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UNILATERAL AND BILATERAL CHARACTERIZATIONS OF INCREASING MAPS

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Abstract. We define and study various properties of lateral increasing for mappings defined between two ordered spaces. After introducing some particular classes of ordered spaces, we formulate unilateral characterizations of the property of increasing. The main theorem characterizes the increasing of a map defined on a complete totally ordered space with bilateral conditions which generalize the classical notions of right or left increasing (cf. [2]).

Keywords: Characterization of isotone (order-preserving, increasing) maps, left and right isotone maps, totally ordered spaces, complete ordered spaces.

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§1. Introduction

The aim of this paper is to give characterizations of isotone (increasing) maps by using some variants of left and right isotone properties. Characterizations of strictly increasing maps can be found in [1].

We first recall some classical definitions and notations to avoid ambiguity. If $\leq$ is a binary relation on E which is reflexive, antisymmetric and transitive, $(E,\leq)$ is an *ordered space*. An ordered space such that any two elements are comparable is a *totally ordered space*. An ordered space such that any nonempty majorized subset admits a supremum (and thus, by theorem, any nonempty minorized subset admits an infimum) is a *complete ordered space* (cf. [3]). We will write $x<y$ when $x\leq y$ and $x\neq y$. The following notations are defined in [2]:

$$[a,\rightarrow[\,=\{x\in E : a\leq x\}, \quad]a,\rightarrow[\,=\{x\in E : a<x\}, \quad]\leftarrow,b]=\{x\in E : x\leq b\},$$
$$]\leftarrow,b[\,=\{x\in E : x<b\}, \quad [a,b]=[a,\rightarrow[\,\cap\,]\leftarrow,b], \quad]a,b[\,=\,]a,\rightarrow[\,\cap\,]\leftarrow,b[,$$
$$[a,b[\,=[a,\rightarrow[\,\cap\,]\leftarrow,b[, \quad]a,b]=\,]a,\rightarrow[\,\cap\,]\leftarrow,b].$$

These subsets are called *intervals*, those of the form $[a,b]$ are called *segments*. Since no confusion is possible when several ordered spaces are considered, we will use the same notations for their order, their intervals, etc.

In the following definitions, $f:(A,\leq)\rightarrow(B,\leq)$ is a map from an ordered space $(A,\leq)$ into an ordered space $(B,\leq)$. The notion of "preservation of the order" is classical:

Definition 1. f is *isotone* or *increasing* if:

$$\forall (x,y)\in A\times A,\ x\leq y \Rightarrow f(x)\leq f(y). \tag{I}$$

We remark immediately that in this definition (I), formulated from each element of the product space $A \times A$, can be expressed by a right unilateral form, formulated from each element of the subset A^+:

$$(I) \Longleftrightarrow (I_R^{\rightarrow}) \Longleftrightarrow (I^{\rightarrow})$$

where, by definition,

$$A^+ := \{x \in A : \exists y \in A,\ x < y\},$$
$$(I_R^{\rightarrow}) \Longleftrightarrow \forall x \in A^+,\ f_{|[x,\rightarrow[}\ \text{ is isotone},$$
$$(I^{\rightarrow}) \Longleftrightarrow \forall x \in A^+,\ f([x,\rightarrow[) \subset [f(x),\rightarrow[.$$

The same is possible with a left unilateral form, formulated from each element of the subset A^-:

$$(I) \Longleftrightarrow (I_R^{\leftarrow}) \Longleftrightarrow (I^{\leftarrow})$$

where, by definition,

$$A^- := \{x \in A : \exists y \in A,\ y < x\},$$
$$(I_R^{\leftarrow}) \Longleftrightarrow \forall x \in A^-,\ f_{|]\leftarrow,x]}\ \text{ is isotone},$$
$$(I^{\leftarrow}) \Longleftrightarrow \forall x \in A^-,\ f(]\leftarrow,x]) \subset]\leftarrow, f(x)].$$

We now define some various (left or right) lateral variants. We remark that, for left or right lateral versions, saying that $f : (A,\leq) \to (B,\leq)$ satisfies the left lateral version means by definition that $f : (A,\geq) \to (B,\geq)$ satisfies the right lateral version. Nevertheless we still define the two versions for more completeness.

