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PERIODIC ORBITS AROUND SIMPLE SHAPED BODIES

Fernando Blesa

Abstract. The study of periodic orbits is a very important discipline in dynamical systems. In this paper we are interested in analyzing the dynamics of a particle around simple planar plates. We present several families of periodic orbits in the plane of these plates.

Keywords: Periodic orbits, analytical continuation, regular polygons.

AMS classification: 65P10, 34K13.

§1. Introduction

The most common method to compute the gravitational field of the celestial bodies is by using expansions in spherical harmonics. But these series do not converge. This fact is important if one plans to study orbits around some irregular asteroids, Kuiper Belt Objects, or natural satellites.

Another way to calculate potentials of bodies with simple geometrical shapes is to use expressions of irregular polyhedra with an arbitrary number of planar faces. The advantage of this method is that it allows to avoid the computation of series.

There are several methods which try to eliminate this problem found in the literature. The most notable, apart from the polyhedra approach we have already mentioned, is the use of "mascons" (point masses placed in a suitable way to reproduce the body's mass distribution). A comparison of these methods can be found on Rossi *et al.* [8].

In this work we are interested in the study of the dynamics of the orbits around homogeneous bodies with simple geometric shapes. We proceed as follows: first, we calculate the gravitational potential of square and triangular plates. Then, by computing the Poincaré surface of section of our problem we are able to obtain good guesses of initial conditions for several periodic orbits in the plane of the plates. Finally, the AUTO [7] software package is used to calculate the stability index of these orbits.

§2. The potential of a plate

First works on the potential of a finite plate of homogeneous density σ were done by Kellog [5] and MacMillan [6]. More recent works are due by Broucke [4] and Werner [9]. These two authors gave an intrinsic formulation of the potential of a general triangular plate. Using this result, they are able to derive an intrinsic expression for the gravitational potential of a polyhedron. Werner used the Gauss divergence theorem to obtain his results whereas Broucke based his derivations on a short table of elementary integrals.

In this work, we will use the potential obtained by Broucke, although we remark that we have also repeated our calculations with Werner expressions and we have obtained exactly the same results.

The potential function of a plate of n vertices in its same plane, i.e. in the xy-plane, is:

$$U = G\sigma \sum_{i=1}^{n} (C_{i,\mathrm{mod}(i+1,n)} L_{i,\mathrm{mod}(i+1,n)} / r_{i,\mathrm{mod}(i+1,n)})$$

where $\mathrm{mod}(i+1,n)$ is the modulus n of $i+1$. The various values are defined as follows:

- r_{ij} are the sides of the plate,
- C_{ij} are the cross products of the position vectors of the vertices,
- d_i are the distances of the point mass to the vertices, and
- L_{ij} are the logarithms defined by:

$$L_{ij} = \log\left(\frac{d_i + d_j + r_{ij}}{d_i + d_j - r_{ij}}\right).$$

It should be noted that the potential of the plate outside the xy-plane (see Broucke [4] or Werner [9]) includes several arc tangent terms and its is much more complicated.

The equations of motion of a point mass around this plate are:

$$\ddot{x} = +G\sigma \sum_{i=1}^{n} (y_i - y_{\mathrm{mod}(i+1,n)}) L_{i,\mathrm{mod}(i+1,n)},$$
$$\ddot{y} = -G\sigma \sum_{i=1}^{n} (x_i - x_{\mathrm{mod}(i+1,n)}) L_{i,\mathrm{mod}(i+1,n)}.$$

With the potential already determined, we study the phase space of the orbits of a point mass around the plates. First, we use the Poincaré surfaces of sections to study the regions around the triangular and square plates. Our aim is to compute families of periodic orbits. The method to continue the orbits in order to obtain the families with the AUTO2000 software package is explained in detail in Blesa and Elipe [3].

In what follows, we will consider the following plates:

- $n = 3$. An equilateral triangle with the origin at the center of mass, its sides $r_{ij} = 1$, and with one of its vertices in the minus x axis.
- $n = 4$. A square with its sides parallel to the axes and of length $r_{ij} = 1$.

Figure 1 shows both potentials with $G\sigma = 1$.

