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COUPLINGS FOR MULTICOMPONENT SYSTEMS

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Abstract. We consider multicomponent systems which can be seen as interacting particle systems. For two systems with state space $\mathbf{X}$ and $\mathbf{Y}$ respectively and a general subset $K \subseteq \mathbf{X} \times \mathbf{Y}$ we give conditions on the rates of the processes to ensure the existence of a $K-$coupling. We apply our results to obtain conditions for the stochastic comparison in terms of workload of tandem Jackson networks.

Keywords: Stochastic comparison, interacting particle systems, coupling, queueing networks.

AMS classification: 60K35, 60J25.

§1. Introduction

We begin with the definition of interacting particle systems to be considered in this paper. Interacting particle systems are Feller processes $\{X_t, t \geq 0\}$ with state space $\mathbf{X} \subseteq \mathbf{V}^{\mathbf{S}}$, where $\mathbf{V}$ is a countable set and $\mathbf{S} \subseteq \mathbb{Z}^d$, $d \geq 1$. $\mathbf{S}$ represents the set of particles and $\mathbf{V}$ represents the set of values each particle can take. Basic notations and definitions of such processes can be seen in [1] and [7]. The evolution of these processes is defined through their transition rates. Given $\eta \in \mathbf{X}$, we consider that there are two kind of changes involving a particle $x \in \mathbf{S}$: changes of particle x from $\eta(x)$ to $a \in \mathbf{V}$ and changes of the pair $(x, x+k)$, with $k \in \mathbf{N}_x$, from $(\eta(x), \eta(x+k))$ to $(r,s) \in \mathbf{V}^2$, where $x + \mathbf{N}_x$ is the set of points that can change paired with x. For $a \in \mathbf{V}$ or $a = (k,(r,s)) \in \mathbf{N}_x \times \mathbf{V}^2$, we denote by $c_a(x,\eta)$ the rate of change of the process from η to η_{xa}, where, for $a \in \mathbf{V}$,

$$\eta_{xa}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ a & \text{if } y = x, \end{cases}$$

and for $a = (k,(r,s)) \in \mathbf{N}_x \times \mathbf{V}^2$,

$$\eta_{xa}(y) = \begin{cases} \eta(y) & \text{if} y \neq x, x+k, \\ r & \text{if} y = x, \\ s & \text{if} y = x+k. \end{cases}$$

Therefore, the generator of the particle system with rates $c_a(x,\eta)$ is defined on Lipschitz functions f on $\mathbf{X}$ as follows:

$$\Omega f(\eta) = \sum_{x\in\mathbf{S}} \left\{ \sum_{a\in\mathbf{V}} c_a(x,\eta)\big(f(\eta_{xa}) - f(\eta)\big) + \sum_{a\in\mathbf{N}_x\times\mathbf{V}^2} \tfrac{1}{2}\, c_a(x,\eta)\big(f(\eta_{xa}) - f(\eta)\big) \right\}, \quad (1)$$

