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A Homogenization Result for Three-Phase Flow through Periodic Heterogeneous Porous Media

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Abstract

We give homogenization results for an immiscible and incompressible three-phase flow model in a heterogeneous petroleum reservoir with periodic structure, including capillary effects. We consider a model which leads to a coupled system of partial differential equations which includes an elliptic equation and two nonlinear degenerate parabolic equations of convection-diffusion types. Using two-scale convergence, we get an homogenized model which governs the global behavior of the flow. The determination of effective properties require the numerical resolution of local problems in a standard cell.

Keywords: Homogenization, Two-Scale Convergence, Three-Phase Flow, Porous Media.

AMS Classification: 35B27, 74Q15, 76M50.

1 Introduction

In problems involving displacement process of multiphase flow through a heterogeneous porous medium, it is often desirable replace the complicated model with an effective model, which gives the global behavior and, allow in numerical reservoir simulation, to disconnect the numerical mesh size from the heterogeneities size in the reservoir itself. Homogenization techniques allow, under some assumptions, to replace problems involving two (or more) very different scales by one macroscopic problem describing the global behavior.

In this paper, we are investigating displacement process of incompressible and immiscible three-phase flow in heterogeneous porous media, including capillary effects. The equations governing these types of flow can be effectively rewritten in a fractional flow formulation; i.e., in terms of a global pressure and saturations as the primary variables. This formulation leads to a coupled system of partial differential equations which includes two nonlinear degenerate parabolic equations of convection-diffusion types and an elliptic equation. The reservoir is assumed to be made of uniformly periodically repeated cells. Each cell being made with different types of porous media, where the porosity, the absolute permeability tensor and the relative permeabilities characterize a porous media type. Using two-scale convergence, we get an homogenized model which governs the global behavior of the flow. The resulting equations are of the same type that the points equations, with effective coefficients. The method allows the determination of these effective parameters from a knowledge of the geometrical structure of the basic cell and its heterogeneities. Numerical computations to obtain the homogenized coefficients of the entire reservoir have been carried out via a finite element method [2]. The extension of the present approach to a model of three-phase flow in a porous medium made of different rock types is described in details in [7]. These types of problems have been addressed by various authors in the field of petroleum engineering (see [6] and the references therein). The outline of the remainder of the paper is as follows. Section 2 contains a short description of the mathematical and physical model used in this study. In section 3, the formulation of a weak solution of the problem is presented. Section 4 is devoted to the presentation of a homogenization result. Here, we have used only the recently developed notion of two-scale convergence. Lastly, some concluding remarks are forwarded.

2 Mathematical Model

We consider saturated, three-phase, incompressible, immiscible flow, the phases being w (water or wetting phase), o (oil or nonwetting phase) and g (gas). We consider the reservoir $\Omega \subset \mathbb{R}^d$ $(1 \leq d \leq 3)$ to be a bounded, connected domain with a periodic structure. More precisely, we shall scale this periodic structure by a parameter ε which represents the ratio of the cell size to the size of the whole region Ω and we assume that $0 < \varepsilon << 1$ in a decreasing sequence tending to zero. For sake of simplicity, without loss of generality, let $Y =]0,1[^d \subset \mathbb{R}^d$ $(1 \leq d \leq 3)$ representing the microscopic domain of the basic cell. The boundary Γ of the domain Ω splits up into three parts such that

$$\overline{\Gamma} = \overline{\Gamma}_e \cup \overline{\Gamma}_i \cup \overline{\Gamma}_s, \quad \overline{\Gamma}_l \cap \overline{\Gamma}_m = \emptyset, \quad l \neq m.$$

$\overline{\Gamma}_e$ is the part of the boundary where the water is injected, $\overline{\Gamma}_i$ is the impervious part of the boundary $\overline{\Gamma}_s$ is the producing part of the boundary. Let $]0,T[$ denotes the time period,

$$\Sigma_j = \Gamma_j \times]0,T[, \quad j = e, s, i.$$

and $\vec{n}$ the outward normal to Γ. Assume that in a such geometrical configuration of the reservoir, porosity and absolute permeability tensor depend on the microscopic variable

$y = x/\varepsilon$ where x is the macroscopic scale. Namely,

$$\Phi^\varepsilon(x) := \Phi\left(\frac{x}{\varepsilon}\right) = \Phi(y) \quad \text{and} \quad K^\varepsilon(x) := K\left(\frac{x}{\varepsilon}\right) = K(y)$$

with Φ and K are Y-periodic functions on y.

In the sequel, we will use a formulation obtained after transformation using the concept of global pressure (see [4]). For sake of simplicity, without loss of generality, we neglect the effect of gravity. The main unknowns are p^ε: the global pressure, $\vec{V_t^\varepsilon}$: the total velocity, p^ε_{cw}: the capillary pressure between water and oil, and p^ε_{cg}: the capillary pressure between gas and oil. We consider the following equations and boundary and initial conditions.

