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L^1-CONTRACTION AND STRONG CONVERGENCE OF APPROXIMATIONS FOR A PSEUDOMONOTONE SPDE

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Abstract. We prove an L^1-contraction principle to the problem

$$\begin{cases} du - \operatorname{div}(|\nabla u|^{p-2}\nabla u + F(u))\,dt = H(u)\,dW & \text{in } \Omega \times Q_T, \\ u = 0 & \text{on } \Omega \times (0,T) \times \partial D, \\ u(0,\cdot) = u_0 \in L^2(D) & \text{in } \Omega \times D, \end{cases}$$

for a cylindrical Wiener process $W(t)$ in $L^2(D)$ with respect to a filtration $(\mathcal{F}_t)$ satisfying the usual assumptions, $p \geq 2$ and $F : \mathbb{R} \to \mathbb{R}^d$ locally Lipschitz continuous. We consider the case of multiplicative noise with $H : L^2(D) \to HS(L^2(D))$, $HS(L^2(D))$ being the space of Hilbert-Schmidt operators, satisfying appropriate regularity conditions. Moreover, we discuss conditions for strong convergence of approximate solutions and adapt the argument of Gyöngy and Krylov.

Keywords: stochastic evolution equation, contraction principle, pathwise uniqueness, strong convergence.

AMS classification: 35K92, 35K55, 60H15.

§1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete, countably generated probability space (for example the classical Wiener space), $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $T > 0$, $Q_T := (0,T) \times D$ and $p \geq 2$. For separable Hilbert spaces $\mathcal{U}, \mathcal{H}$, we denote the space of Hilbert-Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$ by $HS(\mathcal{U};\mathcal{H})$. We are interested in

$$\begin{cases} du - \operatorname{div}(|\nabla u|^{p-2}\nabla u + F(u))\,dt = H(u)\,dW & \text{in } \Omega \times Q_T, \\ u = 0 & \text{on } \Omega \times (0,T) \times \partial D, \\ u(0,\cdot) = u_0 \in L^2(D) & \text{in } \Omega \times D, \end{cases} \tag{1}$$

for $F : \mathbb{R} \to \mathbb{R}^d$ locally Lipschitz continuous. $H : L^2(D) \to HS(L^2(D))$ satisfies the following assumption: For an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(D)$ and $u \in L^2(D)$,

$$H(u)(e_n) := \{x \mapsto h_n(u(x))\},$$

where, for all $n \in \mathbb{N}$, $h_n : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$(H1)\ \sum_{n=0}^{\infty} |h_n(\lambda) - h_n(\mu)|^2 \leq C_1 |\lambda - \mu|^2$$

holds for a constant $C_1 \geq 0$. $W(t)$ is a cylindrical Wiener process with values in $L^2(D)$ with respect to a filtration $(\mathcal{F}_t)$ satisfying the usual assumptions. More precisely: Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(D)$ and $(\beta_n(t))_{n\in\mathbb{N}}$ a sequence of independent, real-valued $\mathcal{F}_t$-Brownian motions. We (formally) define

$$W(t) := \sum_{n=1}^{\infty} e_n\beta_n(t). \tag{2}$$

It is easy to see that the sum on the right-hand side of (2) does not converge in $L^2(D)$. It has to be understood in the following sense (see, e.g. [3]): For $u = \sum_{n=1}^{\infty} u_n e_n$ and $v = \sum_{n=1}^{\infty} v_n e_n$

$$(u, v)_U := \sum_{n=1}^{\infty} \frac{u_n v_n}{n^2}$$

is a scalar product on $L^2(D)$. Let U be the linear space obtained by completion of $L^2(D)$ with respect to the norm $\|\cdot\|_U$ induced by $(\cdot,\cdot)_U$. It follows immediately that $(U, (\cdot,\cdot)_U)$ is a Hilbert space and (ne_n) is an orthonormal basis of U. Since

$$W(t) = \sum_{n=1}^{\infty} e_n\beta_n(t) = \sum_{n=1}^{\infty} \frac{1}{n}(ne_n)\beta_n(t), \tag{3}$$

$W(t)$ can be interpreted as Q-Wiener process with covariance Matrix $Q = \mathrm{diag}(\frac{1}{n^2})$ and values in U. Since $Q^{\frac{1}{2}}(U) = L^2(D)$, for all square integrable and predictable $\Phi : \Omega \times (0, T) \to HS(L^2(D))$ the stochastic integral with respect to the cylindrical Wiener process $W(t)$ is well-defined.