satisfying the usual existence and uniqueness conditions (see [1]). We will work with two processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$, with the same set of particles $\mathbf{S}$ but with possibly different set of values $\mathbf{V}$ and $\mathbf{W}$. Therefore, the state spaces of (X_t) is $\mathbf{X} = \mathbf{V}^{\mathbf{S}}$ and the state space of (Y_t) is $\mathbf{Y} = \mathbf{W}^{\mathbf{S}}$. A coupling of these processes is defined as a process on $\mathbf{X} \times \mathbf{Y}$ whose marginals are the original processes. When the coupled process is Markovian, it is called Markovian coupling. If $\mathbf{X} = \mathbf{Y}$ and it is endowed with a partial order $\leq$ a coupling of the processes is said to be order-preserving if whenever $X_0 \leq Y_0$ then $P(X_t \leq Y_t) = 1$ for all $t \geq 0$. Let $K \subseteq \mathbf{X} \times \mathbf{Y}$. A $K-$coupling of the processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ is a process $\{Z_t, t \geq 0\}$ on $\mathbf{X} \times \mathbf{Y}$ whose marginals are the original processes and such that if $(X_0, Y_0) \in K$ then $P\big((X_t, Y_t) \in K\big) = 1$ for all $t \geq 0$. Note that, if $\mathbf{X} = \mathbf{Y}$, a order-preserving coupling is a $K-$coupling with $K = \{(\eta, \xi) \in \mathbf{X} \times \mathbf{Y} : \eta \leq \xi\}$. In general, when $\mathbf{X} = \mathbf{Y}$, if the relation $\leq_K$ on $\mathbf{X}$ given by $\eta \leq_K \xi$ if $(\eta, \xi) \in K$ is a partial order, then a $K-$coupling is an order-preserving coupling with respect to the partial order $\leq_K$, but this is not always the case. In [8] the authors give necessary and sufficient conditions on the transition rates of two continuous-time Markov chains with countable state spaces for the existence of a $K-$coupling, thus solving a problem stated in [4]; they also show that the coupling can be chosen to be Markovian and they simplify the construction of such a coupling when the state spaces of the chains are the same. In [2] we gave conditions for the stochastic comparability of two interacting particle systems, whose rates verify a certain separability condition and used this results to improve and extend the results on queueing networks given in [5] and [6]. In this work we find general sufficient conditions for the existence of a Markovian $K-$coupling of the processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ with generators Ω_1 and Ω_2 as in (1), respectively (for the proofs of the results, see [3]). We remark here that we do not impose any condition on the set K or the rates of the processes c^1, c^2.

§2. Results

Let K be an arbitrary subset of $\mathbf{X} \times \mathbf{Y}$ and c^1, c^2 the rates of the processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ with respective state spaces $\mathbf{X}$ and $\mathbf{Y}$. Define, for each $(\eta, \xi) \in K$, and $x \in \mathbf{S}$, the sets

$$E^x = \{a \in \mathbf{V} \cup (\mathbf{N}_x \times \mathbf{V}^2) : c^1_a(x, \eta) > 0\},$$

$$F^x = \{a \in \mathbf{W} \cup (\mathbf{N}_x \times \mathbf{W}^2) : c^2_a(x, \xi) > 0\},$$

$$G^x = E^x \cup F^x,$$

$$R^x = \Big\{a \in E^x : (\eta_{xa}, \xi) \notin K \text{ and } \Big((\eta_{xa_x}, \xi) \notin K \text{ or } \big[a = (k, (r,s)) \in \mathbf{N}_x \times \mathbf{V}^2, \\ (\eta_{xa_x}, \xi) \in K \text{ and } (\eta_{x+ka_{x+k}}, \xi) \in K\big]\Big)\Big\},$$

$$S^x = \Big\{a \in F^x : (\eta, \xi_{xa}) \notin K \text{ and } \Big((\eta, \xi_{xa_x}) \notin K \text{ or } \big[a = (k, (r,s)) \in \mathbf{N}_x \times \mathbf{W}^2, \\ (\eta, \xi_{xa_x}) \in K \text{ and } (\eta, \xi_{x+ka_{x+k}}) \in K\big]\Big)\Big\},$$

where, if $a \in \mathbf{V}$ or $a \in \mathbf{W}$, then $a_x = a$ and if, $a = (k, (r,s)) \in \mathbf{N}_x \times \mathbf{V}^2$ or $a = (k, (r,s)) \in \mathbf{N}_x \times \mathbf{W}^2$ then $a_x = r$. Consider a network whose nodes are the points of G^x. Define the arrows of the network as follows: there is an arrow from b to a (that is, $(b, a) \in \mathscr{A}^x$) if

- $(\eta_{xa}, \xi_{xb}) \in K$, for $a \in E^x$, $b \in F^x$,
- for all $c \in F^x$ such that $(\eta_{xb}, \xi_{xc}) \in K$, it holds $(\eta_{xa}, \xi_{xc}) \in K$, for $a, b \in R^x$,
- for all $c \in E^x$ such that $(\eta_{xc}, \xi_{xa}) \in K$, it holds $(\eta_{xc}, \xi_{xb}) \in K$, for $a, b \in S^x$.

