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Some remarks on the evaluation of linear recurrences*

R. Barrio[†], B. Melendo[‡] and S. Serrano[§]

Abstract

Some remarks on the numerical evaluation of recurrence relations are presented. Results concerning to rounding error bounds of the numerical scheme are given and the results are illustrated with some numerical examples. In particular, it is analyzed the case of perturbed Gegenbauer polynomials and the limit case of Jacobi-Sobolev polynomials. In these examples the theoretical bounds give sharp relative rounding error estimations.

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1 Introduction

Linear recurrences play a significant role in different areas of science. For example, they appear on the evaluation of classical orthogonal polynomials, some families of Sobolev type orthogonal polynomials, on the evaluation of special functions, on numerous numerical algorithms, and so on. Therefore, it is not strange that linear recurrences have been extensively studied and theoretical analysis and studies of the asymptotics appear frequently in the literature (see [11] and references herein).

When we want to evaluate numerically a linear recurrence we have several alternatives. For example, we can use directly the recurrence or try to obtain explicit solutions of the recurrence as it is done in [7]. This alternative is very useful in theoretical studies, but, as the solution involves the evaluation of products and sums with variable number of indexes, it is very difficult to program and very expensive computationally. So, numerical

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[†]GME, Departamento de Matemática Aplicada. Universidad de Zaragoza, E-50009 Zaragoza, Spain. E-mail: `rabarrio@posta.unizar.es`

[‡]GME, Depto. Matemática Aplicada, Centro Politécnico Superior, Universidad de Zaragoza, María de Luna 3, E–50015 Zaragoza, Spain. E-mail: `bmelendo@posta.unizar.es`

[§]GME, Depto. Matemática Aplicada, Centro Politécnico Superior, Universidad de Zaragoza, María de Luna 3, E–50015 Zaragoza, Spain. E-mail: `sserrano@posta.unizar.es`

evaluation of linear recurrence relations are usually done just by direct substitution on the recurrence. In [1] it was studied the numerical stability of the evaluation of the three-term recurrence relations that permit to evaluate finite linear combinations of classical orthogonal polynomials. In this paper we focus our attention to the numerical stability of the evaluation of general order linear recurrences by using the direct substitution algorithm. We present forward error bounds that give us sharp estimates of the behaviour of the linear recurrence. These bounds are illustrated in two examples that are taken from recent topics: perturbed Gegenbauer polynomials, Sobolev orthogonal polynomials.

2 Preliminaries

Let be a $(m+1)$-order linear recurrence relation

$$\begin{aligned} &l_0 = c_0, \qquad l_s = \sum_{i=1}^{s} a_{s,i}\, l_{s-i} + c_s, \quad s = 1\ldots, m-1 \\ &l_r = \sum_{i=1}^{m} a_{r,i}\, l_{r-i} + c_r, \quad r \geq m. \end{aligned} \tag{1}$$

It is interesting to remark that any polynomial series $\sum c_i\, p_i(x)$ where the polynomials $\{p_i(x)\}$ verify an homogeneous recurrence relation can be evaluated by means of another recurrence relation [3].

As one of the goals of the present paper is to study the numerical stability of the evaluation of linear recurrences, we introduce some basics of rounding error analysis. In the paper we assume that the computations are carried out in a floating–point arithmetic that obeys the models [5]

$$\mathrm{fl}(x \,\mathrm{op}\, y) = (x \,\mathrm{op}\, y)\,(1+\rho), \quad \mathrm{fl}(x \,\mathrm{op}\, y) = \frac{(x \,\mathrm{op}\, y)}{1+\alpha}, \qquad |\rho|\,,\, |\alpha| \leq u, \tag{2}$$

where $\mathrm{op} \in \{+,-,\times,\div\}$ and u is the unit roundoff. Also, we denote $\gamma_n := n\,u/(1-n\,u) = n\,u + \mathcal{O}(u^2)$ and we assume the notation $\widehat{a}$ and $\mathrm{fl}(a)$ for the computed value of a.

Let $R \in \mathbb{R}^{(n+1)\times(n+1)}$ be the upper triangular matrix:

$$R = \begin{pmatrix} 1 & -a_{n,1} & \ddots & -a_{n,m} & & & & & \\ & 1 & -a_{n-1,1} & \ddots & -a_{n-1,m} & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots & -a_{m,m} \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & 1 & -a_{1,1} \\ & & & & & & & & 1 \end{pmatrix},$$

then the algorithm is equivalent to solve the $(m+1)$ diagonal upper triangular linear system $R\boldsymbol{l} = \boldsymbol{c}$ where $\boldsymbol{l}, \boldsymbol{c} \in \mathbb{R}^{n+1}$ are the vectors $\boldsymbol{l}^{\mathrm{T}} = (l_n, l_{n-1}, \ldots, l_0)$ and $\boldsymbol{c}^{\mathrm{T}} = (c_n, c_{n-1}, \ldots, c_0)$. This matrix formulation of the recurrence relation (1) permits us to use classical rounding error bounds for linear systems.

