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Anti-maximum principle for cooperative system involving Schrödinger operator in $I\!R^N$

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Abstract

We obtain a result concerning the anti-maximum principle for weak solutions $U = (u, v) \in I\!R^N$ of the following cooperative elliptic system:

$$(S) \begin{cases} \mathcal{A}u(x) := (-\Delta + q(|x|))u(x) = \lambda u(x) + bv(x) + f(x) \\ \mathcal{A}u(x) := (-\Delta + q(|x|))v(x) = cu(x) + \lambda v(x) + g(x) \\ u(x) \to 0, \quad v(x) \to 0, \quad \text{as } |x| \to +\infty. \end{cases}$$

Here $\mathcal{A} = -\Delta + q(x)$ in $L^2(I\!R^2)$ is the Schrödinger operator. We assume that the potential $q(x) \equiv q(|x|)$, is strictly positive, locally bounded, and has superquadratic growth as $|x| \to \infty$; b, c are strictly positive constants. We show that there exists a simple eigenvalue Λ_1 of (S) with positive eigenfunction Φ_1. Then we prove an anti-maximum principle in the following form: Let f, $g \in L^2(I\!R^2)$ be positive functions which are "sufficiently smooth" perturbations of a radially symmetric function, then there exists $\delta = \delta(f, g, b, c) > 0$ such that for $\lambda \in (\Lambda_1, \Lambda_1 + \delta)$ the weak solution U satisfies $U = (u, v) \leq -C\Phi_1$ where $C = C(f, g, \lambda)$ is a positive constant.

Keywords: Schrödinger operator, maximum principle, anti-maximum principle.
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1 Schrödinger equation in $I\!R^N$

We first recall recent results obtained for the scalar case in [1, 2].
We recall the Schrödinger equation in $I\!R^N$;

$$\mathcal{A}u := (-\Delta + q)u = \lambda u + f \quad \text{in } I\!R^N, \quad f \in L^2(I\!R^N) \tag{1}$$

The potential q is assumed to be continuous in $I\!R^N$ satisfying $q \in L^1_{loc}(I\!R^N)$, $q \geq C > 0$ and $q(x) \longrightarrow \infty$ when $|x| \longrightarrow \infty$. λ is a real parameter.

1.1 Maximum principle: φ_1–positivity

Hypothesis 1.1 *The potential* $q\colon I\!R_+ \to I\!R$ *is locally essentially bounded,* $q(r) \geq \text{const} > 0$ *for* $r \geq 0$*, and there exists a constant* $c_1 > 0$ *such that*

$$c_1 Q(r) \leq q(r) - \frac{1}{4r^2} \quad \textit{for} \;\; R_0 \leq r < \infty. \tag{2}$$

where $Q(r)$ *is a function of* $r \equiv |x|$, $R_0 \leq r < \infty$*, for some* $R_0 > 0$*:*

$$\begin{cases} Q(r) > 0, \;\; Q \textit{ is locally absolutely continuous,} \\ Q'(r) \geq 0, \quad \textit{and} \quad \int_{R_0}^{\infty} Q(r)^{-1/2}\, dr < \infty. \end{cases} \tag{3}$$

Theorem 1.2 *[2] Let the hypothesis* 1.1 *be satisfied. Assume that* $u \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}u - \lambda u = f \in L^2(I\!R^2)$, $\lambda \in I\!R$*, and* $f \geq 0$ *a.e. in* $I\!R^2$ *with* $f > 0$ *in some set of positive Lebesgue measure. Then, for every* $\lambda \in (-\infty, \lambda_1)$*, there exists a constant* $c > 0$ *(depending upon* f *and* λ*) such that*

$$u \geq c\varphi_1 \quad \textit{in} \;\; I\!R^2. \tag{4}$$

1.2 Anti-maximum principle: φ_1–negativity

We get an anti-maximum principle for systems (S) involving some potentials with a superquadratic growth at infinity. Here we study 2×2 systems; analogous results can be obtained for the case of N equations- see [3].

We define $X^{1,2}$ the space of Lebesgue measurable functions $f\colon I\!R^2 \to I\!R$ having the following properties:

$$\frac{\partial f}{\partial \theta}(r, \bullet) \in L^2(-\pi, \pi) \quad \text{for all} \;\; r \geq 0$$

and there is a constant $C \geq 0$ such that

$$|f(r,\theta)| + \left(\oint \left| \frac{\partial f}{\partial \theta}(r, \vartheta) \right|^2 d\vartheta \right)^{1/2} \leq C\varphi_1(r) \tag{5}$$

for almost every $r \geq 0$ and $\theta \in [-\pi, \pi]$.