For $x \in \mathbf{S}$, $k \in \mathbf{N}_x$ and $(\eta, \xi) \in K$, define the sets $B_{x,k}$ and $C_{x,k}$ by

$$B_{x,k} = \{(r,s) \in \mathbf{V}^2 \cup \mathbf{W}^2 : (k,(r,s)) \in R^x \cup S^x\}$$

and

$$\begin{aligned} C_{x,k} = &\big\{(r,s) \in \mathbf{V}^2 : (k,(r,s)) \in E^x,\ \exists b \in S^x \text{ with arrow from } b \text{ to } (k,(r,s))\big\} \\ &\cup \big\{(r,s) \in \mathbf{W}^2 : (k,(r,s)) \in F^x,\ \exists a \in R^x \text{ with arrow from } (k,(r,s)) \text{ to } a\big\}. \end{aligned}$$

$B_{x,k}$ can be interpreted as the set of paired changes involving x and $x+k$ for one of the configurations (η or ξ) which take the coupling out of K, whereas $C_{x,k}$ can be interpreted as the set of paired changes involving x and $x+k$ than can be used in the construction of the coupling to avoid an exit from K for x, due to a change of the other configuration. We note that all the sets E^x, F^x, G^x, R^x, S^x, $\mathscr{A}^x$, $B_{x,k}$ and $C_{x,k}$ depend on the configurations η and ξ, although this fact will not be reflected in notation for the sake of simplicity. We define a network flow problem in the network $(G^x, \mathscr{A}^x)$, for fixed $x \in \mathbf{S}$ and $(\eta, \xi) \in K$, and we will show in Proposition 1 that the feasibility of the problem implies the existence of the generator of a Markovian $K-$coupling of the processes. We also give the explicit form of such a coupling in the proof of the proposition. The statement of the network flow problem is as follows: First of all we define t_a^x for $a \in G^x$ as $t_a^x = 1$ if $a \in E^x \cap \mathbf{V}$ or $a \in F^x \cap \mathbf{W}$ and, for $a = (k,(r,s)) \in E^x \cap (\mathbf{N}_x \times \mathbf{V}^2)$ or $a = (k,(r,s)) \in F^x \cap (\mathbf{N}_x \times \mathbf{W}^2)$, as

$$t_a^x = \begin{cases} 1 & \text{if } (r,s) \in B_{x,k}, (s,r) \notin B_{x+k,-k}, \\ 0 & \text{if } (r,s) \notin B_{x,k}, (s,r) \in B_{x+k,-k}, \\ 1 & \text{if } (r,s) \notin B_{x,k},\ (s,r) \notin B_{x+k,-k},\ (r,s) \in C_{x,k}, (s,r) \notin C_{x+k,-k}, \\ 0 & \text{if } (r,s) \notin B_{x,k},\ (s,r) \notin B_{x+k,-k},\ (r,s) \notin C_{x,k}, (s,r) \in C_{x+k,-k}, \\ 1/2 & \text{otherwise.} \end{cases}$$

The offer of the points $b \in F^x$ is $o_b^x = t_b^x c_b^2(x,\xi)$ (and must be sent if they are in S^x) and the demand of the points $a \in E^x$ is $d_a^x = t_a^x c_a^1(x,\eta)$ (and must be satisfied if they are in R^x). We can state this network flow problem equivalently as: there exists a function (flow) $f^x : \mathscr{A}^x \to \mathbb{R}$ verifying

$$\begin{aligned} 0 \le \sum_{a \in G^x} f_{ba}^x - \sum_{a \in G^x} f_{ab}^x &\le o_b^x \quad \forall b \in F^x \setminus S^x, \\ 0 \le \sum_{b \in G^x} f_{ba}^x - \sum_{b \in G^x} f_{ab}^x &\le d_a^x \quad \forall a \in E^x \setminus R^x, \\ \sum_{a \in G^x} f_{ba}^x - \sum_{a \in G^x} f_{ab}^x &= o_b^x \quad \forall b \in S^x, \\ \sum_{b \in G^x} f_{ba}^x - \sum_{b \in G^x} f_{ab}^x &= d_a^x \quad \forall a \in R^x, \\ f_{ba}^x &\ge 0 \quad \forall a, b \in G^x. \end{aligned} \tag{2}$$