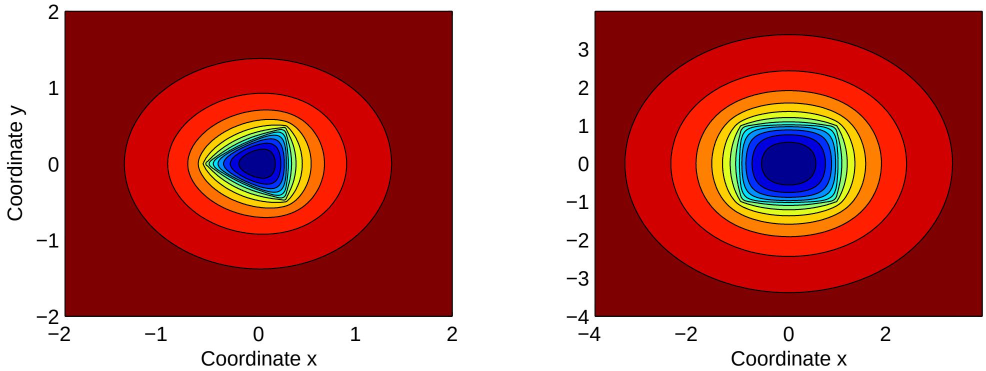

Figure 1: Level curves of the potential of a triangle and a square

§3. Triangular plate

In this section we consider a triangular plate which vertex coordinates are

$$(\sqrt{3}/6, 1/2), (\sqrt{3}/3, 0), (\sqrt{3}/6, -1/2).$$

All the computer results have been done in a 2.8 Ghz Linux PC. As a numerical integrator we use a variable step, variable order Taylor method [1] to compute our results. We have integrated many orbits with the same value of the Energy constant, and we have plotted $(x_0, \dot{x}_0)$ when $y = 0$ to obtain the Poincaré surface of section. The limit curves are not symmetric with respect to the y axis due to the geometry of the triangle.

As an example, in Figure 2 we plot the Poincaré surface of section for an energy $E = -0.1$. We may observe two fixed points which correspond to a stable periodic orbit (a direct orbit on the left and retrograde orbit on the right). Around them we have quasi-periodic orbits. Finally we have an apparently empty region. For this value of the energy, this empty region between the limit curves and the quasi-periodic orbits correspond to collision orbits. There are no points there since these orbits collide before we reach $y = 0$. For higher values of the energy there could be escape orbits as well, but they did not cut the x axis either. So the curves of this surface of section we are interested in are direct or retrograde orbits. It is also possible to find out some islands, but they are too small to be shown on this figure. They correspond to multiple periodic orbits. Figure 3 shows on the left a direct periodic orbit and on the right a direct orbit belonging to the islands which are around the first one. We reproduce their initial conditions in Table 1. It should be noted that since the orbits are symmetrical, the values of y_0 and $\dot{x}_0$ are equal to zero.

Once we have determined the orbits from the Poincaré surface of section, we have chosen to continue the family of the left orbit of Figure 3.

Using the AUTO software package, we are able to compute the eigenvalues of the monodromy matrix of the orbit. Its trace allows us to obtain its stability index k, which is drawn on Figure 4. It takes the critic values $k = +2$ and $k = -2$ for two values of x_0 close to the triangular plate. If we increase x_0, the stability grows until it arrives to a limit of $k = +2$ for big values of x_0.

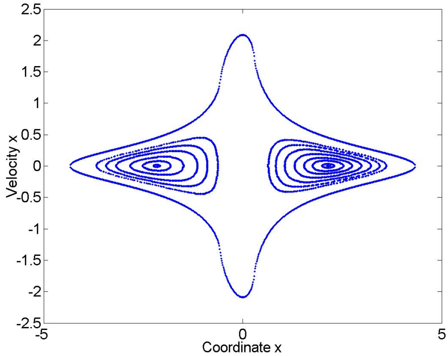

Figure 2: Poincaré surface of section. $E = -0.1$.

