Monografías del Semin. Matem. García de Galdeano. **27:** 65–71, (2003).

Limit Behaviour of Loss Networks

Andjel, E.[1], López, F.J.[2] and Sanz, G.[2]

[1] Laboratoire d'Analyse, Topologie et Probabilités. Université de Provence. Marseille.

[2]Departamento de Métodos Estadísticos, Universidad de Zaragoza.

Abstract

Resource sharing systems are used to model situations where several servers must complete jobs using some resources which must be shared with other servers. This kind of situations appear, for example, in computer and communications networks. In this paper we consider a Markovian class of processes of this type, loss networks, where the servers are communication stations arranged along a line which can be identified with $\mathbf{Z}$. For these processes, the uniqueness of equilibrium measure is obtained and, as a consequence, the ergodicity of loss networks with calls of finite length is proved.

Keywords: Loss networks, Resource sharing systems, Reversibility, Interacting particle systems.

AMS Classification: 60K35.

1 Introduction

Loss networks play an essential role in modeling and analysing complex systems as telephone networks, database structures or communications systems (Kelly [8]). Loss networks can be seen as K stations arranged along a cable. These stations must transmit calls along the cable. Each call requests a fraction C^{-1} of the cable between the origin station and the destination station of the call. The call is rejected, "lost", if past any point between both stations C calls are in progress. Calls arrive (from outside) to any station according to a Poisson process, independent for each station. Also, the completion times of each call are exponential times. All exponential times are taken independent. A description of the invariant measure can be seen in Kelly [7]. When the number of stations is countable and the length of each call (distance between origin and destination stations) is bounded, loss networks can be seen as resource sharing systems ([2], [4], [6]).

Our purpose in this work is to study the ergodicity of one-dimensional loss networks (stations arranged on $\mathbf{Z}$). The main difficult to analyse the ergodicity of these processes is that they are not attractive and, then, the usual stochastic comparison techniques through couplings cannot be used.

As it is said before all arrivals and completion times of the calls are taken independent, although the completion time of a call can depend on its length (bounded), but not on the origin and destination stations.

In this work we prove the ergodicity of one-dimensional loss networks. To get the ergodicity we first prove (Theorem 1) the existence of reversible measure (then invariant) for the loss network and, using a result of López [10], the uniqueness of that measure is obtained, under a mild positivity condition. Then (Theorem 3), using entropy arguments, the equivalence between invariant and reversible measures is proved and, finally the ergodicity of one-dimensional loss networks is obtained (Theorem 4).

2 Main results

Loss networks can be seen as Interacting Particle Systems on $X = W^{\mathbf{Z}}$, with $W = \{0, 1, \dots, C\}^k$, where the sites $x \in \mathbf{Z}$ denote the stations and the state of the station x in the configuration $\eta \in X$ (denoted by $\eta(x)$) will be a k-tuple whose ith component (denoted by $\eta(x, i)$) is the number of calls in progress from the station x to the station $x+i$ in the configuration η. Notations and basic results on Interacting Particle Systems can be seen in Liggett [10]. The evolution of the process is the following: each station $x \in \mathbf{Z}$, tries to call the station $x + i$ ($i = 1, \dots, k$) after an exponential time with parameter λ_i. The call is rejected (and lost) if past any point $x, \dots, x + i$ there are already C calls in progress. If the call is accepted, it lasts an exponential time with parameter δ_i. All the exponential times are taken independent. As it was said before, the rates are translation invariant, that is the arrival and completion times of a call only depend on its length, but not on the origin and destination stations; moreover its length is bounded, that is, there is a maximum number k such that no station tries to make a call to any station located to a distance bigger than k. Under these conditions (translation invariant rates and bounded length of calls) the process is well defined and as the state space is compact the process has at least an invariant measure (see chap. 1 in Liggett [9]).

We are going to prove that every one-dimensional loss network , under the aforementioned conditions is ergodic, that is the process has only one invariant measure and it converges to that measure from any initial measure. In Mountford [11] it is proved that every one-dimensional process with finite range and translation invariant rates having an unique invariant measure is ergodic. So, it suffices to prove that any loss network has only one invariant measure. We prove that the loss networks under consideration have one unique reversible measure and that every invariant measure is also reversible, and therefore the result will be proved.

