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STOKES AND NAVIER-STOKES EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS AND PRESSURE LOSS

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Abstract. The object of this present work is to show the existence and uniqueness results for the Stokes and Navier-Stokes equations which model the laminar flow of an incompressible fluid inside a two-dimensional plane channel with periodic sections. The data of the pressure loss coefficient in the channel enables us to establish a relation on the pressure and to thus formulate an equivalent problem.

Keywords: Stokes problem, Navier-Stokes equations, incompressible fluid, periodic boundary conditions, pressure loss.

§1. Introduction

The problem which one proposes to study here is that modelling a laminar flow inside a two-dimensional plane channel with periodic section. Let Ω be an open bounded connected lipschitzian set of $\mathbb{R}^2$ (see figure hereafter), and its boundary Γ is $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where $\Gamma_0 = \{0\} \times]-1,1[$ and $\Gamma_1 = \{1\} \times]-1,1[$. One defines the space

$$V = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega)\ ;\ \operatorname{div}\mathbf{v} = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_2,\ \mathbf{v}|_{\Gamma_0} = \mathbf{v}|_{\Gamma_1} \right\}.$$

Here, there are not external forces and viscosity is equal to 1. Thus for $\pi \in \mathbb{R}$ given, one considers the problem

$$(\mathscr{S}) \begin{cases} \text{Find } \mathbf{u} \in V \text{ such that} \\ \forall \mathbf{v} \in V,\ \int_\Omega \nabla\mathbf{u}.\nabla\mathbf{v}\, d\mathbf{x} = \pi \int_{-1}^{+1} v_1(1,y)\, dy. \end{cases}$$

§2. Stokes problem $(\mathscr{S})$

With an aim of drawing up the suitable functional framework of the problem, firstly one proposes to study the problem $(\mathscr{S})$.

Theorem 1. *Problem $(\mathscr{S})$ has an unique solution $\mathbf{u} \in V$. Moreover, there exists a constant depending only on Ω, $C(\Omega) > 0$, such that:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq \pi C(\Omega). \tag{1}$$

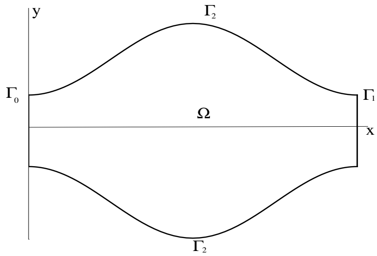

Figure 1: Geometry of channel

Proof. Let us note initially that space V provided the norm $H^1(\Omega)^2$ being a closed subspace of $H^1(\Omega)^2$ is thus an Hilbert space. Let us set

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}.\nabla\mathbf{v} d\mathbf{x}, \qquad l(\mathbf{v}) = \pi \int_{-1}^{+1} v_1(1,y)\, dy.$$

It is clear, thanks to the Poincaré inequality, that the bilinear continuous form is V-coercive. It is easy to also see that $l \in V'$. One deduces from Lax-Milgram Theorem the existence and uniqueness of $\mathbf{u}$ solution of $(\mathscr{S})$. Moreover,

$$\int_\Omega |\nabla\mathbf{u}|^2 d\mathbf{x} \le \pi\sqrt{2}\left(\int_{-1}^{+1} |u_1(1,y)|^2 dy\right)^{1/2},$$

i.e.

$$\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \le \pi\sqrt{2}\,\|\mathbf{u}\|_{L^2(\Gamma)} \le \pi\sqrt{2}\,\|\mathbf{u}\|_{H^{1/2}(\Gamma)}.$$

Thanks to the trace Theorem properties, finally one gets

$$\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \le \pi C_1(\Omega)\,\|\mathbf{u}\|_{H^1(\Omega)},$$

which implies the estimate (1). □

§3. Equivalent formulation of problem $(\mathscr{S})$

We now will give an interpretation of the problem $(\mathscr{S})$. One introduces the space

$$\mathscr{V} = \left\{\mathbf{v} \in \mathscr{D}(\Omega)^2 \;;\; \operatorname{div}\mathbf{v} = 0\right\}.$$

Let $\mathbf{u}$ be the solution of $(\mathscr{S})$. Then, for all $\mathbf{v} \in \mathscr{V}$, one has

$$\langle -\Delta\mathbf{u}, \mathbf{v}\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)} = 0.$$