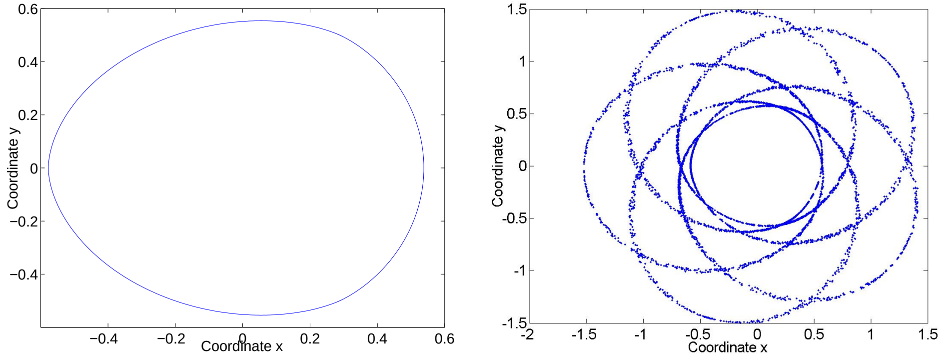

Figure 3: Periodic orbits around a triangle. On the left we plot a stable direct orbit with $k = -2$, $E = -0.315$ and period $p = 1$. On the right we have a multiple periodic orbit ($p = 6$) with $E = -0.2$.

Orbits	x_0	y_0	$\dot{x}_0$	$\dot{y}_0$
1	-0.583500E+00	0.000000E+00	0.000000E+00	0.120021E+01
2	-0.598450E+00	0.000000E+00	0.000000E+00	0.105375E+01

Table 1: Initial conditions of the periodic orbits in Figure 3.

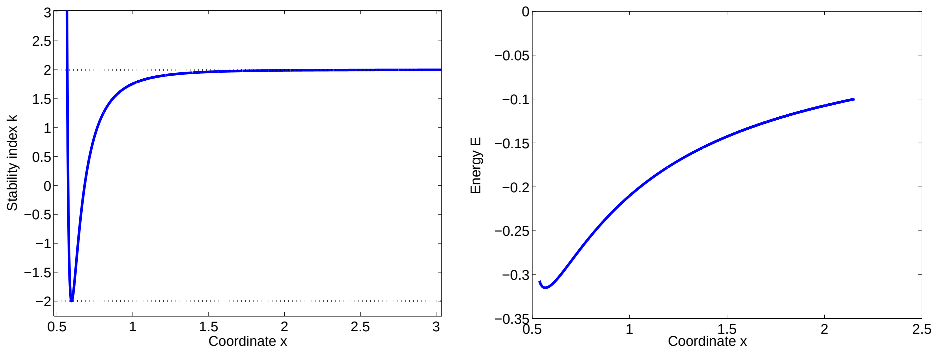

Figure 4: Stability index k and energy E of the stable orbit in Figure 3.

In Figure 4 on the right, we have plotted the variation of the energy as a function of x_0 for the family corresponding to the stable orbit on the left of Figure 3.

The critic value $k = -2$ is the origin of another branch of periodic orbits: the retrograde orbits. Their stability presents a similar evolution to the direct orbits so we do not plot it (for more details, see Blesa [2]).

§4. Square plate

The vertex coordinates of the square plate are

$$(1,1),\ (-1,1),\ (-1,-1),\ (1,-1).$$

For this plate we repeat the previous procedure in a similar way. We obtain the Poincaré surfaces of sections. The limit curves of these surfaces of sections of the square plate are symmetrical with respect to the y axis, contrary to the triangle limit curves, due to its different geometry. In Figure 5 we plot a zoom of the direct orbits surface of section in order to show more clearly the details around the stable orbit, but, similarly to the triangular plate, there is another symmetric surface of section in the positive x axis, which belongs to the retrograde orbits.

In Figure 5, we can appreciate fixed points and islands. The former belong to periodic orbits of simple period and the latter are multiple period orbits. Also on the islands, we have stable and unstable orbits. These orbits are symmetric periodic orbits and they are plotted in Figure 6. We give their initial conditions in Table 2. The value of the energy for these orbits is $E = -0.8$. From top to bottom and from left to right, we have first the stable orbit which also is on the center of Figure 5. Then we have the outer islands (multiple period p=3) and the inner islands (multiple period p=5). Finally, we plot an unstable periodic orbit of period $p = 4$ which is found between both chains of islands.

