]$,*

$$2(n+1)\left|D^kK_nf(x)-D^kf(x)\right|\le M\frac{1}{(1-x)^k}+o(1),\ x\in(a,b),$$

if and only if

$$\frac{e_1}{(e_0-e_1)^{k+1}}D^{k+1}f+k(e_0-e_1)^{k+1}D^kf\in Lip_M1\ \textit{on}\ (a,b).$$

Remark 2 In this paper we have not mentioned anything about the trivial classes for the saturation problems we are dealing with. For this matter we refer the reader to [4] where, roughly speaking, it was stated that these classes were formed for the spaces of solutions of the differential equation $Tg = D^k(qD^1g + pD^2g) \equiv 0$ which can be easily obtained from z_0 and z_1, solutions of the reduced equation $Lz \equiv 0$.

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