So that thanks to De Rham Theorem, there exists $p \in \mathscr{D}'(\Omega)$ such that

$$-\Delta \mathbf{u} + \nabla p = 0 \text{ in } \Omega. \tag{2}$$

Moreover, since $\nabla p \in H^{-1}(\Omega)^2$, it is known that there exists $q \in L^2(\Omega)$ such that (see [1])

$$\nabla q = \nabla p \text{ in } \Omega. \tag{3}$$

The open set Ω being connected, there exists $C \in \mathbb{R}$ such that $p = q + C$, what means that $p \in L^2(\Omega)$. Let us recall that (see [1])

$$\inf_{K\in\mathbb{R}} \|p+K\|_{L^2(\Omega)} \leq C \|\nabla p\|_{H^{-1}(\Omega)^2}.$$

One deduces from the estimate (1) and from (2) that

$$\inf_{K\in\mathbb{R}} \|p+K\|_{L^2(\Omega)} \leq C \|\Delta \mathbf{u}\|_{H^{-1}(\Omega)^2} \leq C \|\mathbf{u}\|_{H^1(\Omega)^2} \leq \pi C(\Omega).$$

Since $\mathbf{u} \in H^1(\Omega)^2$ and $\mathbf{0} = -\Delta \mathbf{u} + \nabla p \in L^2(\Omega)^2$, it is shown that $-\partial \mathbf{u}/\partial \mathbf{n} + p\mathbf{n} \in H^{-1/2}(\Gamma)^2$ and one has the Green formula: for all $\mathbf{v} \in V$,

$$\int_\Omega (-\triangle \mathbf{u} + \nabla p) . \mathbf{v} \, d\mathbf{x} = \int_\Omega \nabla \mathbf{u} . \nabla \mathbf{v} \, d\mathbf{x} + \left\langle -\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n}, \mathbf{v} \right\rangle, \tag{4}$$

where the bracket represents the duality product $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Moreover, as $p \in L^2(\Omega)$ and $\triangle p = 0$ in Ω, one has $p \in H^{-1/2}(\Gamma)$. Consequently, one has therefore $\partial \mathbf{u}/\partial \mathbf{n} \in H^{-1/2}(\Gamma)^2$. The function $\mathbf{u}$ being solution of $(\mathscr{S})$, for all $\mathbf{v} \in V$, one has according to (2) and (4):

$$\left\langle \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}, \mathbf{v} \right\rangle = \pi \int_{-1}^{+1} v_1(1,y)\, dy, \tag{5}$$

i.e.

$$\left\langle \frac{\partial \mathbf{u}}{\partial x} - p\mathbf{e}_1, \mathbf{v} \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial \mathbf{u}}{\partial x} + p\mathbf{e}_1, \mathbf{v} \right\rangle_{\Gamma_0} = \langle \pi \mathbf{e}_1, \mathbf{v} \rangle_{\Gamma_1}, \tag{6}$$

where $\{\mathbf{e}_i\}$ is the orthonormal basis.

i) Let $\mu \in H_{00}^{1/2}(\Gamma_1)$ and let us set

$$\mu_2 = \begin{cases} \mu, & \text{on } \Gamma_0 \cup \Gamma_1, \\ 0, & \text{on } \Gamma_2, \end{cases} \quad \text{and} \quad \boldsymbol{\mu} = \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix},$$

where (see [2])

$$H_{00}^{1/2}(\Gamma_1) = \left\{ \varphi \in \mathbf{L}^2(\Gamma_1)\,;\, \exists \mathbf{v} \in H^1(\Omega), \text{ with } \mathbf{v}|_{\Gamma_2} = \mathbf{0},\ \mathbf{v}|_{\Gamma_0\cup\Gamma_1} = \varphi \right\}.$$

It is checked easily that

$$\boldsymbol{\mu} \in H^{1/2}(\Gamma)^2 \quad \text{and} \quad \int_\Gamma \boldsymbol{\mu} . \mathbf{n} \, d\sigma = 0.$$

So that there exists $\mathbf{v} \in H^1(\Omega)^2$ satisfying (see [3])

$$\operatorname{div}\mathbf{v} = 0 \text{ in } \Omega \qquad \text{and} \qquad \mathbf{v} = \boldsymbol{\mu} \text{ on } \Gamma.$$

