Study of the scalar Oseen equation

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Abstract

This paper is devoted to the scalar Oseen equation, a linearized form of the Navier-Stokes equations. Because of the various decay properties in various directions of $\mathbb{R}^N$, the problem is set in Sobolev spaces with anisotropic weights. In a first step, some weighted Hardy-type inequalities are obtained, which yield some norm equivalences. In a second step, we establish existence results.

Keywords: Oseen equations, anisotropic weights, Hardy inequality, Sobolev spaces, Exterior domains

AMS Classification: 76D05, 26D15, 46D35

1 Introduction.

Let $\Omega$ be an exterior domain of $\mathbb{R}^N$, $N \geq 2$. We consider the following system:

$$
\begin{cases}
-\nu \Delta u + \rho u_0 \cdot \nabla u + \nabla P = f & \text{in } \Omega \\
\text{div } u = 0 & \text{in } \Omega \\
u = u_* & \text{on } \partial \Omega \\
\lim_{|x| \to \infty} u(x) = u_\infty.
\end{cases}
$$

(1)

C. W. Oseen [7] obtained (1) by linearising the Navier-Stokes equations, describing the flow of a viscous and incompressible fluid past several obstacles, around a nonzero constant solution $u_0$. Thus, the result offers a better approximation than that of Stokes. The viscosity $\nu$, the density $\rho$, the external force $f$, and the boundary values $u_*$ on $\partial \Omega$ are given. The unknown velocity field $u$ is assumed to converge to a constant vector $u_\infty$, and the scalar $P$ denotes the unknown pressure. Among the works devoted to the system (1), which is called the Oseen equations, we can cite Finn [5], and more recently Farwig [4],
Galdi [6]. The purpose of this paper is to study a simplified case of \((1)\), the scalar Oseen equation:

\[
-\nu\Delta u + k \frac{\partial u}{\partial x_1} = f \quad \text{in } \mathbb{R}^N, \quad k > 0.
\]

To prescribe the growth or the decay properties of functions at infinity, the problem is set in weighted Sobolev spaces. Since the fundamental solution \(E(x)\) of \((2)\),

\[
E(x) = \frac{1}{4\pi \nu r} e^{-ks/2\nu}, \quad r = |x|, \quad s = r - x_1,
\]

has anisotropic decay properties, we will deal with the anisotropic weights introduced by Farwig [3, 4]. The case \(k = 0\) yields the Laplace’s equation studied by Amrouche-Girault-Giroire [1] in weighted Sobolev spaces. In a first step, we establish anisotropically weighted Poincaré-type inequalities and, in a second part, we present some existence results.

2 Notations

In this paper, we will use the following notations:

\[
r = r(x) = |x| = (x_1^2 + x_2^2 + \ldots + x_N^2)^{1/2}, \quad x \in \mathbb{R}^N
\]

\[
s = s(x) = r - x_1, \quad \rho = \rho(x) = (1 + r^2)^{1/2}.
\]

For the anisotropic weights, we set

\[
\eta_{\alpha}^\beta = (1 + r)^{\alpha/2}(1 + s)^{\beta/2}.
\]

We will use the following spaces, \(\alpha \in \mathbb{R}, \, 1 < p < +\infty\),

\[
W^{1,p}_\alpha(\Omega) = \{ v \in \mathcal{D}'(\Omega), \rho^{-1} v \in L^p(\Omega), \rho^\alpha \nabla v \in L^p(\Omega) \} \quad \text{if } n/p + \alpha \neq 1,
\]

with its natural norm

\[
\|v\|_{W^{1,p}_\alpha(\Omega)} = \left( \|\rho^{-1} v\|_{L^p(\Omega)} + \|\rho^\alpha \nabla v\|_{L^p(\Omega)} \right)^{1/p},
\]

and semi-norm

\[
|v|_{W^{1,p}_\alpha(\Omega)} = \|\rho^\alpha \nabla v\|_{L^p(\Omega)}.
\]

