Anti-maximum principle for cooperative system involving Schrödinger operator in $\mathbb{R}^N$

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Abstract

We obtain a result concerning the anti-maximum principle for weak solutions $U = (u, v) \in \mathbb{R}^N$ of the following cooperative elliptic system:

\[
\begin{aligned}
\mathcal{A}u(x) &:= (-\Delta + q(|x|))u(x) = \lambda u(x) + bv(x) + f(x) \\
\mathcal{A}v(x) &:= (-\Delta + q(|x|))v(x) = cu(x) + \lambda v(x) + g(x)
\end{aligned}
\]

$u(x) \to 0$, $v(x) \to 0$, as $|x| \to +\infty$.

Here $\mathcal{A} = -\Delta + q(x)$ in $L^2(\mathbb{R}^2)$ is the Schrödinger operator. We assume that the potential $q(x) \equiv q(|x|)$, is strictly positive, locally bounded, and has superquadratic growth as $|x| \to \infty$; $b, c$ are strictly positive constants. We show that there exists a simple eigenvalue $\Lambda_1$ of $(S)$ with positive eigenfunction $\Phi_1$. Then we prove an anti-maximum principle in the following form: Let $f, g \in L^2(\mathbb{R}^2)$ be positive functions which are “sufficiently smooth” perturbations of a radially symmetric function, then there exists $\delta = \delta(f, g, b, c) > 0$ such that for $\lambda \in (\Lambda_1, \Lambda_1 + \delta)$ the weak solution $U$ satisfies $U = (u, v) \leq -C\Phi_1$ where $C = C(f, g, \lambda)$ is a positive constant.

Keywords: Schrödinger operator, maximum principle, anti-maximum principle.

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1 Schrödinger equation in $\mathbb{R}^N$

We first recall recent results obtained for the scalar case in [1, 2].

We recall the Schrödinger equation in $\mathbb{R}^N$:

\[
\mathcal{A}u := (-\Delta + q)u = \lambda u + f \quad \text{in } \mathbb{R}^N, \quad f \in L^2(\mathbb{R}^N)
\]

The potential $q$ is assumed to be continuous in $\mathbb{R}^N$ satisfying $q \in L^1_{loc}(\mathbb{R}^N)$, $q \geq C > 0$ and $q(x) \to \infty$ when $|x| \to \infty$. $\lambda$ is a real parameter.
1.1 Maximum principle: $\varphi_1$-positivity

**Hypothesis 1.1** The potential $q: IR_+ \to IR$ is locally essentially bounded, $q(r) \geq \text{const} > 0$ for $r \geq 0$, and there exists a constant $c_1 > 0$ such that

$$c_1 Q(r) \leq q(r) - \frac{1}{4r^2} \quad \text{for} \quad R_0 \leq r < \infty.$$  

where $Q(r)$ is a function of $|x|$, $R_0 \leq r < \infty$, for some $R_0 > 0$:

$$
\begin{align*}
Q(r) &> 0, \quad Q \text{ is locally absolutely continuous}, \\
Q'(r) &\geq 0, \quad \text{and} \quad \int_{R_0}^{\infty} Q(r)^{-1/2} \, dr < \infty.
\end{align*}
$$

**Theorem 1.2** [2] Let the hypothesis 1.1 be satisfied. Assume that $u \in \mathcal{D}(A)$, $Au - \lambda u = f \in L^2(IR^2)$, $\lambda \in IR$, and $f \geq 0$ a.e. in $IR^2$ with $f > 0$ in some set of positive Lebesgue measure. Then, for every $\lambda \in (-\infty, \lambda_1)$, there exists a constant $c > 0$ (depending upon $f$ and $\lambda$) such that

$$u \geq c\varphi_1 \quad \text{in} \quad IR^2.$$  

1.2 Anti-maximum principle: $\varphi_1$-negativity

We get an anti-maximum principle for systems (S) involving some potentials with a superquadratic growth at infinity. Here we study $2 \times 2$ systems; analogous results can be obtained for the case of $N$ equations - see [3].

We define $X^{1,2}$ the space of Lebesgue measurable functions $f: IR^2 \to IR$ having the following properties:

$$\frac{\partial f}{\partial \theta}(r, \bullet) \in L^2(-\pi, \pi) \quad \text{for all} \quad r \geq 0$$

and there is a constant $C \geq 0$ such that

$$\left| f(r, \theta) \right| + \left( \int_{\theta} \left| \frac{\partial f}{\partial \theta}(r, \theta) \right|^2 \, d\theta \right)^{1/2} \leq C\varphi_1(r)$$

for almost every $r \geq 0$ and $\theta \in [-\pi, \pi]$.