Due to the term $-\operatorname{div} F(u)$ the equation (1) is pseudomonotone. Therefore the classical results of well-posedness in [10] do not apply to (1). In order to show existence and uniqueness of solutions to (1), one can use an implicit time discretization. Well-posedness of (1) is the subject of a forthcoming research article.
In this contribution, we want to present some partial results: Firstly, we prove an L^1-contraction principle which, in particular, implies pathwise uniqueness of (1).
Further results are devoted to the question of strong convergence of approximate solutions (u_N) for (1), e.g., the approximate solutions constructed from an implicit Euler scheme. In the deterministic case, the a-priori estimates provide only weak convergence of (u_N) and strong convergence is obtained by a compactness argument. In the case of a stochastic PDE with multiplicative noise we apply Skorokhod's theorem: We change the probability space in order to pass to the limit in all nonlinear expressions. The solution obtained in this way is a martingale solution on a different stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{P})$ with respect to a different, $\hat{\mathcal{F}}_t$-Wiener process $\hat{W}(t)$.
The argument of Gyöngy and Krylov (see [4]) is based on a result from Yamada and Watanabe (see [12]) and states, roughly speaking, that existence of a martingale solution together with pathwise uniqueness of an SPDE implies the convergence in probability of the (Euler) approximation on any probability space $(\Omega, \mathcal{F}, P)$. Recall that (1) is called pathwise unique, if whenenver $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{P}, \hat{W}(t), u^1)$, $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{P}, \hat{W}(t), u^2)$ are solutions to (1) with respect

to the same initial value u_0, then $u_1(t) = u_2(t)$ a.s. in $\hat{\Omega}$ for any $t \in [0, T]$. The crucial step in the argument of Gyöngy and Krylov is the construction of two martingale solutions u^i, $i = 1, 2$ without changing the quantities $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{P}, \hat{W}(t))$. To this end, one can adapt a direct martingale identification argument developed in [9], [2] and avoid the use of the martingale representation theorem. The ideas of [9], [2] have been generalized and applied to stochastic differential equations in [6] and [7]. In [5], the direct martingale representation argument has been applied in combination with the Gyöngy-Krylov argument to degenerate parabolic SPDEs. Recently, the technique of [5] has been adapted to the stochastic isentropic Euler equations (see [1]).

§2. L^1 contraction principle

Proposition 1. *Assume that $W(t)$ is a cylindrical Wiener process in $L^2(D)$ with respect to the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and u_1, u_2 are solutions to (1) with respect to the initial values u_{01} and u_{02} in $L^2(D)$ respectively on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Then, we have*

$$E \int_D |u_1(t) - u_2(t)|\, dx \leq \int_D |u_{01} - u_{02}|\, dx \tag{4}$$

for all $t \in [0, T]$.

Proof. For $\delta > 0$, let η_δ be an approximation of the absolute value, i.e.

$$\eta_\delta(r) = \begin{cases} -r & \text{if } r < -2\delta, \\ \dfrac{r^2}{2\delta} & \text{if } -2\delta \leq r \leq 2\delta, \\ r & \text{if } r > 2\delta. \end{cases}$$

Using the Itô formula, it follows that

$$I_1 = I_2 + I_3 + I_4 + I_5 \tag{5}$$

for all $t \in [0, T]$ a.s. in Ω, where

$$\begin{aligned} I_1 &= \int_D \eta_\delta(u_1 - u_2)(t)\, dx - \int_D \eta_\delta(u_{01} - u_{02})\, dx, \\ I_2 &= -\int_0^t \int_D (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2) \cdot \nabla(u_1 - u_2)\eta_\delta''(u_1 - u_2)\, dx\, ds, \\ I_3 &= -\int_0^t \int_D (F(u_1) - F(u_2)) \cdot \nabla(u_1 - u_2)\eta_\delta''(u_1 - u_2)\, dx\, ds, \\ I_4 &= \int_0^t (\eta_\delta'(u_1 - u_2), H(u_1) - H(u_2)\, dW)_2, \\ I_5 &= \frac{1}{2}\int_0^t \eta_\delta''(u_1 - u_2)\|H(u_1) - H(u_2)\|^2_{HS(L^2(D))}\, ds. \end{aligned} \tag{6}$$