For the anisotropically weighted Sobolev spaces, we set

\[
H^{1,p}_{\alpha,\beta}(\Omega) = \{ v \in \mathcal{D}'(\Omega), \eta_{\beta}^{-1} v \in L^p(\Omega), \eta_{\beta}^\alpha \nabla v \in L^p(\Omega) \},
\]

\[
X^{1,p}_{\alpha,\beta}(\Omega) = \{ v \in \mathcal{D}'(\Omega), \eta_{\beta}^{-2} v \in L^p(\Omega), \eta_{\beta}^\alpha \nabla v \in L^p(\Omega) \},
\]

\[
W^{1,p}_{\alpha,\beta}(\Omega) = \{ v \in \mathcal{D}'(\Omega), \eta_{\beta}^{-1} v \in L^p(\Omega), \eta_{\beta}^\alpha \nabla v \in L^p(\Omega) \},
\]

\[
W_{\alpha,\beta}(\Omega) = \{ v \in W^{1,p}_{\alpha,\beta}(\Omega), v = 0 \text{ on } \partial \Omega \},
\]

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equipped with their natural norms.
The dual of \( W_{-\alpha,\beta}^{1,p}(\Omega) \) is noted \( W_{-\alpha,\beta}^{1,p'}(\Omega) \), with \( 1/p' + 1/p = 1 \). If \( \Omega = \mathbb{R}^N \), we have \( W_{-\alpha,\beta}^{1,p}(\Omega) = W_{\alpha,\beta}^{1,p}(\mathbb{R}^N) \).

Let \( j = \min\{[-1/2 - N/p - \alpha/2], [-1 - N/p - (\alpha + \beta)/2]\} \), we have \( P_j \subset H_{\alpha,\beta}^{1,p}(\Omega) \). \( P_j \) stands for the space of polynomials of degree lower than \( j \) and \([a]\) for the integer part of \( a \). We set \( B_R = B(0,R) \) and \( B_R' = \mathbb{R}^N \setminus B_R \). Finally, in what follows, by \( f \sim g \) in \( U \), we mean the following: there exists \( C_1, C_2 > 0 \), such that

\[
\forall x \in U, \quad C_1 f(x) \leq g(x) \leq C_2 f(x).
\]

3 Weighted Hardy-type inequalities.

A fundamental property of the weighted Sobolev spaces \( W_{-\alpha,\beta}^{1,p}(\Omega) \) is that their elements satisfy Hardy-type inequalities. Amrouche-Girault-Giroire [2] proved that, for \( \alpha, \beta \in \mathbb{R} \),

(i) the semi-norm \( \| \cdot \|_{W_{-\alpha,\beta}^{1,p}(\Omega)} \) defines on \( W_{-\alpha,\beta}^{1,2}(\Omega)/P_{j'} \) a norm which is equivalent to the quotient norm, where \( j' = \inf(j,0) \).

(ii) The semi-norm \( \| \cdot \|_{W_{-\alpha,\beta}^{1,p}(\Omega)} \) defines on \( W_{\alpha,\beta}(\Omega) \) a norm which is equivalent to the full norm \( \| \cdot \|_{W_{\alpha,\beta}^{1,p}(\Omega)} \).

We shall establish similar results in the case of anisotropically weighted Sobolev spaces.

We choose to consider the particular case \( N = 3, p = 2 \), but the results can be generalised to \( N \geq 2 \) and \( p \geq 2 \).

We consider the sector

\[
S = S_{R,\lambda} = \{ x \in \mathbb{R}^3; r \geq R, 0 \leq s \leq \lambda r \}, \quad R > 0, 0 < \lambda < 1.
\]  

(4)

In \( \mathbb{R}^3 \setminus S \), we have \( r \sim s \). Therefore, the spaces \( H_{\alpha,\beta}(\mathbb{R}^3 \setminus S) \) and \( W_{(\alpha+\beta)/2}^{1,2}(\mathbb{R}^3 \setminus S) \) coincide algebraically and topologically. It follows that, in \( \mathbb{R}^3 \setminus S \), the previous results hold. Thus, it is enough to prove anisotropically weighted Hardy-type inequalities in \( S \).