First of all, we give an estimation of the invariant measures (later we will see that there is only one) of this process. For $\mu \in P(X)$ (the probability measures on X) and $\xi \in X_n = W^{\{-n,\ldots,n\}}$ or $\xi \in X$ let

$$\mu_n(\xi) = \mu\{\eta : \eta(x) = \xi(x),\ \forall |x| \leq n\};$$

we also define the configuration $\tilde{\xi}_n \in X$ as equal to ξ in $\{-n, \ldots, n\}$ and 0 outside.

We have the following

Theorem 1.- Let $\lambda_i,\ \delta_i$ the rates of a loss network with the aforementioned conditions and μ an invariant measure of the process. Then, there exists $\alpha > 0$ such that

$$\mu_{n+k}(\tilde{\xi}_n) \geq \alpha \mu_n(\xi) \tag{1}$$

for all $n \geq k$ and $\xi \in X_n$.

Before we give the sketch of the proof, let us comment the meaning of it. The event on the right side of the inequality means that there is a configuration of calls going out from $\{-n, \ldots, n\}$ to its right side, while the event on the left side of the inequality implies that, besides these calls, there is no call going out from the frontier points of $\{-n, \ldots, n\}$, that is the points $\{-n-k, \ldots, -n-1\}$ and $\{n+1, \ldots, n+k\}$. Then, writing the inequality in terms of conditional probabilities, the result asserts that, under the invariant measure, the conditional probability of no calls going out from the frontier points of $\{-n, \ldots, n\}$ given the calls going out from $\{-n, \ldots, n\}$ is bounded below by a constant which does not depend on n nor the number and type of the calls inside. This bound will be very useful when we use the relative entropy techniques.

Sketch of the proof. The proof is based in the construction of a coupling that proves the existence of $\alpha > 0$ such that, for all $n \geq k,\ \xi \in X_n$ and $\eta_0 \in X$,

$$P^{\eta_0}\big\{ \mathbb{1}_{\tilde{\xi}_n}^{n+k}(\eta_1) = 1 \big\} \geq \alpha P^{\eta_0}\big\{ \mathbb{1}_{\xi}^{n}(\eta_1) = 1 \big\}, \tag{2}$$

where P^{η_0} denotes the probability when the initial configuration is η_0, and η_1 denotes the process at time $t = 1$ and $\mathbb{1}_{\xi}^{m}(\eta)$ is the indicator function that takes the value 1 if the configuration η is equal to ξ in $\{-m, \ldots, m\}$. To see that this suffices to prove the theorem, note that (2) can be written as

$$S(1)\mathbb{1}_{\tilde{\xi}_n}^{n+k}(\eta_0) \geq \alpha S(1)\mathbb{1}_{\xi}^{n}(\eta_0),$$

where S is the semigroup of the process. As the inequality holds for all $\eta_0 \in X$, we have

$$\int S(1)\mathbb{1}_{\tilde{\xi}_n}^{n+k} d\mu \geq \alpha \int S(1)\mathbb{1}_{\xi}^{n} d\mu$$

which, if μ is invariant, is equivalent to (1).

¿From now on, let $n \geq k$, $\eta_0 \in X$ and $\xi \in X_n$ be fixed. The idea to prove (2), is to construct a coupled process, denoted $(\eta_t, \eta_t^{'})$, with the following properties:

a) Both components $(\eta_t, \eta_t^{'})$ are versions of the process under study.

b) Starting from (η_0, η_0) the conditional probability of $\{\mathbb{1}_{\tilde{\xi}_n}^{n+k}(\eta_1') = 1\}$ given $\{\mathbb{1}_{\xi}^{n}(\eta_1) = 1\}$ is bounded below by a constant α.

The construction of the coupling can be seen in [1].

Now, we are going to study the invariant measures that are also reversible for the loss network. A measure is reversible for the process with semigroup S if

$$\int fS(t)gd\mu = \int gS(t)fd\mu$$

for all continuous functions $f,\ g$ on X (Definition II.5.1 in Liggett [9]).