We first introduce, in an abstract form, a notion defined in [2] (for a real-valued function defined on an open interval of the real field):

Definition 2. f is *left isotone* if:

$$\forall x \in A^-,\ \exists a \in]\leftarrow,x[,\ f([a,x]) \subset]\leftarrow, f(x)]. \qquad (I^-)$$

f is *right isotone* if:

$$\forall x \in A^+, \exists b \in]x,\rightarrow[,\ f([x,b]) \subset [f(x),\rightarrow[. \qquad (I^+)$$

It is obvious that

$$(I) \Longrightarrow (I^-) \text{ and } (I^+). \qquad (1)$$

The converse implication is false in general (an example is given in § 3). The purpose of §3 is to determine some sufficient conditions on the ordered spaces so that the isotone property can be obtained by bilateral conditions such as (I^-) and (I^+) or by more general conditions that will be defined below.

We first define some notions close to the previous ones by taking into account all possible “directions” (for the left or right side); in other words, we formulate some “star” versions.

Definition 3. f is *left star-isotone* if:

$$\forall x \in A^-,\ \forall c \in]{\leftarrow},x[,\ \exists a \in [c,x[,\ f([a,x]) \subset]{\leftarrow}, f(x)] \qquad (I^{-\star})$$

f is *right star-isotone* if:

$$\forall x \in A^+,\ \forall c \in]x,{\rightarrow}[,\ \exists b \in]x,c],\ f([x,b]) \subset [f(x),{\rightarrow}[\qquad (I^{+\star})$$

These notions appear in the theorem of isotone maps under unilateral conditions in §2. It is easy to prove $(I^{-\star}) \Rightarrow (I^-)$ and $(I^{+\star}) \Rightarrow (I^+)$; converse implications are false. For instance, (I^-) is satisfied but $(I^{-\star})$ is not for the map $(0,1) \mapsto 1, (1,0) \mapsto 3, (1,1) \mapsto 2$ where $\mathbb{R}^2$ is endowed with the product order (the partial order associated with the cone $[0,+\infty[^2$). Nevertheless, when the domain A is totally ordered, we get equivalent properties: $(I^{-\star}) \Leftrightarrow (I^-)$ and $(I^{+\star}) \Leftrightarrow (I^+)$. Proofs are obvious.

§2. Unilateral characterization of isotone maps

We now define some other lateral variants from properties $(I_R^{\rightarrow})$ and $(I_R^{\leftarrow})$. Firstly, some "local" versions.

Definition 4. f is *isotone by left restriction* if:

$$\forall x \in A^-,\ \exists a \in]{\leftarrow},x[,\ f_{|[a,x]} \text{ is isotone.} \qquad (I_R^-)$$

f is *isotone by right restriction* if:

$$\forall x \in A^+,\ \exists b \in]x,{\rightarrow}[,\ f_{|[x,b]} \text{ is isotone.} \qquad (I_R^+)$$

Secondly, the corresponding "star" versions.

Definition 5. f is *star-isotone by left restriction* if:

$$\forall x \in A^-,\ \forall c \in]{\leftarrow},x[,\ \exists a \in [c,x[,\ f_{|[a,x]} \text{ is isotone.} \qquad (I_R^{-\star})$$

f is *star-isotone by right restriction* if:

$$\forall x \in A^+,\ \forall c \in]x,{\rightarrow}[,\ \exists b \in]x,c],\ f_{|[x,b]} \text{ is isotone.} \qquad (I_R^{+\star})$$

The following table gives some implications between the notions of right lateral increasing that are defined previously.