A backward error bound can be easily obtained [5]:

Lemma *The computed value $\widehat{l}_n$ of the nth-term of the $(m+1)$-order linear recurrence relation (1) satisfies*

$$\widehat{l}_r = \sum_{i=1}^{m} (a_{r,i} + \delta a_{r,i})\, \widehat{l}_{r-i} + c_r, \quad r \geq m$$

where $|\delta a_{r,i}| \leq u \cdot m\, |a_{r,i}|$.

By using a matrix formulation the above result gives:

$$(R + \Delta R)\, \widehat{\boldsymbol{l}} = \boldsymbol{c}, \qquad |\Delta R| \leq u \cdot m\, |R|$$

Following the matrix approach it is possible to use the backward error bound in order to obtain forward error bounds for banded triangular linear systems [5], that is,

$$\frac{\|\boldsymbol{l} - \widehat{\boldsymbol{l}}\|_\infty}{\|\boldsymbol{l}\|_\infty} \leq \gamma_m\, \kappa(R), \tag{3}$$

where $\kappa(R) = \|\, R^{-1} \,\|\, \|\, R \,\|$ is the matrix condition number. This bound will give, in general, greater error bounds that the forthcoming analysis.

3 Forward error bound

By using a direct approach [1, 8] that considers the recurrence and not the matrix formulation it is possible to obtain sharper rounding error bounds:

Theorem *The error in the evaluation of the nth-term (l_n) of a $(m+1)$-order linear recurrence verifies*

$$|\widehat{l}_n - l_n| \leq u \cdot \sum_{s=0}^{n} \rho_s\, |c_s| + \mathcal{O}(u^2), \tag{4}$$

where

$$\begin{cases} \rho_0 = \displaystyle\sum_{j=1}^{n} \Delta_{j,0}\, |L_{n-j}| \\ \rho_s = (m+2)\, |L_{n-s}| + \displaystyle\sum_{j=s+1}^{n} \Delta_{j,s}\, |L_{n-j}|, \qquad \textit{for} \quad s = 1, \ldots, n-1 \\ \rho_n = (m+2)\, |L_0| = (m+2) \end{cases} \tag{5}$$

and, for $j = s+1, \ldots, n$,

$$\Delta_{j,s} = 2\, |r^{-1}_{n-j+1,n-s+1}| + \sum_{t=1}^{\min\{m-1,j-1\}} (m+2-t)\, |a_{j,t}|\, |r^{-1}_{n-j+t+1,n-s+1}|, \tag{6}$$

where $\{L_i\}$ are given by the "reverse" homogeneous linear recurrence

$$\begin{aligned} &L_0 = 1, \qquad L_s = \sum_{i=1}^{s} a_{n+i-s,i}\, L_{s-i}, \quad s = 1, \ldots, m-1 \\ &L_r = \sum_{i=1}^{m} a_{n+i-r,i}\, L_{r-i}, \quad r = m, \ldots, n. \end{aligned} \tag{7}$$

and $r_{j,s}^{-1}$ are the elements of R^{-1}:

$$R^{-1} = (r_{ij}^{-1}), \qquad r_{ij}^{-1} = \begin{cases} 0, & j < i, \\ 1, & j = i, \\ \displaystyle\sum_{t=1}^{\min\{m,n-i+1\}} a_{n-i+1,t} \cdot r_{i+t,j}^{-1}, & j > i. \end{cases} \tag{8}$$

PROOF See [2].

