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A P^1 bubble/P^1 finite element velocity-pressure solution of a pseudo homogeneous flow model simulating a two phase flow: Application to lake eutrophication remedial by air injection

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Abstract

The climatological variations, mainly and specifically thermal variations, generate in lakes an unsteady dynamic process making water quality decrease progressively; when the dissolved oxygen concentration reaches a low level, the lake is considered in eutrophication. The dynamic aeration process is one of the best remedial techniques against eutrophication. It consists in injecting air in the bottom of the lake.

A good understanding of the eutrophication treatment through aeration requires a modeling based on a two phase flow (water-air bubbles) leading to many difficulties. In this paper, we describe the implementation of the aeration effect correction in a modified one-phase model to restitute as much as possible an equivalent behavior of the two-phase model. Preliminary studies of a mixed velocity-pressure formulation for 2D flows with variable density are presented in [1]. Using the characteristics method to approximate the total derivative convection term, the generic system is of Quasi-Stokes type. For the space discretization, we use the P^1 bubble-P^1 finite element method. Numerical experiments are then performed for scenario analysis.

Keywords: Eutrophication, aeration process, two-phase flow, finite elements method.

AMS Classification: 76T10, 76M10, 35Q30.

1 Position of the problem

The eutrophication process is a highly complex phenomenon related to water quality degradation in reservoirs. Practically the eutrophication in a water basin is characterized mainly by a poor dissolved oxygen concentration in water: less than 3 mg/l (Phelippot [11]). Furthermore, this phenomenon is accompanied by a stratification process dividing the water volume, during a large period of the year, into three distinct layers as illustred in figure 1. Many engineering techniques were used to combat oxygen depletion.

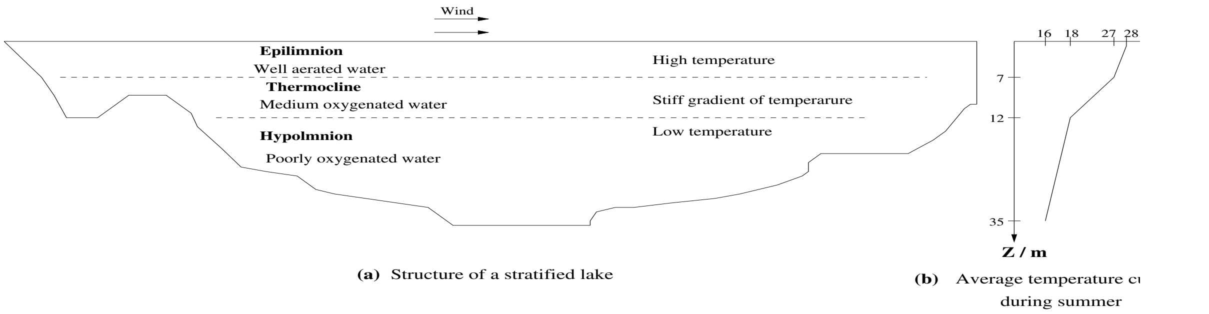

Figure 1: (a): Structure of a stratified lake, (b): average temperature curve during summer

Aeration is considered as the cheapest and the best remedial and/or preventive action against eutrophication. This process consists in injecting compressed air in the reservoir bottom (Figure 2). The resulting buoyant plume causes local mixing dismantling