Pressure equation:

$$\operatorname{div} \vec{V_t^\varepsilon} = 0, \tag{1}$$

$$\vec{V_t^\varepsilon} = -K^\varepsilon M^\varepsilon \nabla p^\varepsilon. \tag{2}$$

Saturation equations:

$$\partial_t \left[\Phi^\varepsilon(x) s_w(p_c^\varepsilon)\right] + \operatorname{div}\left[\upsilon_w^\varepsilon \vec{V_t^\varepsilon} - K^\varepsilon(\overline{M}_g^\varepsilon \nabla p_{cw}^\varepsilon - \overline{M}_o^\varepsilon(\nabla p_{cg}^\varepsilon - \nabla p_{cw}^\varepsilon))\right] = 0, \tag{3}$$

$$\partial_t \left[\Phi^\varepsilon(x) s_g(p_c^\varepsilon)\right] + \operatorname{div}\left[\upsilon_g^\varepsilon \vec{V_t^\varepsilon} - K^\varepsilon(\overline{M}_w^\varepsilon \nabla p_{cg}^\varepsilon - \overline{M}_o^\varepsilon(\nabla p_{cw}^\varepsilon - \nabla p_{cg}^\varepsilon))\right] = 0. \tag{4}$$

Boundary and initial conditions:

$$\begin{cases} \vec{V_t^\varepsilon} \,.\, \vec{n} = \vec{V_w^\varepsilon} \,.\, \vec{n} = -q_e, \ \vec{V_g^\varepsilon} \,.\, \vec{n} = \vec{V_o^\varepsilon} \,.\, \vec{n} = 0 \ \text{ on } \ \Sigma_e, \\ \hfill p_{cw}^\varepsilon = 0 \ p_{cg}^\varepsilon = 0 \ \text{ on } \ \Sigma_e. \end{cases} \tag{5}$$

$$\vec{V_t^\varepsilon} \,.\, \vec{n} = \vec{V_w^\varepsilon} \,.\, \vec{n} = \vec{V_g^\varepsilon} \,.\, \vec{n} = \vec{V_o^\varepsilon} \,.\, \vec{n} = 0 \ \text{ on } \ \Sigma_i. \tag{6}$$

$$\begin{cases} \hfill \vec{V_t^\varepsilon} \,.\, \vec{n} = q_s \ \text{ on } \ \Sigma_s, \\ (\upsilon_o^\varepsilon + \upsilon_g^\varepsilon)\upsilon_w^\varepsilon \nabla p_{cw}^\varepsilon \,.\, \vec{n} - \upsilon_g^\varepsilon \upsilon_w^\varepsilon \nabla p_{cg}^\varepsilon \,.\, \vec{n} = 0, \ \text{ on } \ \Sigma, \\ (\upsilon_o^\varepsilon + \upsilon_w^\varepsilon)\upsilon_g^\varepsilon \nabla p_{cg}^\varepsilon \,.\, \vec{n} - \upsilon_g^\varepsilon \upsilon_w^\varepsilon \nabla p_{cw}^\varepsilon \,.\, \vec{n} = 0 \ \text{ on } \ \Sigma \end{cases} \tag{7}$$

$$\int_{\Gamma_e} q_e \, d\gamma_e - \int_{\Gamma_s} q_s \, d\gamma_s = 0. \tag{8}$$

$$\begin{cases} s_w^{\varepsilon,0}(x) = s_w^\varepsilon(x,0) = s_w(p_{cw}^{\varepsilon,0}, p_{cg}^{\varepsilon,0}), \ \text{ with } \ 0 \le s_w^{\varepsilon,0} \le 1 \ \text{ a.e. in } \Omega, \\ s_g^{\varepsilon,0}(x) = s_g^\varepsilon(x,0) = s_g(p_{cw}^{\varepsilon,0}, p_{cg}^{\varepsilon,0}), \ \text{ with } \ 0 \le s_g^{\varepsilon,0} \le 1 \text{ a.e. in } \Omega, \\ \hfill s_w^{\varepsilon,0} + s_g^{\varepsilon,0} \le 1. \end{cases} \tag{9}$$

where

s_η is the saturation of the phase $\eta = w, o, g$;

$\vec{V_\eta^\varepsilon}$ is the velocity of the phase $\eta = w, o, g$,

M^ε is the total mobility,

$\overline{M}_\eta^\varepsilon$ is the mobility of the phase $\eta = w, o, g$,

$\upsilon_\eta^\varepsilon$ is the fractional flow of the phase $\eta = w, o, g$,

$p_c^\varepsilon := (p_{cw}^\varepsilon, p_{cg}^\varepsilon)$.