Define the relationship $\preceq$ in G^x as $a \preceq b$ whenever there is an arrow from b to a in the network $(G^x, \mathscr{A}^x)$. It is easy to see that $\preceq$ is a pre-ordering in G^x but not necessarily a partial order, since $a \preceq b$, $b \preceq a$ do not imply $a = b$. A set Γ is called $\preceq -$increasing if $a \in \Gamma$ and $a \preceq b$ imply $b \in \Gamma$. In the same way, a set Γ is $\preceq -$decreasing if $a \in \Gamma$ and $b \preceq a$ imply $b \in \Gamma$.

Proposition 1. *If for any $x \in S$ and $(\eta, \xi) \in K$ the network flow problem (2) is feasible, then there exists a Markovian $K-$coupling of the processes which can be constructed in terms of the solution f^x of (2).*

We now give conditions on the rates of the processes for the existence of a solution of the network flow problem.

Proposition 2. *The network flow problem (2) is feasible if the next two conditions are satisfied:*

(a) if Γ is a $\preceq -$increasing subset of G^x, then

$$\sum_{a \in \Gamma \cap R^x} d_a^x \leq \sum_{a \in \Gamma \cap F^x} o_a^x, \tag{3}$$

(b) if Γ is a $\preceq -$decreasing subset of G^x, then

$$\sum_{a \in \Gamma \cap S^x} o_a^x \leq \sum_{a \in \Gamma \cap E^x} d_a^x. \tag{4}$$

We now state the main result of the section:

Theorem 3. *Consider two interacting particle systems with rates c^1 and c^2. If for every $x \in \mathbf{S}$, $(\eta, \xi) \in K$, the rates verify the following conditions:*

(i) for all $\preceq -$increasing $\Gamma \subseteq G^x$ such that $\forall a \in \Gamma \cap E^x$ we have $(\eta_{xa}, \xi) \notin K$, it holds:

$$\sum_{a \in \Gamma} t_a^x c_a^1(x, \eta) \leq \sum_{a \in \Gamma} t_a^x c_a^2(x, \xi), \tag{5}$$

(ii) for all $\preceq -$decreasing $\Gamma \subseteq G^x$ such that $\forall a \in \Gamma \cap F^x$ we have $(\eta, \xi_{xa}) \notin K$, it holds:

$$\sum_{a \in \Gamma} t_a^x c_a^1(x, \eta) \geq \sum_{a \in \Gamma} t_a^x c_a^2(x, \xi) \tag{6}$$

Then, there exists a Markovian $K-$coupling of the processes.

§3. Application

In this section we use Theorem 3 to give sufficient conditions for the stochastic comparison of tandem Jackson networks. We begin with the definition of such networks. We will work with tandem Jackson networks defined as follows. There are M stations located on a row, each one having two substations; x_i and y_i, stand for substations 1 and 2 of station i, $i = 1, \ldots, M$. The evolution of the network is as follows:

– customers arrive from outside at x_i at rate β_i, $i = 1,\dots,M$;

– the service time at substation x_i is exponentially distributed with parameter δ_1^i, $i = 1,\dots,M$;

– after being served at substation x_i customers either go to substation y_i with probability $p_{x_iy_i}$ or leave the network with probability $1 - p_{x_iy_i}$, $i = 1,\dots,M$;

– the service time at substation y_i is exponentially distributed with parameter δ_2^i, $i = 1,\dots,M$;

– after being served at substation y_i customers either go to substation x_i with probability $p_{y_ix_i}$, go to substation x_{i+1} with probability $p_{y_ix_{i+1}}$ or leave the network with probability $1 - p_{y_ix_i} - p_{y_ix_{i+1}}$, $i = 1,\dots,M$, $x_{M+1} \equiv x_1$;

– all random quantities above are independent.