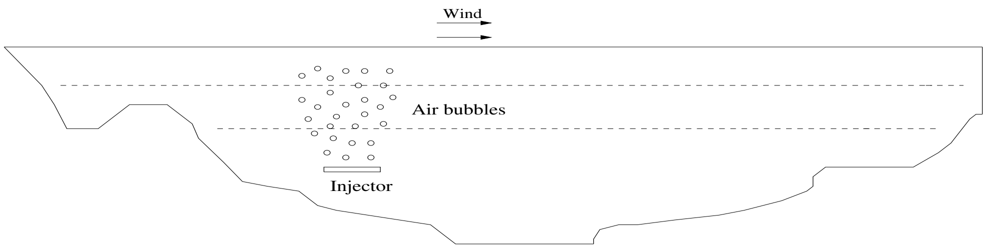

Figure 2: Aeration process

the density structure. To our knowledge, this process has not been studied thoroughly and no mathematical modeling has been introduced so far to find the distribution of air and the extents of the resulting plume so that eutrophication effects are minimized or completely eliminated. In addition, the position of injectors is based on engineering judgement and on trial and errors methods leading very often to non optimal solutions. Therefore, procedures to find the optimum location of injectors and also the quantity of air are highly required. The complexity of the process and of the related modeling issues using two phase flow [2] has led us to suggest an approximation based on a modified one-phase model. Indeed, this amplified model is taking into account the aeration effect by a source term integrating the effect of the momentum released by the bubbles to retitute as much as possible an equivalent behavior of the two-phase model. The technique consists in introducing an averaged void fraction for the material points by prescribing a local boundary condition for the velocity on the injectors in order to simulate the initial motion of the bubbles released in the water. This leads to consider a medium described by a velocity, pressure, density like variables of a material point represented by a weighted combination of the variables of each phase by the void fraction variable mentioned above. Then, we add a source term equivalent to the dynamic effect of the injected bubbles mainly dominated by the Archimede forces. In this case, the governing equations can be solved in a pseudo-homogeneous domain with the void fraction variable indicating the concentration of each phase. The technique proposed can be understood as a correction to the water velocity field and consequently to the convected variables. The time discretization by means of the characteristics method [12] for the convection approximation, is used. For the discrete problem, we use the P^1 bubble/P^1 finite element method. Only 2D fluid flow configurations are considered in this paper to demonstrate the feasibility of the methodology developed. Numerical experiments are then conducted for scenario analysis and understanding of the process on a cross section of the Bouregreg lake in Morocco to complete our learning process of the phenomena. Though the results are somehow qualitative, they are quite encouraging and with the help of validation and calibration, a cheap computational alternative is offered for studying the highly complicated aeration process.

2 One phase model with source term

A general model for such a two phase flow, constituted by the union of single phase (water or air) regions with free interfaces, is given by the "local instantaneous formulation" which describes the local behavior of each phase. Starting from this form and using the conventional averaging techniques, one can derive the so-called average formulation of a two phase flow (Ishii [10], Abdelwahed [1], Abdelwahed and Dabaghi [4]). However, due to the high number of unknowns and other inherent difficulties related to coefficients/parameters and boundary conditions, simpler alternatives have to be devised. We consider an approach based on a pseudo-homogeneous flow in which we integrate the effect of the momutum released by the bubbles.

Let Ω be an open two dimensional domain representing the water basin and Γ the boundary of Ω, $\mathbf{n}$ is the normal unitary vector on Γ, $T \in \mathbb{R}^*$, $Q_T = \Omega \times]0, T]$, $\Sigma_T = \Gamma \times [0, T]$ and $\Sigma(\mathbf{u}_d) = \{(\mathbf{x}, t) \in \Sigma_T | \mathbf{u}_d.\mathbf{n} < 0\}$.

We introduce the following variables in table 1:

where $k \in \{L, G\}$ denotes respectively the liquid and the gaz phase.

In such a case, the suggested model can be written as follows [1][5]:

Given : $\alpha^0 : \Omega \to \mathbb{R}^+$ such that $\alpha^0(1-\alpha^0) = 0$, $\mathbf{f} : Q_T \to \mathbb{R}^2$, $\mathbf{u}^0 : \Omega \to \mathbb{R}^2$ and $\mathbf{u}_d : \Sigma_T \to \mathbb{R}^2$.

α	bubble void fraction	variable	scalar function
ρ_k	density	variable	scalar function
$\mathbf{u}$	liquid velocity	variable	vector function
p	liquid pressure	variable	scalar function
μ_k	viscosity coefficient	given	scalar constant

Table 1: Notations

Find : $\alpha : Q_T \to \mathbb{R}^+$, $\rho_L : Q_T \to \mathbb{R}^+$, $\mathbf{u} : Q_T \to \mathbb{R}^2$ and $p : Q_T \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} + \mathbf{u}.\nabla\alpha = 0 \quad , \quad \dfrac{\partial \rho_L}{\partial t} + \mathbf{u}.\nabla\rho_L = 0 & \text{in } Q_T \\ \alpha(.,t)_{|t=0} = \alpha^0(.) \quad , \quad \rho_L(.,t)_{|t=0} = {\rho_L}^0(.) & \text{in } \overline{\Omega} \\ \alpha \text{ given} \quad , \quad \rho_L \text{ given} & \text{on } \Sigma(\mathbf{u}_d) \end{cases} \tag{1}$$

$$\begin{cases} \rho\left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u}\right) - \mathrm{div}(\mu\,\varepsilon(\mathbf{u})) + \nabla p = \mathbf{f} & \text{in } Q_T \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } Q_T \\ \mathbf{u} = \mathbf{u}_d & \text{on } \Sigma_T \\ \mathbf{u}(.,t)_{|t=0} = \mathbf{u}^0(.) & \text{in } \overline{\Omega} \end{cases} \tag{2}$$