We first deal with the case \( \beta > 0 \).

**Lemma 1**  Let \( \alpha, \beta \in \mathbb{R} \) satisfy \( \beta > 0 \). Then there exists a constant \( C > 0 \), such that

\[
\forall u \in H_{\alpha,\beta}^{1,2}(S), \quad \| u \|_{H_{\alpha,\beta}^{1,2}(S)} \leq C |u|_{H_{\alpha,\beta}^{1,2}(S)}
\]

(5)

**Idea of the proof.** We first prove the inequality for \( u \in \mathcal{D}(S) \), then by density, we prove it for all \( u \) in \( H_{\alpha,\beta}(S) \). Since \( \beta > 0 \), it is enough to prove

\[
I = \int_S (1 + r)^{\alpha-1} s^{\beta-1} |u|^2 dx \leq C \int_S (1 + r)^{\alpha} s^{\beta} |\nabla u|^2 dx.
\]

(6)
Using polar coordinates with \( u(x) = v(r, \theta, \varphi) \), (6) is equivalent to the following inequality

\[
I = \int_0^{2\pi} \int_R^{+\infty} \int_0^{\theta_0} (1 + r)^{\alpha-1}(r - r \cos \theta)^{\beta-1}r^2 \sin \theta |v|^2 d\theta dr d\varphi \\
\leq C \int_0^{2\pi} \int_R^{+\infty} \int_0^{\theta_0} (1 + r)^{\alpha}(r - r \cos \theta)^{\beta} \sin \theta \frac{\partial v}{\partial \theta}^2 d\theta dr d\varphi,
\]

with

\( \theta_0 \) such that \( \cos \theta_0 = 1 - \lambda, \ 0 < \lambda < 1 \).

We set

\[
J = \int_0^{\theta_0} (1 - \cos \theta)^{\beta-1} \sin \theta |v|^2 d\theta.
\]

An integration by parts yields

\[
J = \frac{1}{\beta} [(1 - \cos \theta)^{\beta} |v|^2]_0^{\theta_0} - \frac{2}{\beta} \int_0^{\theta_0} (1 - \cos \theta)^{\beta} \frac{\partial v}{\partial \theta} |v| d\theta.
\]

Since \( \beta > 0 \) and \( v \in \mathcal{D}(S) \), we have

\[
J \leq \frac{2}{\beta} \int_0^{\theta_0} (1 - \cos \theta)^{\beta} \frac{\partial v}{\partial \theta} |v| d\theta.
\]

Using the Cauchy-Schwarz inequality, we get

\[
J \leq \frac{4}{\beta^2} \int_0^{\theta_0} (1 - \cos \theta)^{\beta+1} \frac{1}{\sin \theta} \frac{\partial v}{\partial \theta}^2 d\theta.
\]

This last inequality allows to have (7). \( \blacksquare \)

**Remark 2** Inequality (5) is not valid for \( \beta \leq 0 \). For \( \beta = 0 \), Farwig [3] gave a counter-example with the case \( \alpha = 0 \). For \( \beta < 0 \), taking as counter-example \( v(r, \theta, \varphi) = v(r) \), we can show that the inequality (7) does not hold.

Nevertheless, for \( \beta \leq 0 \), we have the analogue of Lemma 1 in the anisotropically weighted Sobolev space \( X^{1,2}_{\alpha,\beta}(S) \).