Since η_δ is convex, it follows that $I_2 \leq 0$ for all $t \in [0,T]$, a.s. in Ω. Moreover, $E[I_4] = 0$ for all $t \in [0,T]$. Therefore, from (5) and (6) it follows that

$$E[I_1] \leq E[I_3] + E[I_5]. \tag{7}$$

Since, for any $t \in [0,T]$, $\eta_\delta(u_1 - u_2)(t)$ converges to $|(u_1 - u_2)(t)|$ for $\delta \to 0^+$ a.e. in $\Omega \times D$, and $|\eta_\delta(u_1 - u_2)(t)| \leq |(u_1 - u_2)(t)|$ for all $\delta > 0$ a.s. in $\Omega \times D$, it follows that

$$\lim_{\delta \to 0^+} E[I_1] = E \int_D |u_1(t) - u_2(t)| \, dx - E \int_D |u_{01} - u_{02}| \, dx \tag{8}$$

for any $t \in [0,T]$. For any $\delta > 0$ we have

$$\eta''_\delta(u_1 - u_2) = \frac{1}{2\delta} \chi_{\{|u_1 - u_2| \leq 2\delta\}}$$

a.s. on $\Omega \times Q_T$, thus for $L \geq 0$ being the Lipschitz constant of F we have

$$\begin{aligned}
|E[I_3]| &\leq \frac{1}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} |F(u_1) - F(u_2)| \, |\nabla(u_1 - u_2)| \, dx \, ds \\
&\leq \frac{L}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} |u_1 - u_2| \, |\nabla(u_1 - u_2)| \, dx \, ds \\
&\leq LE \int_{\{|u_1 - u_2| \leq 2\delta\}} |\nabla(u_1 - u_2)| \, dx \, ds.
\end{aligned} \tag{9}$$

Simirlarly, by $(H1)$

$$\begin{aligned}
|E[I_5]| &\leq \frac{1}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} \sum_{n=1}^{\infty} |h_n(u_1) - h_n(u_2)|^2 \, dx \, ds \\
&\leq \frac{C_1}{2\delta} E \int_{\{|u_1 - u_2| \leq 2\delta\}} |u_1 - u_2|^2 \, dx \, ds \\
&\leq 2\delta C
\end{aligned} \tag{10}$$

where $C \geq 0$ is a constant not depending on $\delta > 0$. Thus from (9) it follows that

$$\lim_{\delta \to 0^+} E[I_3] = E \int_{\{u_1 = u_2\}} |\nabla(u_1 - u_2)| \, dx \, ds = 0 \tag{11}$$

and from (10) it follows that $\lim_{\delta \to 0^+} E[I_5] = 0$. □

§3. Pathwise uniqueness and strong convergence

If u_1 and u_2 are solutions to (1) with respect to the same initial value $u_0 \in L^2(D)$ and $\mu_{1,2}$ is the joint law of (u_1, u_2) on $L^2(0,T; L^2(D)) \times L^2(0,T; L^2(D))$, by Proposition 1 it follows that

$$\mu_{1,2}(\{(\xi, \zeta) \in L^2(0,T; L^2(D)) \times L^2(0,T; L^2(D)) \mid \xi = \zeta\}) = \int_{\Omega \times \Omega} \chi_{\{u_1 = u_2\}} \, dP \otimes dP = 1,$$

hence the support of $\mu_{1,2}$ is contained in the diagonal of $L^2(0,T; L^2(D)) \times L^2(0,T; L^2(D))$. The concept of pathwise uniqueness is linked to the concept of existence of strong solutions via the following Lemma (see [4], Lemma 1.1, p.144 and 145):