Although in the loss networks the transition rates of each particle depend on the situation of its k neighbours on the left and the right sides, we can make a transformation of the process such that the rates of each particle only depend on its left and right neighbour. That transformation is basically to define an equivalent process by considering $k-$dimensional particles with state space W^k, each one corresponding to k adjacent particles of the original process (that is, particle x of the collapsed process is the k tuple of particles $(kx, kx+1, \ldots, kx+(k-1)$ in the original process).

Theorem 2.- The loss network has only one reversible measure.

Sketch of the proof.- If we consider the collapsed process, the rates of the particles only depend on the left neighbour and on the right neighbour. In this case, Lemma 11.18 of Chen [3] can be extended from spin systems to our situation in such way that a measure μ is reversible for an interacting particle process on $\mathbf{Z}$ with rates $c_{ab}(x,\eta)$ (which represents the change between a and b of the particle x) iff

$$c_{ab}(x,\eta)\mu_n(\eta) = c_{ab}(x,\eta_{xab})\mu_n(\eta_{xab}) \qquad \forall\ a,b,|x|<n,$$

for all $n \geq 1$ and every configuration η admissible on X_n, where

$$\eta_{xab}(y) = \begin{cases} \eta(y) & if \quad y \neq x\ , \\ \eta(x) & if \quad y = x\ ,\ \eta(x) \neq a \quad and \quad \eta(x) \neq b \quad , \\ b & if \quad y = x \qquad\qquad\quad and \quad \eta(x) = a \quad , \\ a & if \quad y = x \qquad\qquad\quad and \quad \eta(x) = b \quad . \end{cases}$$

To prove the existence of a reversible measure for the process, it suffices to consider the process on $\{-n, \ldots, n\}$ with all particles outside staying in state 0. It is easy to see that this finite process is reversible, therefore the invariant measure satisfies the above equation. Taking in consideration the sequence of invariant measures for each n, as the space is compact, there is a convergent sub-sequence that also verifies the reversibility condition.

The next step is to prove that the reversible measure is unique. This is a consequence of Corollary 3.11 of López [10] that asserts that, any particle process on **Z** that verifies some positivity conditions has, at most, a reversible measure. The positivity conditions say, basically, that on the admissible configurations (in the loss networks the admissible configurations are all with no more than C calls past any point of the cable) the system has to be irreducible.

It is known that each reversible measure for a process is invariant for it, but the converse is not true. So, even if we had proved that the loss networks have only one reversible measure, it could exist other invariant measures that were not reversible. The following result asserts that this is not possible.

Theorem 3.- Each invariant measure of a loss network is reversible.

Sketch of the proof.- We use the relative entropy technique that has been used before for processes with two states and strictly positive rates as the Ising model (Holley and Stroock [5]). The fact that our processes have null rates and non-admissible configurations make the use of this technique harder than in other situations considered in the literature.

Given $n \geq 0$, $\xi \in E_n$, a probability measure μ, $a, b \in W$, $|x| \leq n$, let us define

$$\Gamma^n_{ab}(x, \xi) = \int c_{ab}(x, \eta) \mathbb{1}^n_\xi(\eta) d\mu.$$

¿From the discussion above on reversible measures, it is easy to see that a measure μ is reversible for the process iff $\Gamma^n_{ab}(x, \xi) = \Gamma^n_{ab}(x, \xi_{xab})$ for all $|x| \leq n, a, b \in W$ and ξ admissible on X_n.