**Lemma 3** Let \( \alpha, \beta \in \mathbb{R} \) satisfy \( \beta \leq 0 \) and \( \alpha + \beta + 2 > 0 \). Then there exists \( C > 0 \), such that

\[
\forall u \in X^{1,2}_{\alpha,\beta}(S), \ |u|_{X^{1,2}_{\alpha,\beta}(S)} \leq C |u|_{X^{1,2}_{\alpha,\beta}(S)}.
\]

**Idea of the proof.** Let \( u \in \mathcal{D}(S) \) and \( u(x) = v(r, \theta, \varphi) \). For \( R > 0 \) sufficiently large, it is enough to prove

\[
I = \int_0^{2\pi} \int_R^{+\infty} \int_0^{\theta_0} r^{\alpha+1}(1 + r - r \cos \theta)^{\beta} \sin \theta |v|^2 d\theta dr d\varphi \\
\leq C \int_0^{2\pi} \int_R^{+\infty} \int_0^{\theta_0} r^{\alpha+3}(1 + r - r \cos \theta)^{\beta} \sin \theta \nabla u|^2 d\theta dr d\varphi.
\]
We set
\[ J = \int_{-\infty}^{+\infty} r^{\alpha+1} (1 + r - r \cos \theta)^{\beta} |v|^2 dr. \]

Since \( \beta \leq 0 \) and \( \alpha + \beta + 2 > 0 \), we have
\[ J \leq \frac{1}{\alpha + \beta + 2} \int_{-\infty}^{+\infty} \frac{\partial}{\partial r} [r^{\alpha+2} (1 + r - r \cos \theta)^{\beta}] |v|^2 dr. \]

An integration by parts and the Cauchy-Schwarz inequality yields
\[ J \leq \frac{4}{(\alpha + \beta + 2)^2} \int_{-\infty}^{+\infty} r^{\alpha+3} (1 + r - r \cos \theta)^{\beta} |\partial_r v|^2 dr, \]
which allows to obtain (8).

By Lemma 1, we have the two following results.

**Lemma 4** Let \( \alpha, \beta, R \in \mathbb{R} \) satisfy \( \beta > 0, \alpha + \beta + 1 \neq 0 \) and \( R > 0 \). Then, there exists a constant \( C_R > 0 \) such that
\[ \forall u \in H^{1,2}_{\alpha,\beta}(B'_R), \ |u|_{H^{1,2}_{\alpha,\beta}(B'_R)} \leq C_R |u|_{H^{1,2}_{\alpha,\beta}(B'_R)}. \]

In other words, the semi-norm \( |.|_{H^{1,2}_{\alpha,\beta}(B'_R)} \) is a norm on \( H^{1,2}_{\alpha,\beta}(B'_R) \) equivalent to the norm of \( H^{1,2}_{\alpha,\beta}(B'_R) \).

**Idea of the proof.** It is enough to consider \( u \in \mathcal{D}(B'_R) \). We use the following partition of unity
\[ \varphi_1, \varphi_2 \in C^\infty(B'_R), \ 0 \leq \varphi_1, \varphi_2 \leq 1, \ \varphi_1 + \varphi_2 = 1 \text{ in } B'_R, \]
with
\[ \varphi_1 = 1 \text{ in } S_{R,\lambda/2}, \ \text{supp}\varphi_1 \subset S_{R,\lambda}. \]

We have
\[ |u|_{H^{1,2}_{\alpha,\beta}(B'_R)} = \|\varphi_1 u + \varphi_2 u\|_{H^{1,2}_{\alpha,\beta}(B'_R)} \leq \|\varphi_1 u\|_{H^{1,2}_{\alpha,\beta}(B'_R)} + \|\varphi_2 u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}. \]

Since \( \beta > 0 \), Lemma 1 yields
\[ \|\varphi_1 u\|_{H^{1,2}_{\alpha,\beta}(B'_R)} = \|\varphi_1 u\|_{H^{1,2}_{\alpha,\beta}(S_{R,\lambda})} \leq C \|\varphi_1 u\|_{H^{1,2}_{\alpha,\beta}(S_{R,\lambda})} = C \|\varphi_1 u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}. \]