Lemma 2. *Let V be a Polish space equipped with the Borel σ-algebra. A sequence of V-valued random variables (X_n) converges in probability if and only if for every pair of subsequences X_n and X_m there exists a joint subsequence (X_{n_k}, X_{m_k}) which converges for $k \to \infty$ in law to a probability measure μ such that*

$$\mu(\{(w,z) \in V \times V \mid w = z\}) = 1.$$

§4. Assumptions and convergence results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be the original stochastic basis for (1) and $W(t)$ the corresponding cylindrical $\mathcal{F}_t$-Wiener process in $L^2(D)$. Assume now that (u_N) is a sequence of square-integrable, left-continuous and $\mathcal{F}_t$-adapted stochastic processes on $(\Omega, \mathcal{F}, P)$ with values in $L^2(D)$ which is tight in $L^2(0,T;L^2(D))$ satisfying

$$E \int_0^T \|\nabla u_N\|^p \, dt \leq C \tag{12}$$

for all $N \in \mathbb{N}$, $p \geq 2$ and some constant $C \geq 0$.
Let (u_M) and (u_L) be a pair of subsequences of (u_N). Since (u_M, u_L, W) is tight on

$$L^2(0,T;L^2(D)) \times L^2(0,T;L^2(D)) \times C([0,T];U),$$

according to Prokhorov's theorem we can extract a joint subsequence $\mu^j := (u_{M_j}, u_{L_j}, W)$ which converges in law to some probability measure μ. Applying the theorem of Skorokhod of [11], Theorem 1.10.4 and Addendum 1.10.5, p.59 to (u_{M_j}, u_{L_j}, W) we find a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, a sequence of measurable functions

$$\Phi_j : (\hat{\Omega}, \hat{\mathcal{F}}) \to (\Omega, \mathcal{F}), \ j \in \mathbb{N}$$

such that $P = \hat{P} \circ \Phi_j^{-1}$ for all $j \in \mathbb{N}$ and measurable functions u^1_∞, u^2_∞, W_∞ having the following properties:

i.) $\hat{u}_{M_j} := u_{M_j} \circ \phi_j \to u^1_\infty$ in $L^2(0,T;L^2(D))$ for $j \to \infty$ a.s. in $\hat{\Omega}$,

ii.) $\hat{u}_{L_j} := u_{L_j} \circ \phi_j \to u^2_\infty$ in $L^2(0,T;L^2(D))$ for $j \to \infty$ a.s. in $\hat{\Omega}$,

iii.) $W_j := W \circ \phi_j \to W_\infty$ in $C([0,T];U)$ for $j \to \infty$ a.s. in $\hat{\Omega}$.

iv.) $\mathcal{L}(u^1_\infty, u^2_\infty, W) = \mu$.

The following Lemma is a direct consequence of (14), (12), the Vitali theorem and the equality of laws of W and W_j for all $j \in \mathbb{N}$:

Lemma 3. *We have the following convergence results for $j \to \infty$:*

i.) $\hat{u}_{M_j} \to u^1_\infty$ *and* $\hat{u}_{L_j} \to u^2_\infty$ *in* $L^2(\hat{\Omega}; L^2(0,T;L^2(D)))$

ii.) $W_j \to W_\infty$ *in* $L^2(\hat{\Omega}; C([0,T];U))$

iii.) $W_j(t) - W_j(s) \to W_\infty(t) - W_\infty(s)$ *in* $L^2(\hat{\Omega}; U)$ *for all* $t \in [0,T]$, $0 \leq s \leq t$

iv.) For all $t \in [0,T]$, $0 \le s \le t$ *and all* $\psi \in C_b(L^2(0,s;L^2(D))^2 \times C([0,T];L^2(D))^2 \times C([0,s];U))$

$$\lim_{j\to\infty} \psi(\hat{u}_{M_j}, \hat{u}_{L_j}, B_{M_j}, B_{L_j}, W_j) = \psi(u^1_\infty, u^2_\infty, B^1_\infty, B^2_\infty, W_\infty) \tag{13}$$

in $L^2(\hat{\Omega})$.

Assume that there exist B^1_∞, B^2_∞ in $L^2(\hat{\Omega}; C([0,T];L^2(D)))$ such that

$$\lim_{j\to\infty} B_{M_j} = B^1_\infty, \ \lim_{j\to\infty} B_{L_j} = B^2_\infty \tag{14}$$

in $L^2(\hat{\Omega}; C([0,T];L^2(D)))$. Moreover, assume that for $i = 1,2$, the function $u^i_\infty : \hat{\Omega}\times[0,T] \to L^2(D)$ is a stochastic process with $u^i_\infty(0) = u_0$ and there exists $G^i \in L^{p'}(\hat{\Omega}\times Q_T)^d$ such that