**Theorem 1.3** Let hypothesis 1.1 be satisfied. Assume that $u \in \mathcal{D}(A)$ satisfies (1), $f \geq 0$, $f \in X^{1,2}$, then there exists a positive number $\delta$ (depending upon $f$) such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, we have

$$u \leq -c \varphi_1 \quad \text{in} \quad IR^2.$$  

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2 Results

2.1 Estimate of the constant

Theorem 2.1 Let the hypothesis (6) be satisfied [3]. Assume that \( u \in \mathcal{D}(A) \), \( Au - \lambda u = f \in L^2(\mathbb{R}^2) \), \( \lambda \in \mathbb{R} \), and \( f \geq 0 \) a.e. in \( \mathbb{R}^2 \) with \( f > 0 \) in some set of positive Lebesgue measure. If \( f \in X^{1,2} \), then there exists a positive number \( \delta = \delta(f) > 0 \) and \( C(f, \lambda) > 0 \) such that, for every \( \lambda \in (\lambda_1, \lambda_1 + \delta) \), on \( a : \)

\[
\begin{align*}
  u &\leq -C(f, \lambda)\varphi_1 \text{ in } \mathbb{R}^2 \\
  \delta &\geq \min(\delta_1, C^{-1} \alpha) \quad (7)
\end{align*}
\]

and

\[
C(f, \lambda) = (\lambda - \lambda_1)^{-1} - \Gamma \alpha \quad (8)
\]

Where;

\[
\begin{align*}
  \delta_1 &= Sp[(A - \lambda)^{-1}]^{-1} \\
  \alpha &= \int_{\mathbb{R}^2} f \varphi_1; \\
  \Gamma &= (2C_f + \|f\|_{X^{1,2}})(2C_g)^{-1/2} \int_{R_1}^{+\infty} Q(r)^{-1/2} + M(R_1)(2C_f M(R_1) R_1^2/2 + \|f\|_{X^{1,2}}); \\
  M(R_1) &= \max_{0 \leq s \leq r \leq R_1} \varphi_1(s)/\varphi_1(r).
\end{align*}
\]

2.2 Anti-maximum Principle for a linear cooperative system \( 2 \times 2 \)

We decouple systems (S)

\[
\begin{pmatrix}
  A & 0 \\
  0 & A
\end{pmatrix} - \begin{pmatrix}
  \sqrt{bc} & 0 \\
  0 & -\sqrt{bc}
\end{pmatrix} PU = \lambda PU + PF \quad (9)
\]

Where \( U = \begin{pmatrix} u \\ v \end{pmatrix} \), \( F = \begin{pmatrix} f \\ g \end{pmatrix} \) and \( P = \begin{pmatrix} 1/2 & 1/2 \gamma \\ 1/2 & -1/2 \gamma \end{pmatrix}; \gamma = \sqrt{c/b} > 0 \)

Theorem 2.2 Assume that \( b > 0 \), \( c > 0 \) (a strictly cooperative system) and \( 0 \leq f, g \in L^2(\mathbb{R}^2) \) with \( f \) and \( g \in X^{1,2} \). Then

There exists an eigenvalue \( \Lambda_1 \) with a positive eigenfunction \( \Phi_1 \) defined by

\[
\begin{align*}
  \Lambda_1 &= \lambda_1 - \sqrt{bc} > 0 \\
  \Phi_1 &= \begin{pmatrix} \sqrt{bc} \\ c \end{pmatrix} \varphi_1
\end{align*}
\]
and there exists a constant $\delta = \delta(f, g, b, c) > 0$ such that, for every $\lambda \in (\Lambda_1, \Lambda_1 + \delta)$, the weak solution $PU = (\tilde{u}, \tilde{v})$ to (9) satisfies

$$PF = \left( \begin{array}{c} 0 \neq \tilde{j} > 0 \\ 0 \neq \tilde{g} > 0 \end{array} \right) \implies PU = \left( \begin{array}{c} \tilde{u} < 0 \\ \tilde{v} > 0 \end{array} \right)$$

The weak solution $U = (u, v)$ to (S) satisfies

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \leq -C\Phi_1$$

Where $C = C(f, g, \lambda) > 0$.

References