Since \( \alpha + \beta + 1 \neq 0 \), using the following Hardy-type inequality
\[ \forall v \in \mathcal{D}(R, +\infty), \ \int_{R}^{+\infty} (1 + t)^{\gamma} t^{\xi} |v(t)|^p dt \leq (\frac{\beta|\gamma + \xi + 1|}{c})^p \int_{R}^{+\infty} (1 + t)^{\gamma + p \xi} |v'(t)|^p dt \]
with \( \gamma, \xi, R \in \mathbb{R} \) such that \( \xi > 0, \gamma + \xi + 1 \neq 0 \) and \( (\gamma + \xi + 1)^2 R + \xi(\gamma + \xi + 1) > 0 \), we get
\[ |\varphi_1 u|_{H^{1,2}_{\alpha,\beta}(B'_R)} \leq C |u|_{H^{1,2}_{\alpha,\beta}(B'_R)}. \]
Thus, we have
\[ \| \varphi_1 u \|_{H^{1,2}_{\alpha,\beta}(B_R)} \leq C |u|_{H^{1,2}_{\alpha,\beta}(B_R)}, \]
and by the same method, we get
\[ \| \varphi_2 u \|_{H^{1,2}_{\alpha,\beta}(B_R)} \leq C |u|_{H^{1,2}_{\alpha,\beta}(B_R)}, \]
which conclude the proof. ■

**Theorem 5** Let \( \alpha, \beta \in \mathbb{R} \) satisfy \( \beta > 0 \) and \( \alpha + \beta + 1 \neq 0 \). Let \( j' = \inf(j, 0) \), where \( j \) is the highest degree of the polynomials contained in \( H^{1,2}_{\alpha,\beta}(\Omega) \). Then the semi-norm \( |\cdot|_{H^{1,2}_{\alpha,\beta}(\Omega)} \) defines on \( H^{1,2}_{\alpha,\beta}(\Omega)/\mathcal{P}_{j'} \) a norm which is equivalent to the quotient norm.

## 4 Weak solutions of the scalar Oseen equation.

In this section, we propose to solve the scalar Oseen equation with \( \nu = k = 1 \), \( N = 3 \):
\[
- \Delta u + \frac{\partial u}{\partial x_1} = f \quad \text{in} \quad \mathbb{R}^3. \quad (10)
\]

We introduce the concept of weak solution.

**Definition 6** A function \( u : \mathbb{R}^3 \to \mathbb{R} \) is called a weak solution to (10) if
(i) \( u \in H^{1,0}_{\text{loc}}(\mathbb{R}^3) \),
(ii) \( u \) satisfies
\[ \forall \varphi \in \mathcal{D}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \, dx - \int_{\mathbb{R}^3} u \frac{\partial \varphi}{\partial x_1} = [f, \varphi]. \quad (11) \]

We are, first, interested in existence of weak solutions when the data \( f \in W^{-1,2}_0(\mathbb{R}^3) \), which is the dual of \( W^{1,2}_0(\mathbb{R}^3) \).

**Theorem 7** Given a function \( f \in W^{-1,2}_0(\mathbb{R}^3) \), the problem (10) has a weak solution \( u \in W^{1,2}_0(\mathbb{R}^3) \) such that
\[
\| \nabla u \|_{L^2(\mathbb{R}^3)} \leq \| f \|_{W^{-1,2}_0(\mathbb{R}^3)}. \quad (12)
\]
More over
\[
\frac{\partial u}{\partial x_1} \in W^{-1,2}_0(\mathbb{R}^3). \quad (13)
\]

**Idea of the proof.** For \( R > 0 \), we consider the following equations
\[
\begin{cases}
- \Delta u + \frac{\partial u}{\partial x_1} = f \quad \text{in} \ B_R \\
\quad u = 0 \quad \text{on} \ \partial B_R,
\end{cases} \quad (14)
\]
Since \( f \in W^{-1,2}_0(\mathbb{R}^3) \), we have \( f \in H^{-1}(B_R) \), thus, by Lax-Milgram theorem, we prove

\[
\| \nabla u_R \|_{L^2(B_R)} \leq \| f \|_{W^{-1,2}_0(\mathbb{R}^3)},
\]

then, it suffices consider a sequence of problems analogous to (14) and to choose a weakly convergent subsequence.}

We now look for weak solutions when the data \( f \in W^{-1,2}_0(\mathbb{R}^3) \).