3 A Weak Formulation of the Problem

We define certain functions spaces and notations. Let

$$\mathcal{V} = \{ \ u \in H^1(\Omega); \ u = 0 \text{ on } \Gamma_e \ \}, \quad I\!H_0 = \{ \ \vec{V} \in (L^2(\Omega))^3; \ \text{div} \ \vec{V} = 0\},$$

$$I\!H = \{\vec{V} \in I\!H_0, \ \vec{V} \ . \ \vec{n} | \ \Gamma_e = -q_e, \ \vec{V} \ . \ \vec{n} | \ \Gamma_i = 0, \ \vec{V} \ . \ \vec{n} | \ \Gamma_s = q_s \ \},$$

and

$$\top = \{(s_w, s_g) \in L^\infty(\Omega) \times L^\infty(\Omega); 0 \leq s_w \leq 1, \ 0 \leq s_g \leq 1 \text{ and } s_w + s_g \leq 1 \text{ a.e. in } \ \Omega\}.$$

Under some reasonable assumptions, some of which are more general than is needed for the physical problem (cf. [5], [7]), we associate to (1)-(9), the following (P_ε) problem in a weak formulation:

$$(s_w^\varepsilon, s_g^\varepsilon) \in \top, \ \ \partial_t \left(\Phi^\varepsilon(x) s_\eta(p_c^\varepsilon)\right) \in L^2 \left(]0, T[; \ \mathcal{V}'\right), \ \eta = w, g,$$

$$(p_{cw}^\varepsilon, p_{cg}^\varepsilon) \in \Big(L^\infty(]0, T[\times\Omega) \bigcap L^2(]0, T[; \ \mathcal{V})\Big)^2,$$

$$\vec{V_t^\varepsilon} \in L^\infty(]0, T[; \ I\!H), \ \ p_\varepsilon \in L^\infty(]0, T[; \ H^1(\Omega)/I\!R),$$

such that $\forall \ (u, v) \in \left(L^2(]0, T[; \ \mathcal{V})\right)^2$:

$$\int_0^T \langle \partial_t \left(\Phi^\varepsilon \left(x\right) s_w(p_c^\varepsilon)\right), \ u\rangle dt - \int_0^T \int_\Omega \upsilon_w(s^\varepsilon) \ \vec{V_t^\varepsilon} \ .\nabla u dx dt$$

$$+ \int_0^T \int_\Omega K^\varepsilon \left(x\right) \overline{M}_g(s^\varepsilon) \nabla p_{cw}^\varepsilon . \nabla u dx dt$$

$$+ \int_0^T \int_\Omega K^\varepsilon \left(x\right) \overline{M}_o(s^\varepsilon) \left(\nabla p_{cw}^\varepsilon - \nabla p_{cg}^\varepsilon\right) . \nabla u dx dt = - \int_0^T \int_{\Gamma_s} \upsilon_w(s^\varepsilon) q_s u d\gamma_s dt, \tag{10}$$

$$\int_0^T \langle \partial_t \left(\Phi^\varepsilon \left(x\right) s_g(p_c^\varepsilon)\right), \ v\rangle dt - \int_0^T \int_\Omega \upsilon_g(s^\varepsilon) \ \vec{V_t^\varepsilon} \ .\nabla v dx dt$$

$$+ \int_0^T \int_\Omega K^\varepsilon \left(x\right) \overline{M}_w(s^\varepsilon) \nabla p_{cg}^\varepsilon . \nabla v dx dt$$

$$+ \int_0^T \int_\Omega K^\varepsilon \left(x\right) \overline{M}_o(s^\varepsilon) \left(\nabla p_{cg}^\varepsilon - \nabla p_{cw}^\varepsilon\right) . \nabla v dx dt = - \int_0^T \int_{\Gamma_s} \upsilon_g(s^\varepsilon) q_s v d\gamma_s dt \tag{11}$$

and $\forall \ w \in H^1(\Omega)/I\!R$:

$$\int_\Omega K^\varepsilon \left(x\right) M(s^\varepsilon) \nabla p_\varepsilon . \nabla w dx = \int_{\Gamma_e} q_e w d\gamma_e - \int_{\Gamma_s} q_s w d\gamma_s. \tag{12}$$

For the existence and uniqueness of the solution of this problem, we refer to [5].

4 A Homogenization Result

In this section, we give a homogenization result for the problem (P_ε). The convergence of the homogenization process is obtained by the technique of two-scale convergence [1].

Let $(p^\varepsilon_{cw}, p^\varepsilon_{cg}, \vec{V^\varepsilon_t}, p_\varepsilon)$ *be the solution of the problem* (P_ε) *then the following result holds:*

1) $(p^\varepsilon_{cw}, p^\varepsilon_{cg})$ *converge in* $(L^2(]0,T[;\ H^1(\Omega)))^2$ *weakly towards* (p^0_{cw}, p^0_{cg}).