In particular $\mathbf{v} \in V$ and according to (6), this yields

$$\left\langle \frac{\partial u_2}{\partial x}, \mu \right\rangle_{\Gamma_1} = \left\langle \frac{\partial u_2}{\partial x}, \mu \right\rangle_{\Gamma_0},$$

which means that

$$\left.\frac{\partial u_2}{\partial x}\right|_{\Gamma_1} = \left.\frac{\partial u_2}{\partial x}\right|_{\Gamma_0}. \tag{7}$$

One deduces now from (6) that, for all $\mathbf{v} \in V$,

$$\left\langle \frac{\partial u_1}{\partial x} - p, v_1 \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial u_1}{\partial x} + p, v_1 \right\rangle_{\Gamma_0} = \langle \pi, v_1 \rangle_{\Gamma_1}. \tag{8}$$

But, $\operatorname{div}\mathbf{u} = 0$ and $u_2|_{\Gamma_1} = u_2|_{\Gamma_0}$, one thus has

$$\left.\frac{\partial u_2}{\partial y}\right|_{\Gamma_1} = \left.\frac{\partial u_2}{\partial y}\right|_{\Gamma_0} \qquad \text{and} \qquad \left.\frac{\partial u_1}{\partial x}\right|_{\Gamma_1} = \left.\frac{\partial u_1}{\partial x}\right|_{\Gamma_0}. \tag{9}$$

Consequently, thanks to (8) one deduces:

$$\langle -p, v_1 \rangle_{\Gamma_1} + \langle p, v_1 \rangle_{\Gamma_0} = \langle \pi, v_1 \rangle_{\Gamma_1} \tag{10}$$

ii) While proceeding as in *i*), one shows that

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \pi \tag{11}$$

where the equality takes place with the $H^{1/2}$ sense. In short, if $\mathbf{u} \in H^1(\Omega)^2$ is solution of $(\mathscr{S})$, then there exists $p \in L^2(\Omega)$, unique up to an additive constant, such that:

$$-\Delta\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in} \quad \Omega, \tag{12}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{13}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_2, \qquad \mathbf{u}|_{\Gamma_1} = \mathbf{u}|_{\Gamma_0}, \tag{14}$$

$$\left.\frac{\partial \mathbf{u}}{\partial x}\right|_{\Gamma_1} = \left.\frac{\partial \mathbf{u}}{\partial x}\right|_{\Gamma_0}, \tag{15}$$

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \pi. \tag{16}$$

It is clear that, if $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ checks (12)–(16), then $\mathbf{u}$ is solution of $(\mathscr{S})$.

Theorem 2. *The problem (12)–(16) has an unique solution $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$, up to an additive constant for p. Moreover, $\mathbf{u}$ verifies $(\mathscr{S})$ and*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leq \pi C(\Omega).$$

Remark 1. The pressure verifies the relation (16), which means that p satisfies the relation of Patankar et al. [5].

§4. Navier-Stokes Equations

One takes again the assumptions of the Stokes problem given above. For $\pi \in \mathbb{R}$ given, one considers the following problem

$$(\mathscr{N}\mathscr{S})\ \begin{cases} \text{Find } \mathbf{u} \in V \text{ such that} \\ \forall \mathbf{v} \in V,\ \displaystyle\int_\Omega \nabla \mathbf{u}.\nabla \mathbf{v}\, d\mathbf{x} + b(\mathbf{u},\mathbf{u},\mathbf{v}) = \pi \int_{-1}^{+1} v_1(1,y)\, dy, \end{cases}$$

with

$$b(\mathbf{u},\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{u}.\nabla)\mathbf{v}.\mathbf{w}\, d\mathbf{x}.$$

With an aim of establishing the existence of the solutions of the problem $(\mathscr{N}\mathscr{S})$, one uses the Brouwer fixed point theorem (see [4, 6]). One will show it.

Theorem 3. *The problem* $(\mathscr{N}\mathscr{S})$ *has at least a solution* $\mathbf{u} \in V$*. Moreover,* $\mathbf{u}$ *satisfies the estimate* (1).