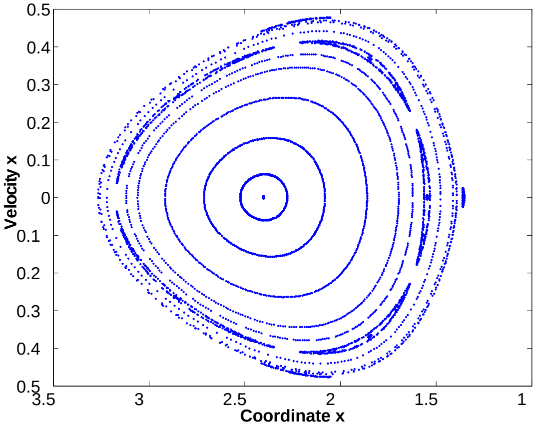

Figure 5: Poincaré surface of section. $E = -0.8$.

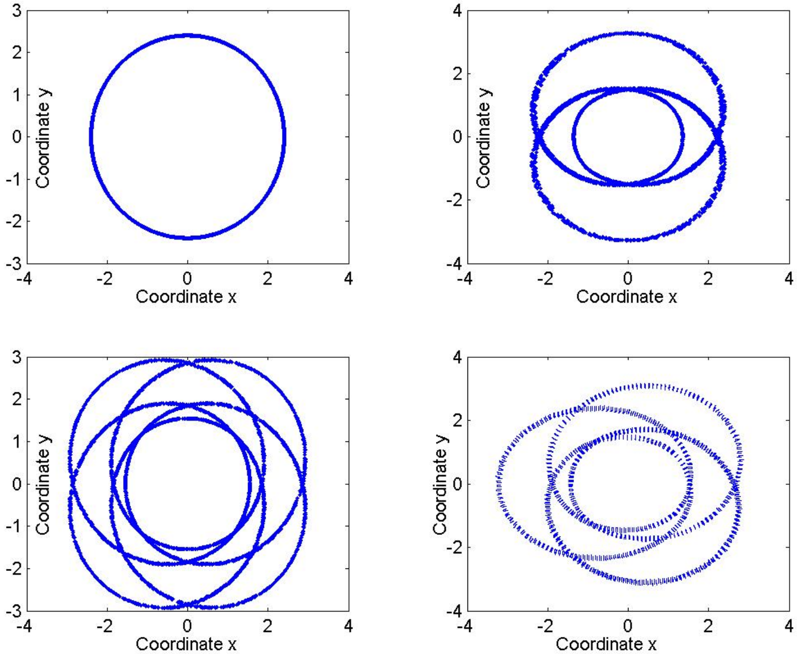

Figure 6: Orbits around the square plate. $E = -0.8$.

Orbits	x_0	y_0	$\dot{x}_0$	$\dot{y}_0$
1	-0.240175E+01	0.000000E+00	0.000000E+00	0.134951E+01
2	-0.135675E+01	0.000000E+00	0.000000E+00	0.218123E+01
3	-0.154650E+01	0.000000E+00	-0.166667E−02	0.197180E+01
4	-0.143375E+01	0.000000E+00	0.000000E+00	0.209084E+01

Table 2: Initial conditions of the periodic orbits in Figure 6. $E = -0.8$.

§5. Conclusions and future work

We have obtained periodic orbits in the plane around a triangular and a square plate. The procedure we used is to calculate their Poincaré surface of sections for several values of the energy, and locate the fixed points and islands around them. Also, with the help of the AUTO continuation package we have found the families of some of the previously calculated periodic orbits. As an example we have computed the stability index and the evolution of the energy of the direct stable orbit around the triangular plate.

Part of the current work is to study[2] the orbits in the *xz*-plane perpendicular to the plates. The potential is more complex since it includes not only the logarithmic terms but also arc tangent terms, see Broucke [4] and Werner [9].

Acknowledgements

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Fernando Blesa
Grupo de Mecánica Espacial
Departamento de Física Aplicada
Universidad de Zaragoza
`fblesa@unizar.es`