2) $(s^\varepsilon_w, s^\varepsilon_g)$ *converge in* $(L^2(Q_T))^2$ *strongly towards* $(s^0_w, s^0_g) := s^0$.

3) $\vec{V^\varepsilon_t}$ *converge in* $(L^2(Q_T))^d$ *weakly towards* $\vec{V^0_t}$.

4) $p^\varepsilon(.,t)$ *converge towards* $p_0(.,t)$ *in* $L^2(\Omega)$ *strongly a.e. in* t.

5) $(p^0_{cw}, p^0_{cg}, \vec{V^0_t}, p_0)$ *is the solution of the following homogenized problem* (P_0)*:*

$$(s^0_w, s^0_g) \in \top, \quad \partial_t(\Phi^* s^0_\eta) \in L^2(]0,T[;\ \mathcal{V}'),\ \eta = w, g,$$

$$(p^0_{cw}, p^0_{cg}) \in \left(L^\infty(]0,T[\times\Omega) \bigcap L^2(]0,T[;\ \mathcal{V})\right)^2,$$

$$\vec{V^0_t} \in L^\infty(]0,T[;\ I\!\!H), \quad p_0 \in L^\infty(]0,T[;\ H^1(\Omega)/I\!\!R),$$

such that $\forall\ (u,v) \in (L^2(]0,T[;\ \mathcal{V}))^2$:

$$\begin{aligned}
&\int_0^T \langle \partial_t(\Phi^* s^0_w),\ u\rangle dt - \int_0^T \int_\Omega \upsilon_w(s^0)\ \vec{V^0_t}.\nabla u dx dt \\
&+ \int_0^T \int_\Omega K^* \overline{M}_g(s^0) \nabla p^0_{cw}.\nabla u dx dt \\
&+ \int_0^T \int_\Omega K^* \overline{M}_o(s^0) \left(\nabla p^0_{cw} - \nabla p^0_{cg}\right).\nabla u dx dt = -\int_0^T \int_{\Gamma_s} \upsilon_w(s^0) q_s u d\gamma_s dt
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\int_0^T \langle \partial_t(\Phi^* s^0_g),\ v\rangle dt - \int_0^T \int_\Omega \upsilon_g(s^0)\ \vec{V^0_t}.\nabla v dx dt \\
&+ \int_0^T \int_\Omega K^* \overline{M}_w(s^0) \nabla p^0_{cg}.\nabla v dx dt \\
&+ \int_0^T \int_\Omega K^* \overline{M}_o(s^0) \left(\nabla p^0_{cg} - \nabla p^0_{cw}\right).\nabla v dx dt = -\int_0^T \int_{\Gamma_s} \upsilon_g(s^0) q_s v d\gamma_s dt
\end{aligned} \tag{14}$$

and $\forall\ w \in H^1(\Omega)/I\!\!R$:

$$\int_\Omega K^* M(s^0) \nabla p_0.\nabla w dx = \int_{\Gamma_e} q_e w d\gamma_e - \int_{\Gamma_s} q_s w d\gamma_s, \tag{15}$$

$$\vec{V^0_t} = -K^* M(s^0) \nabla p_0, \tag{16}$$

where K^* represents the effective absolute permeability tensor given by

$$(K^*)_{ij} = \int_Y \left[K_{ij}(y) + K_{ik}(y) \frac{\partial w_j}{\partial y_k}(y)\right] dy \quad 1 \le i, j \le d, \tag{17}$$

with w_j, $1 \le j \le d$, is the solution of the so-called local or cell problem defined by

$$w_j \in H^1_p(Y)/I\!\!R,$$

$$\int_Y K(y)\nabla_y w_j \nabla_y \varphi \, dy = -\int_Y K_{ij}(y)\frac{\partial \varphi}{\partial y_i} dy, \ \forall \, \varphi \in H_p^1(Y) \tag{18}$$

and

$$\Phi^* = \int_Y \Phi(y)dy. \tag{19}$$

The proof of this result (for details see [7]) is based on a priori estimates and two-scale convergence results (cf. [1] see also [3]). This result could also be obtained formally by the technique of two-scale asymptotic expansions [2].

It can be clearly seen that the effective permeability tensor is symmetric and positive definite but in general cases, even with each type of porous medium being isotropic, we may have non-diagonal tensor. The determination of the effective coefficients requires the knowledge of the functions w_j appearing in the local problems (18). Numerical computations for the two dimensional case using a finite element method have been performed to solve the local problems and obtain the effective permeability tensor [2].

5 Conclusion

In this paper we have presented a model of three-phase flow in a heterogeneous porous medium where the porosity and the absolute permeability tensor characterize the porous medium type. We have presented a weak formulation of the coupled system and finally, we have presented a homogenization result for this problem. The extension of the result to a model with different rock types is presented in [7] following the ideas developed in [3].

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