4 Numerical tests

4.1 Perturbed Gegenbauer polynomials

As a first trial problem we consider the 5th-order linear recurrence:

$$\begin{aligned} &p_0^\lambda(x) = 1, \qquad p_{-j}^\lambda(x) = 0, \quad j = 1, 2, 3, 4 \\ &p_i^\lambda(x) = a_{i,1}(x)\, p_{i-1}^\lambda(x) - a_{i,2}\, p_{i-2}^\lambda(x) + a_{i,3}\, p_{i-3}^\lambda(x) - a_{i,4}\, p_{i-4}^\lambda(x), \end{aligned}$$

where

$$a_{i,1}(x) = 2x\,\frac{i+\lambda-1}{i}, \quad a_{i,2} = \frac{i+2\lambda-2}{i}, \quad a_{i,3} = \frac{2}{i^2}, \quad a_{i,4} = \frac{2}{i^3},$$

that, as $\lim_{i\to\infty} a_{i,3} = 0$ and $\lim_{i\to\infty} a_{i,4} = 0$, we call *perturbed Gegenbauer polynomials* (in the case of $a_{i,3} = 0$ and $a_{i,4} = 0$ we obtain the three-term recurrence of the Gegenbauer orthogonal polynomials [6]). Taking into account such a recurrence we analyze the errors in the evaluation of the polynomial of degree n written as the linear combination of $\{p_i^\lambda(x)\}$ given by $p_n = \sum_{i=0}^{n} 1/(i+1)^2\, p_i^\lambda(x)$. For the evaluation we use an extension [3] of the Clenshaw's algorithm for the evaluation of linear combination of functions that follow a $(m+1)$-order recurrence.

First, just as an indicative of the behavior of the recurrence, we formulate the recurrence as the solution of the linear system

$$R_n^{\lambda,x}\, \boldsymbol{l} = \boldsymbol{c}$$

where $\boldsymbol{l} = (l_n, \ldots, l_0)^{\mathrm{T}}$ and $\boldsymbol{c}(1, \ldots, 1/(i+1)^2, \ldots, 1/(n+1)^2)^{\mathrm{T}} \in \mathbb{R}^{(n+1)}$ and $R_n^{\lambda,x} \in \mathbb{R}^{(n+1)\times(n+1)}$ and we plot in Figure 1 the condition numbers $\kappa(R_n^{\lambda,x})$ for several values of

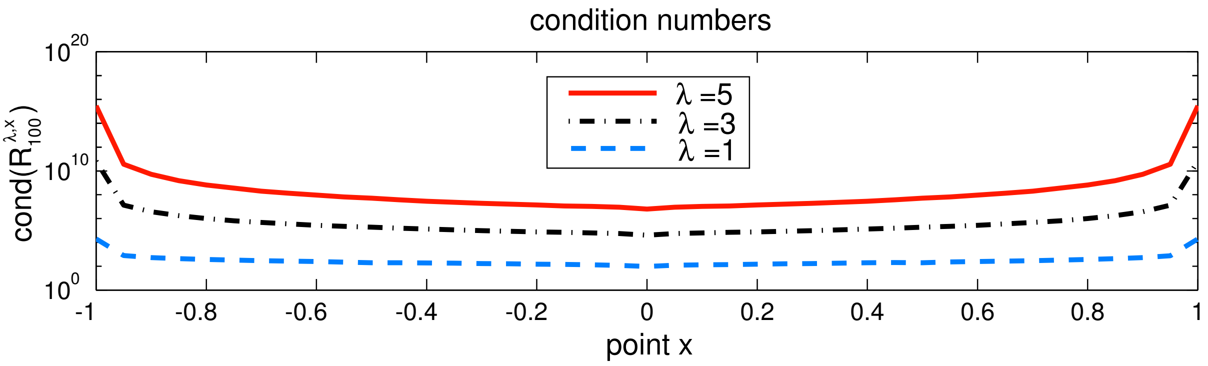

Figure 1: Condition numbers depending on x for some perturbed Gegenbauer recurrence matrices ($n = 100$).

the parameter λ along the interval of definition $[-1, 1]$. Besides, we show in Figure 2 the pseudospectra Λ_ϵ [10] of the recurrence matrices $R_n^{\lambda,x}$ with $n = 100$ for several values of the parameter λ of the polynomials, where

$$\Lambda_\epsilon(R_n^{\lambda,x}) = \{z \in \mathbb{C} \,:\, \|(z\,\mathbb{I} - R_n^{\lambda,x})^{-1}\| \geq \epsilon^{-1}\}.$$

In Figure 2, we observe that when λ grows the level curves also increase in size, giving instability problems.