$$\begin{array}{ccccc}
(I) & \Leftrightarrow & (I_R^{\rightarrow}) & \Leftrightarrow & (I^{\rightarrow}) \\
 & & \Downarrow & & \Downarrow \\
 & & (I_R^{+\star}) & \Rightarrow & (I^{+\star}) \\
 & & \Downarrow & & \Downarrow \\
 & & (I_R^{+}) & \Rightarrow & (I^{+})
\end{array}$$

Generally, properties $(I_R^{+\star})$, $(I^{+\star})$, (I_R^+) or (I^+) are not sufficient to ensure the isotone property. Even if A is countable, totally ordered and complete (and also well ordered);

consider for instance $A := \left\{-n^{-1} : n \in \mathbb{N}^*\right\} \cup \{0\}$, $B := \mathbb{R}$ and f defined by $f(0) = 0$ and $f(-n^{-1}) = n$ (for $n \in \mathbb{N}^*$): $(I_R^{+\star})$ is satisfied without f being isotone.

The purpose of this section is the study of some converse implications relative to the isotone property (I).

We thus introduce the following properties which concern ordered spaces.

Definition 6. An ordered space $(A, \leq)$ is with *finite segments* if:

$$\forall (x,y) \in A \times A,\ x \leq y \Rightarrow [x,y] \text{ is a finite subset.}$$

If A is a finite subset, $(A, \leq)$ is with finite segments. The converse is false: $\mathbb{N}$ and $\mathbb{Z}$ are with finite segments. We also remark that this property is not preserved for the closure of a subset of $\mathbb{R}$ (consider $I := \{n^{-1} : n \in \mathbb{N}^*\}$ and its closure $I \cup \{0\}$).

Definition 7. An ordered space $(A, \leq)$ is with *finite connections* if:

$$\begin{aligned} &\forall (x,y) \in A \times A,\ x \leq y \Rightarrow \exists N \in \mathbb{N},\ \exists (a_i)_{i=0}^{N+1} \subset A, \\ &\qquad x = a_0 \leq a_i \leq a_{i+1} \leq a_{N+1} = y \text{ and }]a_i, a_{i+1}[\, = \emptyset \text{ for } i = 0, 1, \ldots, N. \end{aligned}$$

If $(A, \leq)$ is with finite segments then it is with finite connections. The converse is false; consider for instance $\mathbb{N} \cup \{-\infty, +\infty\}$ endowed with the order defined by: $-\infty$ (resp. $+\infty$) is the minimum (resp. maximum). When the order is total, $(A, \leq)$ is with finite segments if and only if it is with finite connections; the proof of this result is straightforward.

A unilateral characterization of increasing is given below.

Theorem 1. *Let f be a map from an ordered space $(A, \leq)$ into another one $(B, \leq)$. If A is with finite connections then*

$$(I) \Longleftrightarrow (I_R^{+\star}) \Longleftrightarrow (I^{+\star}) \Longleftrightarrow (I_R^{-\star}) \Longleftrightarrow (I^{-\star}).$$

Proof. If $x \leq y$ then, since A is with finite connections, there exists a finite number n of points a_i such that $x \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq y$ and $]a_i, a_{i+1}[\, = \emptyset$ for $i = 0, 1, \ldots, n$, where $a_0 := x$ and $a_{n+1} := y$. By $(\mathrm{I}^{+\star})$, $f(a_i) \leq f(a_{i+1})$ for $i = 0, 1, \ldots, n$, since a_{i+1} is the only point in $]a_i, a_{i+1}]$ and thus $f([a_i, a_{i+1}]) \subset [f(a_i), \rightarrow[$. We get $f(x) \leq f(y)$ by transitivity. Thus, the first two equivalences are proved. The last two ones can be deduced since (I) is equivalent to the increasing of f from $(A, \geq)$ into $(B, \geq)$. $\square$

When the space A is totally ordered, $(I_R^{+\star}) \Leftrightarrow (I_R^{+})$ and also $(I^{+\star}) \Leftrightarrow (I^{+})$. As a direct consequence, we have the following characterizations.