**Theorem 8** Let \( \alpha, \beta \in \mathbb{R} \) satisfy \( \beta > 0 \) and \( \beta > |\alpha| \). Then for a function \( f \in W^{-1,2}_{\alpha,\beta}(\mathbb{R}^3) \), there exists a weak solution \( u \in W^{1,2}_{\alpha,\beta}(\mathbb{R}^3) \) to (10) such that

\[
\| u \|_{W^{1,2}_{\alpha,\beta}(\mathbb{R}^3)} \leq C \| f \|_{W^{-1,2}_{\alpha,\beta}(\mathbb{R}^3)}.
\]

**Idea of the proof.** Let \( R > 0 \) be given and let \( u_R \in H^1_0(B_R) \) be the unique weak solution of (14). We need to prove the uniform estimate

\[
\| u_R \|_{W^{1,2}_{\alpha,\beta}(B_R)} \leq C \| f \|_{W^{-1,2}_{\alpha,\beta}(\mathbb{R}^3)},
\]

which allows to end the proof as in the previous Theorem. In the variationnal equation

\[
\forall \varphi \in H^1_0(B_R), \quad \int_{B_R} \nabla u_R \cdot \nabla \varphi \, dx + \int_{B_R} \frac{\partial u_R}{\partial x_1} \varphi \, dx = [f, \varphi],
\]

we use the test function \( \varphi = \eta^{2\alpha}_{2\beta} u_R \), thus, by an integration by parts, we get

\[
\int_{B_R} \eta^{2\alpha}_{2\beta} |\nabla u_R|^2 \, dx + \int_{B_R} u_R \nabla u_R \cdot \nabla \eta^{2\alpha}_{2\beta} - \frac{1}{2} \int_{B_R} |u_R|^2 \frac{\partial \eta^{2\alpha}_{2\beta}}{\partial x_1} \, dx = [f, \eta^{2\alpha}_{2\beta} u_R].
\]

The Young inequality implies that

\[
\int_{B_R} \eta^{2\alpha}_{2\beta} |\nabla u_R|^2 \, dx + \frac{1}{2} \int_{B_R} \left( - \frac{\partial \eta^{2\alpha}_{2\beta}}{\partial x_1} - \frac{|\nabla \eta^{2\alpha}_{2\beta}|^2}{\eta^{2\alpha}_{2\beta}} \right) |u_R|^2 \, dx \leq [f, \eta^{2\alpha}_{2\beta} u_R].
\]

Introducing the equivalent anisotropic weight functions

\[
\eta^{\alpha}_{\beta} = (1 + \delta r)^{\alpha/2} (1 + \varepsilon s)^{\beta/2}
\]

with sufficiently small positive constants \( \delta \) and \( \varepsilon \), Farwig [3] proved that if \( \alpha, \beta \in \mathbb{R} \) satisfy \( \beta > 0 \) and \( |\alpha| < \beta \), then there are positive numbers \( c_1(\delta, \varepsilon) = O(\delta) + O(\varepsilon) \), \( c_2(\delta) = O(\delta) \), such that

\[
- \frac{\partial \eta^{2\alpha}_{2\beta}}{\partial x_1} - \frac{|\nabla \eta^{2\alpha}_{2\beta}|^2}{\eta^{2\alpha}_{2\beta}} \geq ( (\beta - |\alpha|) - c_1(\delta, \varepsilon) ) \delta \varepsilon s(x) - c_2(\delta) \eta^{2\alpha - 2}_{2\beta}(x), \quad x \in \mathbb{R}^3.
\]

This result with Theorem 5 yield (17).
References