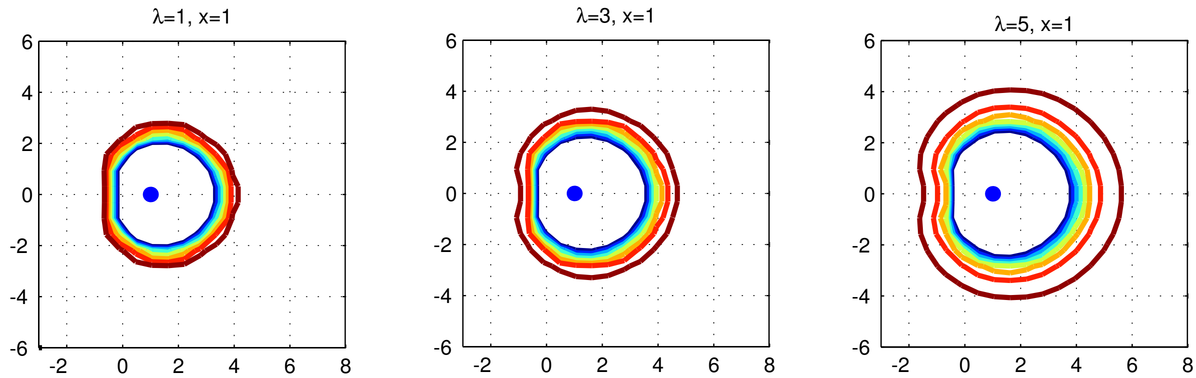

Figure 2: Pseudospectra of the recurrence matrices $R_{100}^{\lambda,x}$ for $x = 1$ depending on the value of the parameter λ.

In Table 1 we present the comparison among the bound given by the Theorem divided by $|p_n|$ and $u \cdot m \cdot \kappa(R_n^{\lambda,x})$. Although the condition numbers are quite large when λ grows and x is near ± 1, the results obtained from the new bound state that the relative errors are accurate enough. This result is not strange, in the non-perturbed case we have already obtained a similar behavior [1].

Table 1: Relative errors and relative error bounds in the evaluation of finite series of perturbed Gegenbauer polynomials. ($\mathrm{E}_{\mathrm{rel}}$ = relative error, new = new bound, bcond = bound based on the condition number $\kappa(R_{100}^{\lambda,x})$)

$n = 100$		$x = -1$	0	0.3	0.6	0.8	1
$\lambda = 1$	$\mathrm{E}_{\mathrm{rel}}$	$5.6 \cdot 10^{-17}$	$1.5 \cdot 10^{-17}$	$3.8 \cdot 10^{-17}$	$9.4 \cdot 10^{-17}$	$2.9 \cdot 10^{-16}$	$8.5 \cdot 10^{-16}$
	new	$9.1 \cdot 10^{-13}$	$3.2 \cdot 10^{-15}$	$4.7 \cdot 10^{-15}$	$6.1 \cdot 10^{-15}$	$8.7 \cdot 10^{-15}$	$8.9 \cdot 10^{-13}$
	bcond	$1.7 \cdot 10^{-11}$	$8.7 \cdot 10^{-14}$	$1.4 \cdot 10^{-13}$	$2.2 \cdot 10^{-13}$	$3.3 \cdot 10^{-13}$	$1.8 \cdot 10^{-11}$
$\lambda = 5$	$\mathrm{E}_{\mathrm{rel}}$	$4.2 \cdot 10^{-15}$	$1.2 \cdot 10^{-16}$	$2.8 \cdot 10^{-16}$	$8.0 \cdot 10^{-16}$	$3.3 \cdot 10^{-13}$	$8.1 \cdot 10^{-15}$
	new	$2.9 \cdot 10^{-11}$	$1.4 \cdot 10^{-12}$	$5.8 \cdot 10^{-12}$	$3.3 \cdot 10^{-12}$	$1.0 \cdot 10^{-10}$	$1.1 \cdot 10^{-12}$
	bcond	$2.5 \cdot 10^{+00}$	$5.5 \cdot 10^{-09}$	$1.7 \cdot 10^{-08}$	$8.2 \cdot 10^{-08}$	$5.8 \cdot 10^{-07}$	$2.6 \cdot 10^{+00}$

4.2 Jacobi-Sobolev orthogonal polynomials

During the last few years, there have been several papers written about polynomials orthogonal with respect to Sobolev inner products [4, 9]. Some of these inner products are of the form

$$(p, q) := \int_I p(x)\, q(x) \mathrm{d}\mu + \sum_{r=1}^{r_1} \alpha_r p(c_r)\, q(c_r) + \sum_{r=1}^{r_2} \beta_r p'(d_r)\, q'(d_r)$$

where I is some interval on the real line, $\alpha_r \geq 0$, $\beta_r \geq 0$ and c_r, d_r are fixed points (not necessarily in I). It is known that these families of polynomials follow $(2s+1)$-order recurrence relations. On our own, we only study the case of adding to the classical inner product the first derivative at $x = 1$, that is $(p, q) = \int_{-1}^{1} p(x)\, q(x) \mathrm{d}\mu + p'(1)\, q'(1)$, and only the case of Jacobi-Sobolev orthogonal polynomials (for more details see [9]). These polynomials are given by the 5-order recurrence:

$$\begin{aligned} &p_0(x) = 1, \qquad p_{-j}(x) = 0, \quad j = 1, 2, 3, 4 \\ &p_i(x) = a_{i,1}\, p_{i-1}(x) - a_{i,2}(x)\, p_{i-2}(x) + a_{i,3}\, p_{i-3}(x) - a_{i,4}\, p_{i-4}(x), \end{aligned}$$

where $\lim_{i\to\infty} a_{i,1} = 2$, $\lim_{i\to\infty} a_{i,2}(x) = (x-1)^2 - 3/2$, $\lim_{i\to\infty} a_{i,3} = 1/2$, and $\lim_{i\to\infty} a_{i,4} = -1/16$. As we are just interested on one test problem we have only considered the limit case (see [3] for more cases), that is:

$$a_{i,1} = 2, \quad a_{i,2}(x) = (x-1)^2 - \frac{3}{2}, \quad a_{i,3} = \frac{1}{2}, \quad a_{i,4} = -\frac{1}{16}.$$

In Table 2 we present the numerical tests comparing the current relative errors and the relative error bounds. The behavior of this limit case is quite extremal, giving very large values near $x = -1$ and low values near $x = 1$. These behavior is reproduced by the condition numbers, but the relative errors are acceptable. In the Figure 3 are represented these condition numbers.

Table 2: Limit Jacobi-Sobolev polynomials.

$n = 100$	$x = -1$	$x = 0$	$x = 0.3$	$x = 0.6$	$x = 0.8$	$x = 1$
$\mathrm{E}_{\mathrm{rel}}$	$1.9 \cdot 10^{-16}$	$8.7 \cdot 10^{-16}$	$2.9 \cdot 10^{-16}$	$1.6 \cdot 10^{-14}$	$9.2 \cdot 10^{-17}$	$2.1 \cdot 10^{-16}$
new	$3.9 \cdot 10^{-14}$	$9.8 \cdot 10^{-14}$	$1.9 \cdot 10^{-13}$	$4.7 \cdot 10^{-13}$	$1.5 \cdot 10^{-14}$	$9.5 \cdot 10^{-15}$
cond	$5.9 \cdot 10^{+31}$	$5.1 \cdot 10^{+12}$	$4.4 \cdot 10^{+4}$	$1.5 \cdot 10^{-6}$	$2.0 \cdot 10^{-13}$	$7.2 \cdot 10^{-14}$

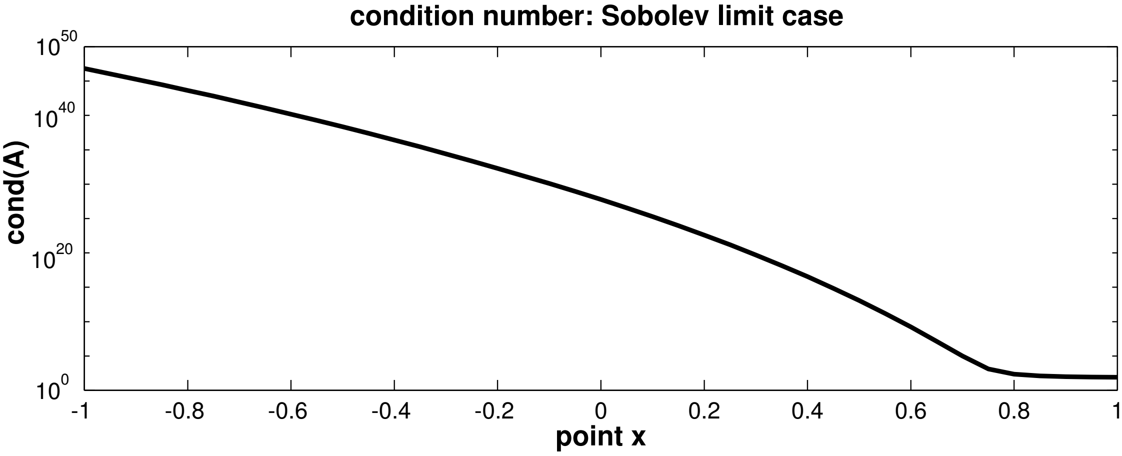

Figure 3: Condition number of recurrence matrices R^x_{100} depending on the point of evaluation $x \in [-1, 1]$.

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