Corollary 2. *If f is a map from a totally ordered space with finite segments $(A, \leq)$ into an ordered space $(B, \leq)$ then*

$$(I) \Longleftrightarrow (I_R^{+}) \Longleftrightarrow (I_R^{-}) \Longleftrightarrow (I^{+}) \Longleftrightarrow (I^{-}).$$

The conclusion of this corollary is not still true under the assumptions of the previous theorem. Indeed, these characterizations are not true if A, even finite, is not totally ordered. Consider for instance the subset A which consists of the points $(0,0)$, $(1,0)$ and $(0,1)$ and is endowed with the order associated with the cone $[0,+\infty[^2$ and the map f defined by $f(0,0) = 1$, $f(1,0) = 2$ and $f(0,1) = 0$: (I_R^+) is satisfied but (I^-) is not (and thus f is not isotone).

This corollary cannot be generalized to the case of a lattice (even finite), i.e. an ordered set whose subsets of two elements possess an infimum and a supremum; we can modify the last example by adding the point $(1,1)$ and by defining $f(1,1) = 2$.

Other unilateral characterizations will be established in the following section as a consequence of the main theorem.

§3. Bilateral characterization of isotone maps

In the main theorem which concerns the converse implication of (1) we will use the following lateral notions of increasing that are more general.

Definition 8. f is *left quasi-isotone* if:

$$\forall x \in A^-,\ \exists a \in \,]\leftarrow,x[\,,\ f(]a,x]) \subset \,]\leftarrow, f(x)]\,. \tag{QI^-}$$

f is *right quasi-isotone* if:

$$\forall x \in A^+,\ \exists b \in \,]x,\rightarrow[\,,\ f([x,b[) \subset [f(x),\rightarrow[\,. \tag{QI^+}$$

The converse implications of $(I^-) \Rightarrow (QI^-)$ and $(I^+) \Rightarrow (QI^+)$ are false: any sequence of real numbers is left and right quasi-isotone.

Definition 9. f is *pointwise left isotone* if

$$\forall x \in A^-,\ \exists a \in \,]\leftarrow,x[\,,\ f(a) \in \,]\leftarrow, f(x)]\,. \tag{PI^-}$$

f is *pointwise right isotone* if:

$$\forall x \in A^+,\ \exists b \in \,]x,\rightarrow[\,,\ f(b) \in [f(x),\rightarrow[\,. \tag{PI^+}$$

Trivially, $(I^-) \Rightarrow (PI^-)$ and $(I^+) \Rightarrow (PI^+)$ but the converse implications are false. For instance, the function defined by $f([-1,0]) = \{0\}$ and $f(x) = x-1$ if $x \in \,]0,1]$ is pointwise right isotone but is not right isotone.

The following notions are also used in the main theorem.

Definition 10. f is *left pseudo-isotone* or *pointwise left star-isotone* if:

$$\forall x \in A^-,\ \forall c \in \,]\leftarrow,x[\,,\ \exists a \in [c,x[\,,\ f(a) \in \,]\leftarrow, f(x)]\,. \tag{$PI^{-\star}$}$$

f is *right pseudo-isotone* or *pointwise right star-isotone* if:

$$\forall x \in A^+,\ \forall c \in \,]x,\rightarrow[\,,\ \exists b \in \,]x,c]\,,\ f(b) \in [f(x),\rightarrow[\,. \tag{$PI^{+\star}$}$$

It is straightforward that $(PI^{-\star}) \Rightarrow (PI^-)$ and $(PI^{+\star}) \Rightarrow (PI^+)$ and that the converse implications are false (consider the previous example). When the domain A is well ordered, these implications are not necessarily equivalences; consider the case where $A = \mathbb{N}$: $(PI^{+\star})$

and (PI^{+}) correspond respectively to the increasing of the sequence and one of its subsequences. We also have the implications $(I^{-\star}) \Rightarrow (PI^{-\star})$ and $(I^{+\star}) \Rightarrow (PI^{+\star})$.