with $\mathbf{f} = \rho_L \mathbf{g} + \mathbf{f}_a$ where $\mathbf{g}$ and $\mathbf{f}_a = \alpha\,(\rho_G - \rho_L)\mathbf{g}$ are respectively the gravitational and the Archimede forces and

$$\mu = (1-\alpha)\mu_L + \alpha\mu_G \quad , \quad \rho = (1-\alpha)\rho_L + \alpha\rho_G \tag{3}$$

represent the properties of the considered pseudo-homogeneous fluid.

3 Numerical analysis

For solving the above one-phase flow model (1)-(2); the numerical formulation is obtained by combining the characteristics method for the time discretization and the finite element method for space approximation. The characteristics method consists in giving an approximation of the total derivative of a variable S as follows (Dabaghi et al. [9], Pironneau [12]) :

$$\frac{dS}{dt}(x,t)|_{(t=t^n)} = \frac{S^{n+1} - S^n o \chi^n}{\Delta t} \tag{4}$$

where Δt denotes the time step, S^n and χ^n the approximation, respectively, of S and χ at time $t^n = n\Delta t$ and $\chi(\mathbf{x}, \tau; t)$ describes the position at time t of a point which was on the position $\mathbf{x}$ at time τ.

Hence using (4), the time discretization of (1)-(2) implies the following Quasi-Stokes type problem:

$$\begin{cases} \dfrac{1}{\Delta t}\,\rho\,\mathbf{u} - \mathrm{div}\,(\mu\,\varepsilon(\mathbf{u})) + \nabla \mathrm{p} = \mathbf{f} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega \\ \Gamma_1 & \\ \mathbf{u} = \mathbf{u}_d & \text{on } \Gamma \end{cases} \tag{5}$$

For the space discretization, we use the mixed P^1 bubble/ P^1 finite element method (Arnold and al. [6], Brezzi and al. [7]).

We show in [1] that the continuous problem resulting from the weak formulation of (5) and the discrete one have a unique solution.

4 Numerical experiments

In the present work, we will consider a "real" application test case on the lake of Bouregreg. The Bouregreg lake is supplying 20% of the total population in Morocco, its averaged characteristics are summarized bellow [8]:

- Volume = 480 − 1000 millions of m^3
- Wetted surface = 32 km^2
- Average depth = 17 m, Maximum depth = 42 m
- Discharge = 9 m^3/s

To avoid eutrophication, the water agency in charge (ONEP) has installed five sets of air injectors. In our present work, for illustration purposes, we limit ourselves to a 2D significant section crossing one air injector axe. The numerical simulations were based on the following assumptions: fixed wind velocity, no slip condition at the cross section delimitation, constant injection velocity and fixed injector position. Several numerical simulations have been conducted under different modelling hypotheses for comparison issues: variable or constant density, with or without bubble effect correction term, etc. The injector velocity order of magnitude is derived from experimental considerations and is in fact related to the compressor air pressure connected to the injector. However for numerical experiments, we have set the injection velocity to 4 m/s. Only results for two characteristic times are presented. They correspond to respectively: the beginning of the aeration process and its stabilization. The optimal effects one should look for correspond to a good oxygenation of the Hypolmnion zone of the cross section with a minimum of dis-strafication effect.

For illustrative purposes, only selected results corresponding to one simulation scenario are presented below.