$$u^i_\infty(t) = B^i_\infty(t) + u_0 + \int_0^t \operatorname{div}(G^i + F(u^i_\infty))\, ds \tag{15}$$

holds in $L^2(D)$ a.s. in $\hat{\Omega}$ for all $t \in [0,T]$. Let us denote the augmentation of the filtration $\sigma(W_j(s))_{0\le s\le t,\ t\in[0,T]}$ by $(\mathcal{F}^j_t)$. It is a direct consequence of Skorokhod's theorem and [3], Theorem 4.4, p. 89 that $W_j(t)$ is a Q-Wiener process in U with respect to $(\mathcal{F}^j_t)$, thus a cylindrical $\mathcal{F}^j_t$-Wiener process in $L^2(D)$ for all $j \in \mathbb{N}$. Moreover $\hat{u}_{M_j} = u_{M_j} \circ \phi_j$, $\hat{u}_{L_j} = u_{L_j} \circ \phi_j$ are left-continuous, $\mathcal{F}^j_t$-adapted processes with values in $L^2(D)$ and therefore the stochastic integrals

$$B_{M_j}(t) := \int_0^t H(u_{M_j})\, dW_j, \ B_{L_j}(t) := \int_0^t H(u_{L_j})\, dW_j, \ t \in [0,T]$$

are well-defined. Note that by assumption (15)

$$B^i_\infty(t) = u^i_\infty(t) - u_0 - \int_0^t \operatorname{div}(G^i + F(u^i_\infty))\, ds \tag{16}$$

for all $t \in [0,T]$. The right-hand side of (16) can not be written as a nice operator applied to $u^i_\infty(t)$. Therefore, it is in general not clear if B^i_∞ is adapted to $\sigma(u^1_\infty(s), u^2_\infty(s), W_\infty(s))_{0\le s\le t}$, $t \in [0,T]$ thus we define $(\mathcal{F}^\infty_t)$ to be the augmentation of the filtration

$$\sigma(u^1_\infty(s), u^2_\infty(s), B^1_\infty(s), B^2_\infty(s), W_\infty(s))_{0\le s\le t}, \ t \in [0,T].$$

§5. Martingale identification argument

Lemma 4. *For* $i = 1,2$, $B^i_\infty(t)$ *is a* $\mathcal{F}^\infty_t$*-martingale with quadratic variation process*

$$\ll B^i_\infty \gg_t = \int_0^t \boldsymbol{H}(u^i_\infty) \circ \boldsymbol{H}^*(u^i_\infty)\, ds \tag{17}$$

for all $t \in [0,T]$, *where we use the notation*

$$\boldsymbol{H}(u) := H(u) \circ Q^{1/2}, \ u \in L^2(D).$$

Proof. Let (e_l) be an orthonormal basis of $L^2(D)$. We fix $t \in [0,T]$, $0 \leq s \leq t$, $\psi \in C_b(L^2(0,s;L^2(D))^2 \times C([0,T];L^2(D))^2 \times C([0,s];U))$ and $n,m \in \mathbb{N}$. Moreover, for $u \in L^2(D)$, and $B(r) \in L^2(D)$, $r \in [0,T]$ we define

$$(B,e_n,e_m)(r) := (B(r),e_n)_2(B(r),e_m)_2,$$

$$\Lambda(s,t,u,e_n,e_m) := \left(\left[\int_s^t \mathbf{H}(u) \circ \mathbf{H}^*(u)\,dr\right](e_n),e_m\right)_2.$$

Since $B_{M_j}(t)$ and $B_{L_j}(t)$ are stochastic integrals, from the convergence results of Lemma 3 and Assumption (14) it follows that

$$\begin{aligned} 0 &= \lim_{j\to\infty} E[(B_{M_j}(t) - B_{M_j}(s),e_n)_2\psi(\hat{u}_{M_j},\hat{u}_{L_j},B_{M_j},B_{L_j},W_j)] \\ &= E[(B^1_\infty(t) - B^1_\infty(s),e_n)_2\psi(u^1_\infty,u^2_\infty,B^1_\infty,B^2_\infty,W_\infty)], \end{aligned} \tag{18}$$