Theorem 1.3 *Let hypothesis* 1.1 *be satisfied. Assume that* $u \in \mathcal{D}(\mathcal{A})$ *satisfies* (1), $f \geq 0$, $\;f \in X^{1,2}$*. then there exists a positive number* δ *(depending upon* f*) such that, for every* $\lambda \in (\lambda_1, \lambda_1 + \delta)$*, we have*

$$u \leq -c\,\varphi_1 \quad \textit{in} \;\; I\!R^2. \tag{6}$$

2 Results

2.1 Estimate of the constant

Theorem 2.1 *Let the hypothesis* (6) *be satisfied [3]. Assume that* $u \in \mathcal{D}(\mathcal{A})$*,* $\mathcal{A}u - \lambda u = f \in L^2(I\!R^2)$*,* $\lambda \in I\!R$*, and* $f \geq 0$ *a.e. in* $I\!R^2$ *with* $f > 0$ *in some set of positive Lebesgue measure. If* $f \in X^{1,2}$*, then there exists a positive number* $\delta = \delta(f) > 0$ *and* $C(f, \lambda) > 0$ *such that, for every* $\lambda \in (\lambda_1, \lambda_1 + \delta)$*, on a :*

$$u \leq -C(f, \lambda)\varphi_1 \quad \text{in } I\!R^2$$

$$\delta = \min(\delta_1, C^{-1}.\alpha) \tag{7}$$

and

$$C(f, \lambda) = ((\lambda - \lambda_1)^{-1} - \Gamma)\alpha \tag{8}$$

Where;

$$\delta_1 = Sp[(\mathcal{A} - \lambda)^{-1})]^{-1};$$
$$\alpha = \int_{I\!R^2} f\varphi_1;$$
$$\Gamma = (2C_f + \|f\|_{X^{1,2}})(2c_q)^{-1/2}\int_{R_1}^{+\infty} \mathcal{Q}(r)^{-1/2} + M(R_1)(2C_f M(R_1)R_1^2/2 + \|f\|_{X^{1,2}});$$
$$M(R_1) = \max_{0\leq s\leq r\leq R_1} \frac{\varphi_1(s)}{\varphi_1(r)};$$

2.2 Anti-maximum Principle for a linear cooperative system 2×2

We decouple systems (S)

$$\left(\begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{pmatrix} - \begin{pmatrix} \sqrt{bc} & 0 \\ 0 & -\sqrt{bc} \end{pmatrix}\right) PU = \lambda PU + PF \tag{9}$$

Where $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $F = \begin{pmatrix} f \\ g \end{pmatrix}$ and $P = \begin{pmatrix} 1/2 & 1/2\gamma \\ 1/2 & -1/2\gamma \end{pmatrix}$, $\gamma = \sqrt{c/b} > 0$

Theorem 2.2 *Assume that* $b > 0$*,* $c > 0$ *(a strictly cooperative system) and* $0 \leq f$*,* $g \in L^2(I\!R^2)$ *with* f *and* $g \in X^{1,2}$*. Then*

There exists an eingenvalue Λ_1 *with a positive eigenfunction* Φ_1 *defined by*

$$\begin{cases} \Lambda_1 = \lambda_1 - \sqrt{bc} > 0 \\ \Phi_1 = \begin{pmatrix} \sqrt{bc} \\ c \end{pmatrix}\varphi_1 \end{cases}$$

and there exists a constant $\delta = \delta(f, g, b, c) > 0$ *such that, for every* $\lambda \in (\Lambda_1, \Lambda_1 + \delta)$*, the weak solution* $PU = (\tilde{u}, \tilde{v})$ *to* (9) *satisfies*

$$PF = \begin{pmatrix} 0 \neq \tilde{f} > 0 \\ 0 \neq \tilde{g} > 0 \end{pmatrix} \Longrightarrow PU = \begin{pmatrix} \tilde{u} < 0 \\ \tilde{v} > 0 \end{pmatrix}$$

The weak solution $U = (u, v)$ *to* (S) *satisfies*

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \leq -C\Phi_1$$

Where $C = C(f, g, \lambda) > 0$.

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