Proof. To show the existence of $\mathbf{u}$, one constructs the approximate solutions of the problem $(\mathscr{N}\mathscr{S})$by the Galerkin method and then thanks to the compactness arguments, one proves by passing to the limits some convergence properties.

i) For each fixed integer $m \geq 1$, one defines an approximate solution $\mathbf{u}_m$ of $(\mathscr{N}\mathscr{S})$ by

$$\begin{aligned} &\mathbf{u}_m = \sum_{i=1}^m g_{im}\mathbf{w}_i, \quad \text{with} \quad g_{im} \in \mathbb{R}, \\ &((\mathbf{u}_m,\mathbf{w}_i)) + b(\mathbf{u}_m,\mathbf{u}_m,\mathbf{w}_i) = \langle \pi\mathbf{n},\mathbf{w}_i\rangle_{\Gamma_1},\ i = 1,\ldots,m \end{aligned} \tag{17}$$

where $V_m = \langle \mathbf{w}_1,\ldots,\mathbf{w}_m\rangle$ is the vector space spanned by the vectors $\mathbf{w}_1,\ldots,\mathbf{w}_m$ and $\{\mathbf{w}_i\}$ is an Hilbertian basis of V which is separable. Let us note that (17) is equivalent to:

$$\forall \mathbf{v} \in V_m,\ ((\mathbf{u}_m,\mathbf{v})) + b(\mathbf{u}_m,\mathbf{u}_m,\mathbf{v}) = \pi \int_{-1}^{+1} v_1(1,y)\, dy. \tag{18}$$

With an aim to establish the existence of the solutions of the problem $\mathbf{u}_m$, the operator as follows is considered

$$\begin{aligned} \mathrm{P}_m :\ & V_m \longrightarrow V_m \\ & \mathbf{u} \ \longmapsto \mathrm{P}_m(\mathbf{u}) \end{aligned}$$

defined by

$$\forall \mathbf{u},\mathbf{v} \in V_m,\ ((\mathrm{P}_m(\mathbf{u}),\mathbf{v})) = ((\mathbf{u},\mathbf{v})) + b(\mathbf{u},\mathbf{u},\mathbf{v}) - \pi \int_{-1}^{+1} v_1(1,y)\, dy.$$

Let us note initially that P_m is continuous and

$$\forall \mathbf{u} \in V,\ b(\mathbf{u},\mathbf{u},\mathbf{u}) = 0.$$

Indeed, thanks to the Green formula, one has

$$b(\mathbf{u},\mathbf{u},\mathbf{u}) = -\frac{1}{2}\int_\Omega |\mathbf{u}|^2 \operatorname{div}\mathbf{u}\, d\mathbf{x} + \frac{1}{2}\int_\Gamma (\mathbf{u}.\mathbf{n})\,|\mathbf{u}|^2\, d\sigma = 0,$$

and one takes into account that $\operatorname{div}\mathbf{u} = 0$ in Ω and

$$\int_{\Gamma} (\mathbf{u}.\mathbf{n})\,|\mathbf{u}|^2\,d\sigma = \int_{\Gamma_0} (\mathbf{u}.\mathbf{n})\,|\mathbf{u}|^2\,d\sigma + \int_{\Gamma_1} (\mathbf{u}.\mathbf{n})\,|\mathbf{u}|^2\,d\sigma.$$

Thanks to Brouwer Theorem, there exists $\mathbf{u}_m$ satisfying (18) and

$$\|\mathbf{u}_m\|_{\mathbf{H}^1(\Omega)} \leq \pi C(\Omega).$$

ii) We can extract a subsequence $\mathbf{u}_\nu$ such that

$$\mathbf{u}_\nu \rightharpoonup \mathbf{u} \quad \text{weakly} \quad \text{in} \quad V,$$

and thanks to the compact imbedding of V in $L^2(\Omega)^2$, we obtain

$$\forall \mathbf{v} \in V,\ ((\mathbf{u},\mathbf{v})) + b\,(\mathbf{u},\mathbf{u},\mathbf{v}) = \pi \int_{-1}^{+1} v_1\,(1,y)\,dy.$$

As for the Stokes problem, one shows the existence of $p \in L^2(\Omega)$, unique except for an additive constant, such that the variational problem $(\mathscr{N}\mathscr{S})$ leads to

$$\begin{cases} -\Delta\mathbf{u} + (\mathbf{u}.\nabla)\,\mathbf{u} + \nabla p = \mathbf{0} & \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \Gamma_2, \\ \mathbf{u}|_{\Gamma_1} = \mathbf{u}|_{\Gamma_0}, & \end{cases}$$

with following boundary conditions

$$\left.\frac{\partial\mathbf{u}}{\partial x}\right|_{\Gamma_1} = \left.\frac{\partial\mathbf{u}}{\partial x}\right|_{\Gamma_0}$$

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \pi. \quad \square$$

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