Let π be the unique reversible measure of the process. To prove that π is the unique invariant measure, we will take μ invariant for the process and will prove that $\mu = \pi$, using the relative entropy of the marginals of μ respect to the marginals of π. Given a finite set S and two probability measures μ and π such as $\pi(x) > 0$ for all $x \in S$, the entropy of μ relative to π is defined as (definition II.4.1 of Liggett [9])

$$H(\mu) = \sum_{x \in S} \mu(x) \log \frac{\mu(x)}{\pi(x)}.$$

From the above definition, it can be proved that the following equality holds

$$\sum_{a,b} \sum_{\xi \in E_n} \sum_{x} (\Gamma^n_{ab}(x, \xi) - \Gamma^n_{ab}(x, \xi_{xab})) \log \frac{\Gamma^n_{ab}(x, \xi)}{\Gamma^n_{ab}(x, \xi_{xab})}$$

$$= \sum_{a,b} \sum_{\xi \in E_n} \sum_{x} (\Gamma^n_{ab}(x, \xi) - \Gamma^n_{ab}(x, \xi_{xab}))$$

$$\times \left(\log \frac{\pi_n(\xi)}{\pi_n(\xi_{xab})} + \log \frac{\Gamma^n_{ab}(x,\xi)}{\mu_n(\xi)} - \log \frac{\Gamma^n_{ab}(x,\xi_{xab})}{\mu_n(\xi_{xab})} \right),$$

where the summation of x is over all values where the change is allowed.

The next step of the proof is to bound the term on the right side. For doing that, we have that if there exists K such that

$$\frac{\int c_{ab}(x,\eta) \mathbb{1}^n_\xi(\eta) d\mu}{\int \mathbb{1}^n_\xi(\eta) d\mu} \geq K, \qquad \frac{\int c_{ab}(x,\eta) \mathbb{1}^n_\xi(\eta) d\pi}{\int \mathbb{1}^n_\xi(\eta) d\pi} \geq K$$

for all $n \geq 1$, $a, b \in W, \xi$ admissible and $|x| \leq n$:

$$\left| \log \frac{\pi_n(\xi)}{\pi_n(\xi_{xab})} + \log \frac{\Gamma^n_{ab}(x,\xi)}{\mu_n(\xi)} - \log \frac{\Gamma^n_{ab}(x,\xi_{xab})}{\mu_n(\xi_{xab})} \right|$$

is 0 if $|x| < n$ and less than γ if $|x| = n$.

Theorem 1 asserts that the condition is satisfied and then the above inequality holds. From that, the equality of functions Γ can be proved and the reversibility of μ is obtained and, therefore, the uniqueness of the invariant measure.

Finally, the main result is

Theorem 4.- One-dimensional loss networks are ergodic.

Proof.- From theorem 3 we get that our one-dimensional loss networks have only one invariant measure. Now, applying theorem 2 of Mountford [11], the result is proved.

Acknowledgements

The work of the second and third authors has been supported by project BFM-2001-2449 of MCYT and CONSI+D project P119/2001 of DGA.

References

[1] Andjel, E.; López, F.J. and Sanz, G. (2002). Ergodicity of one-dimensional resource sharing systems. *To appear in Stoch. Proc. Appl.*

[2] Bolch, G.; Greine, S.; de Meer, H. and Trivedi, K.S. (1998). Queueing networks and Markoc chains. Wiley. New York.

[3] Chen, M.F. (1992). From Markov chains to non-equilibrium particle Systems . World Scientific.

[4] Courcoubetis, C.A.; Reiman, M.I. and Simon, B. (1984). Stability of a concurrent service and locking. *SIAM J. Comp.* 16; 169-178.

[5] Holley, R.A. and Stroock D.W. (1977). In one and two dimensions, every stationary measure for a stochastic Ising model is a Gibss state. *Comm. Math. Phys.* 55; 37-45.

[6] Kelly, F. (1985). Stochastic models for computer communications systems. *J. Roy. Statis. Soc. Ser. B* 85; 379-395.

[7] Kelly, F. (1987). One-dimensional circuit-switched networks. *Ann. Probab.* 15; 1166-1179.

[8] Kelly, F. (1991). Loss networks. *Ann. Appl. Probab.* 1; 319-378.

[9] Liggett, T.M. (1985). Interacting particle systems. Springer. New York.

[10] López, F.J. (1988). Procesos de partículas interactivas con más de dos estados. Ph.D. Thesis. University of Zaragoza.

[11] Mountford, T.S. (1995). A coupling of infinite particle systems. *J. Math. Kyoto Univ.* 35; 43-52.