The following table contents some links between several properties of right lateral increasing; the symbol $\updownarrow$ means the equivalence of the properties when the domain A is totally ordered.

$$\begin{array}{ccccccc}
(I) & \Rightarrow & (I_R^{+\star}) & \Rightarrow & (I^{+\star}) & \Rightarrow & (PI^{+\star}) \\
 & & \Downarrow\updownarrow & & \Downarrow\updownarrow & & \Downarrow \\
 & & (I_R^{+}) & \Rightarrow & (I^{+}) & \Rightarrow & (PI^{+}) \\
 & & & & \Downarrow & & \\
 & & & & (QI^{+}) & &
\end{array}$$

We can state the main theorem.

Theorem 3. *Let $(A,\leq)$ be a complete totally ordered space, $(B,\leq)$ an ordered space and $f:(A,\leq)\to(B,\leq)$ a map which is right pseudo-isotone and left quasi-isotone. Then f is isotone.*

Proof. Assume that f is not isotone: there exist $x_1 < x_2$ in A such that the relation $f(x_1) \leq f(x_2)$ is not satisfied in the ordered space B. Consider the subset

$$P := \{x \in [x_1,x_2] : f(x) \nleq f(x_2)\}. \tag{2}$$

On the one hand, $P \neq \emptyset$ for $x_1 \in P$: $f(x_1) \nleq f(x_2)$. On the other hand, P is a majorized subset: by definition, $P \subset \,]\leftarrow, x_2]$. Denote $x_0 := \sup P$ (P is a nonempty majorized subset of the complete ordered space A).

We have in particular ($x_1 \in P$ and x_2 majorizes P):

$$x_1 \leq x_0 \leq x_2. \tag{3}$$

We also get $P \subset [x_1,x_0]$ (for $P \subset [x_1,\rightarrow[$ by definition, and x_0 majorizes P). Now we justify the property:

$$f(x_0) \nleq f(x_2). \tag{4}$$

By assuming $f(x_0) \leq f(x_2)$, two cases appear. If $x_1 = x_0$ then $f(x_0) = f(x_1) \nleq f(x_2)$ (for $x_1 \in P$) which contradicts the assumption. Otherwise, $x_1 < x_0$ and then, since f is left quasi-isotone at x_0, there exists $a < x_0$ such that $f(]a,x_0]) \subset \,]\leftarrow, f(x_0)]$ and then, from the assumption, $f(x) \leq f(x_2)$ for all $x \in \,]a,x_0]$. We deduce that $P \subset \,]\leftarrow,x_0] \setminus]a,x_0]$ and thus, since A is totally ordered, $P \subset \,]\leftarrow,a]$. Consequently, a majorizes P and thus $x_0 \leq a$ which gives a contradiction. Thus, property (4) is satisfied.

We remark that $x_0 \in P$ (by (3)-(4)) and then $x_0 = \max P$. We get:

$$x_1 \leq x_0 < x_2. \tag{5}$$

Indeed, by (3), it is sufficient to justify that $x_0 \neq x_2$; this follows from $f(x_0) \neq f(x_2)$ (by (4)).

Since f is right pseudo-isotone at x_0, and $x_0 < x_2$ (by (5)), there exists $b \in \,]x_0,x_2]$ such that $f(b) \geq f(x_0)$. In particular, $b \in [x_1,x_2]$ but $b \notin P$ (for $x_0 < b$) which implies that $f(b) \leq f(x_2)$. Thus, $f(x_0) \leq f(x_2)$ which gives a contradiction with (4). $\square$

We can formulate the following bilateral characterizations of the increasing of a map when the order is total and complete on the domain:

Corollary 4. *Let $(A,\leq)$ be a complete totally ordered space, $(B,\leq)$ an ordered space and $f:(A,\leq)\to(B,\leq)$. Then:*

$$(I)\iff(I^-)\text{ and }(I^+)\iff(QI^-)\text{ and }(PI^{+\star})\iff(PI^{-\star})\text{ and }(QI^+).$$

Obviously, some other equivalences can be formulated above (by changing (QI^-) or $(PI^{-\star})$ by (I^-) and also (QI^+) or $(PI^{+\star})$ by (I^+)).