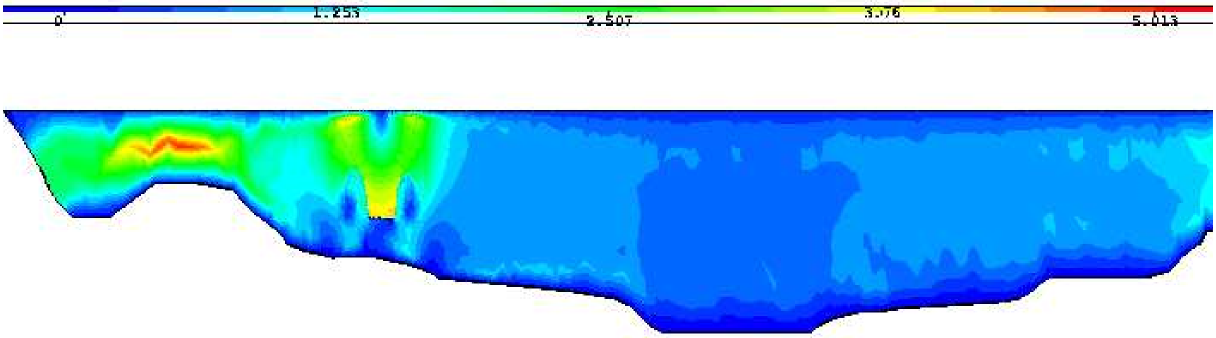

Figure 3: velocity isovalues with injection, Time=10s

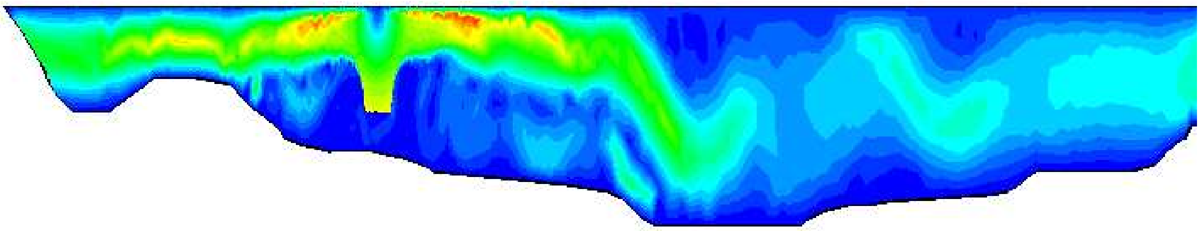

Figure 4: velocity isovalues with injection, Time=10mn

5 Conclusion

A modified pseudo-homogeneous one phase flow model governed by conventional Navier-Stokes equations with variable density and air-bubble momentum correction terms has been used to simulate virtually the aeration process required to combat oxygen depletion and consequently the eutrophication. The preliminary results using the P^1 bubble/P^1 finite element combined to the characteristics method have permitted to illustrate the effect of the mechanical aeration process. High quality qualitative results were obtained permitting us to learn more about the process using numerical experiments. It is probably early to speak about "model validation or calibration", however, the difficulties posed by a full two-fluid model for studying the two-phase air bubbles-water flow encouraged us to follow this direction.