Let η be a configuration of a network, that is $(\eta(x_i),\eta(y_i))$ is the number of customers at substations x_i, y_i of station i. The *workload* of station i in a configuration η is defined as $T_i(\eta) = \eta(x_i)(\delta_1^i + \delta_2^i) + \eta(y_i)\delta_2^i$. We say that a network η has smaller workload than another network η' whenever $T_i(\eta) \le T_i(\eta')$ for all $i = 1,\dots,M$. Our objective is to give conditions on the rates of two stations (whose evolutions will be written (η_t) and (η'_t) respectively) which assure the stochastic dominance of (η_t) by (η'_t) in terms of the workload. In other words, if $T_i(\eta_0) \le T_i(\eta'_0)$ for all $i = 1,\dots,M$, then

$$P\{T_i(\eta_t) \le T_i(\eta'_t) \text{ for all } i = 1,\dots,M\} = 1 \qquad \forall\, t > 0.$$

In this case, we write $(\eta_t) \le_{workload} (\eta'_t)$. We apply Theorem 3 to (η_t), (η'_t) with the set $K = \{(\eta,\eta') : T_i(\eta) \le T_i(\eta'),\ \forall\, i = 1,\dots,M\}$.

Theorem 4. *Let (η_t), (η'_t) be two tandem Jackson networks such that $\delta_1^i \ge \delta_2^i$ for $i = 1,\dots,M$. The following conditions are sufficient for $(\eta_t) \le_{workload} (\eta'_t)$: for all η,η' with $T_i(\eta) \le T_i(\eta')$, for all $i = 1,\dots,M$, we have*

(i) for all i such that $\delta_1^i + \delta_2^i > \Delta T_i \ge \delta_2^i$ (where $\Delta T_i = T_i(\eta') - T_i(\eta)$ and $y_0 \equiv y_M$):

$$\delta_2^{i-1}(p_{y_{i-1}x_i} - p'_{y_ix_i})^+ + \delta_2^i p_{y_{i-1}x_i}\mathbf{1}_{\{\Delta T_i < \delta_1^i\}} + \beta_i \le \delta_2^{i-1}(p'_{y_{i-1}x_i} - p_{y_{i-1}x_i})^+\mathbf{1}_{\{\Delta T_{i-1} \ge \delta_2^{i-1}\}} + \delta_2^i p'_{y_ix_i} + \beta'_i,$$

(ii) for all i such that $\Delta T_i < \delta_2^i$ (where $\Delta T_i = T_i(\eta') - T_i(\eta)$ and $x_{M+1} \equiv x_1$):

$$\beta_i + \delta_2^{i-1}(p_{y_{i-1}x_i} - p'_{y_{i-1}x_i})^+ + \delta_2^i(p_{y_ix_i} - p'_{y_ix_i})^+\mathbf{1}_{\{\Delta T_i < \delta_1^i\}} \le \beta'_i + \delta_2^{i-1}(p'_{y_{i-1}x_i} - p_{y_{i-1}x_i})^+\mathbf{1}_{\{\Delta T_{i-1} \ge \delta_2^{i-1}\}},$$

$$p_{x_iy_i} \le p'_{x_iy_i}$$

and, if $\Delta T_{i+1} > \delta_1^{i+1} + \delta_2^{i+1}$,

$$p_{y_ix_i} \le p'_{y_ix_i}$$

and, if $\Delta T_{i+1} \leq \delta_1^{i+1} + \delta_2^{i+1}$,

$$(p_{y_i x_{i+1}} \leq p'_{y_i x_{i+1}},\ p_{y_i x_i} \leq p'_{y_i x_i}) \quad or \quad (p_{y_i x_{i+1}} > p'_{y_i x_{i+1}},\ p_{y_i x_i} + p_{y_i x_{i+1}} \leq p'_{y_i x_i} + p'_{y_i x_{i+1}}).$$

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