When the domain is $\mathbb{N}$ or $\mathbb{Z}$ we can simplify the previous characterizations. The set $\mathbb{N}$ is "discrete" (i.e., cf. [4], well ordered and such that all elements except the first one has a predecessor) but $\mathbb{Z}$ is not; nevertheless $\mathbb{Z}$ satisfies the following more general definitions.

Definition 11. An ordered space $(Z,\leq)$ is *left quasi-discrete* if:

$$\forall x\in Z^-,\ \exists a\in\,]\!\leftarrow,x[,\ ]a,x[\,=\emptyset;$$

$(Z,\leq)$ is *right quasi-discrete* if:

$$\forall x\in Z^+,\ \exists b\in\,]x,\rightarrow\![,\ ]x,b[\,=\emptyset.$$

$(Z,\leq)$ is *quasi-discrete* if it is left and right quasi-discrete.

A countable totally ordered space is not necessarily left or right quasi-discrete. The sets $I:=\{n^{-1}:n\in\mathbb{N}^*\}$, $J:=\{-n^{-1}:n\in\mathbb{N}^*\}$ and $I\cup J$ are quasi-discrete; $I\cup\{0\}$ is left but not right quasi-discrete, $J\cup\{0\}$ is right but not left quasi-discrete, $I\cup J\cup\{0\}$ is not left nor right quasi-discrete.

An ordered space with finite connections is quasi-discrete. The converse is false: $I\cup J$ is quasi-discrete but there exist no finite connections between an element of I and another one of J.

It is immediate that any map from a left (resp. right) quasi-discrete ordered space $(Z,\leq)$ into an ordered space $(B,\leq)$ is left (resp. right) quasi-isotone. We deduce the following corollaries giving unilateral characterizations of the isotone property.

Corollary 5. *Let $(Z,\leq)$ be a left quasi-discrete and complete totally ordered space, $(B,\leq)$ an ordered space and $f:(Z,\leq)\to(B,\leq)$. Then:*

$$(I)\iff(I_R^+)\iff(I^+)\iff(PI^{+\star}).$$

In the above corollary, property (PI^+) is not equivalent to the isotone property (consider the map defined on the finite subset $\{-1,0,1\}$ by $f(-1)=0=f(1)$ and $f(0)=-1$).

Corollary 6. *Let $(Z,\leq)$ be a right quasi-discrete and complete totally ordered space, $(B,\leq)$ an ordered space and $f:(Z,\leq)\to(B,\leq)$. Then:*

$$(I)\iff(I_R^-)\iff(I^-)\iff(PI^{-\star}).$$

Property (PI^-) is not equivalent to the isotone property even if the domain is finite and well ordered.

The equivalences of one of these two corollaries are not valid under the assumptions of the other one. Consider $Z = I \cup \{0\}$ which is a left quasi-discrete and complete totally ordered space and f given by $f(0) = 1$ and $f(i) = i$ for $i \in I$; this function satisfies (I_R^-) (and thus (I^-), $(PI^{-\star})$) but is not isotone.

When Z is a quasi-discrete and complete totally ordered space then all properties that appear in the two previous corollaries are equivalent. The fact that the order is total is crucial. Consider $Z = 2^{\mathbb{N}}$ with the inclusion for order; Z is a quasi-discrete and complete partially ordered space. Consider also the function $f : Z \to \{0,1\}$ given by $f(A) = 1$ if A is a finite subset of $\mathbb{N}$ and $f(A) = 0$ otherwise. This function is left and right isotone (in fact $(I_R^{-\star})$ and $(I_R^{+\star})$ hold) but is not isotone.

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