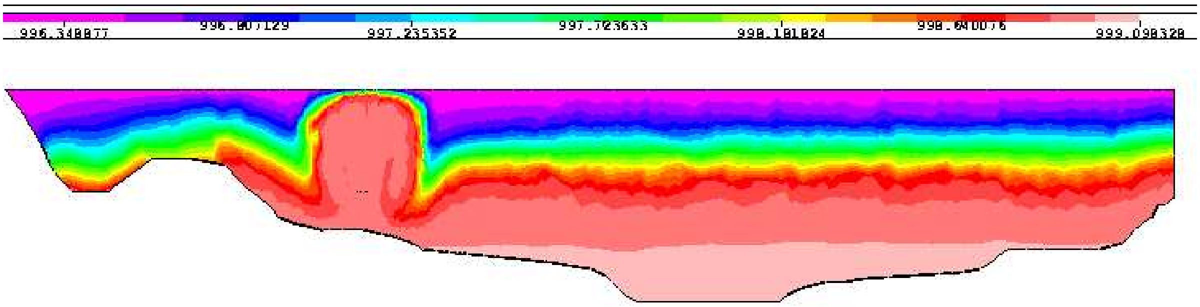

Figure 5: density isovalues with injection, Time=10s

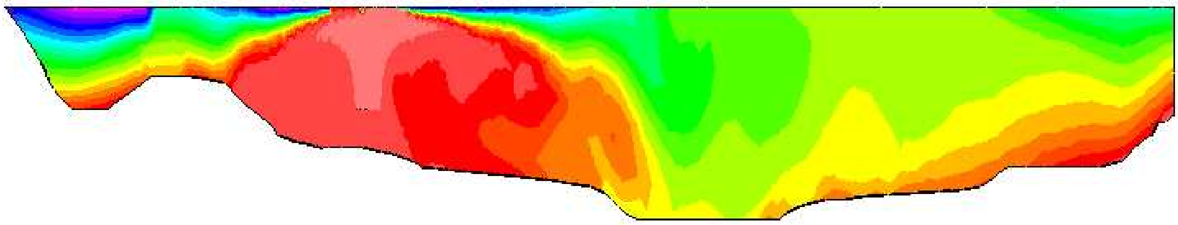

Figure 6: density isovalues with injection, Time=10mn

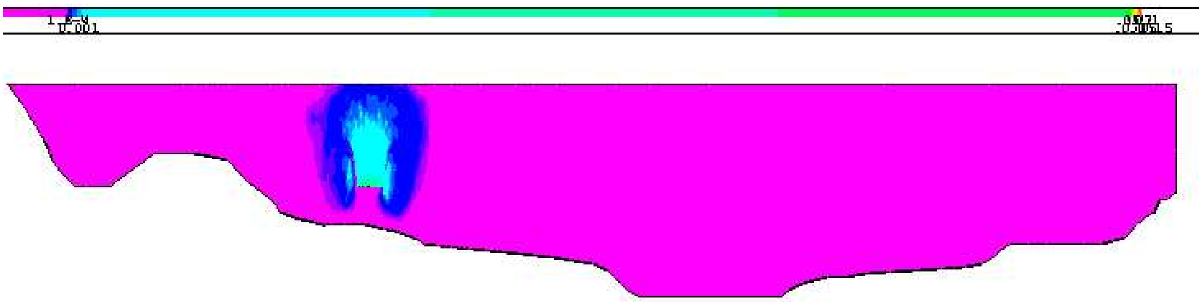

Figure 7: void fraction isovalues with injection, Time=10s

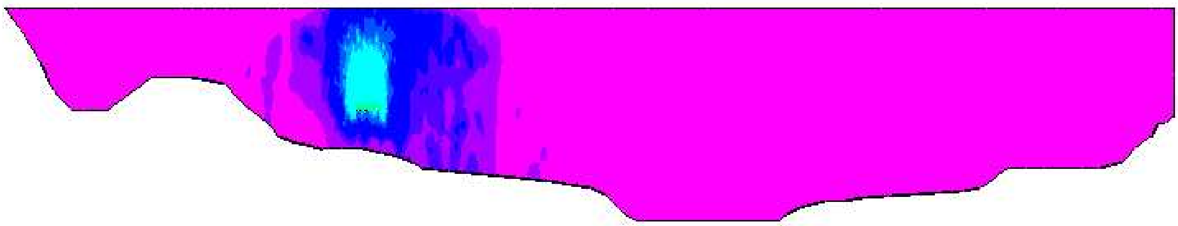

Figure 8: void fraction isovalues with injection, Time=10mn

References

[1] Abdelwahed M. *Modélisation et simulation numérique d'écoulement diphasique*, Thèse de l'université de Pau 2002.

[2] Abdelwahed M., Dabaghi F. and Ouazar D., *A virtual numerical simulator for aeration effects in lake eutrophication*, Accepted for publication in International Journal of Computational Fluid Dynamics, 2002.

[3] Abdelwahed M., Amara M., Dabaghi F. and Hassine M., *A numerical modelling of a two phase flow for water eutrophication problems*, Proceeding of the european Congress on Computational Methods in Applied Sciences and Engineering, Barcelona 2000.

[4] Abdelwahed M., Dabaghi F., *Two phase flow formulation (2D)*; Report D17, DGXIII - European Commission, ESIMEAU Contract, March 2000.

[5] Abdelwahed M., Dabaghi F., *Two phase flow solver (2D)*; Report D25, DGXIII - European Commission, ESIMEAU Contract, January 2001.

[6] Arnold D., Brezzi F., Fortin M. *A stable finite element for the Stokes equations* , Calcoclo **21(4)** 337-344. 1984.

[7] Brezzi F. and Fortin M. *Mixed and hybrid finite element method*, Springer Series in Computational Mathematics, Springer Verlag New York, 1991.

[8] Dabaghi F. & Ouazar D. *Validation and calibration through the experimental basin*, report D9.1, DGXIII - European Commission, ESIMEAU Contract, 1999.

[9] Dabaghi F. and Saiac J.H. *Characteristics and time dependant methods for solving the 3D incompressible Euler equations by a stream vector-vorticity formulation*, Int. J. for Hydraulics, Hydrology and Hydrodynamics in Eng., **2**, N∘.4, 1989.

[10] Ishii M. *Thermo-fluid dynamic theory of two-phase flow*, Collection de la direction des études de recherche d'électricité de france, 1975.

[11] Phelippot S. *L'eutrophisation des lacs*, Rapport de l'Association Francaise pour l'Etude des Eaux, 1975.

[12] Pironneau O. *On the transport diffusion algorithm and its applications to the Navier-Stokes equations*, Numer Math. **38** 309-332, 1982.