$$\begin{aligned} 0 &= \lim_{j\to\infty} E[(B_{L_j}(t) - B_{L_j}(s),e_n)_2\psi(\hat{u}_{M_j},\hat{u}_{L_j},B_{M_j},B_{L_j},W_j)] \\ &= E[(B^2_\infty(t) - B^2_\infty(s),e_n)_2\psi(u^1_\infty,u^2_\infty,B^1_\infty,B^2_\infty,W_\infty)], \end{aligned} \tag{19}$$

for $j \to \infty$. Moreover,

$$\begin{aligned} 0 &= E[((B_{M_j},e_n,e_m)(t) - (B_{M_j},e_n,e_m)(s) - \Lambda(s,t,\hat{u}_{M_j},e_n,e_m))\psi(\hat{u}_{M_j},\hat{u}_{L_j},B_{M_j},B_{L_j},W_j)] \\ &\to E[((B^1_\infty,e_n,e_m)(t) - (B^1_\infty,e_n,e_m)(s) - \Lambda(s,t,u^1_\infty,e_n,e_m))\psi(u^1_\infty,u^2_\infty,B^1_\infty,B^2_\infty,W_\infty)], \end{aligned} \tag{20}$$

$$\begin{aligned} 0 &= E[((B_{L_j},e_n,e_m)(t) - (B_{L_j},e_n,e_m)(s) - \Lambda(s,t,\hat{u}_{L_j},e_n,e_m))\psi(\hat{u}_{M_j},\hat{u}_{L_j},B_{M_j},B_{L_j},W_j)] \\ &\to E[((B^2_\infty,e_n,e_m)(t) - (B^2_\infty,e_n,e_m)(s) - \Lambda(s,t,u^2_\infty,e_n,e_m))\psi(u^1_\infty,u^2_\infty,B^1_\infty,B^2_\infty,W_\infty)]. \end{aligned} \tag{21}$$

for $j \to \infty$. □

Lemma 5. *$W_\infty(t)$ is a $\mathcal{F}^\infty_t$-martingale.*

Proof. By definition of $(\mathcal{F}^\infty_t)$, W_∞ is adapted to $(\mathcal{F}^\infty_t)$. We fix $t \in [0,T]$, $0 \leq s \leq t$, $\psi \in C_b(L^2(0,s;L^2(D))^2 \times C([0,T];L^2(D))^2 \times C([0,s];U))$ and $h \in U$. Since $\hat{u}_{M_j}$ and $\hat{u}_{L_j}$ are $\mathcal{F}^j_t$-adapted and B_{M_j}, B_{L_j} are stochastic integrals with respect to W_j for all $j \in \mathbb{N}$, we have

$$E[(W_j(t) - W_j(s),h)_U\psi(\hat{u}_{M_j},\hat{u}_{L_j},B_{M_j},B_{L_j},W_j)] = 0 \tag{22}$$

for all $j \in \mathbb{N}$. Using the convergence results of Lemma 3, we may pass to the limit with $j \to \infty$ in (22) and find that

$$E[(W_\infty(t) - W_\infty(s),h)_U\psi(u^1_\infty,u^2_\infty,B^1_\infty,B^2_\infty,W_\infty)] = 0. \tag{23}$$

□

Lemma 6. *$W_\infty(t)$ is a $\mathcal{F}^\infty_t$-Wiener process.*

Remark 1. In particular, Lemma 6 implies that $W_\infty(t)$ is a cylindrical Wiener process in $L^2(D)$ with increments $W(t) - W(s)$, $0 \le s \le t \le T$, independent of $\mathcal{F}_s^\infty$.

Proof. From Lemma 5 it follows that $W_\infty(t)$ is a $\mathcal{F}_t^\infty$-martingale with $W_\infty(0) = 0$. According to [3], Theorem 4.4, p. 89 it is left to show that

$$\ll W_\infty \gg_t = tQ \text{ for all } t \in [0,T]. \tag{24}$$

Let (g_l) be an orthonormal basis of U. We fix $t \in [0,T]$, $0 \le s \le t$, $\psi \in C_b(L^2(0,s;L^2(D))^2 \times C([0,T];L^2(D))^2 \times C([0,s];U))$ and $n,m \in \mathbb{N}$. Since $\ll W_j \gg_t = tQ$ for all $t \in [0,T]$ and all $j \in \mathbb{N}$, from the convergence results of Lemma 3 it follows that

$$\begin{aligned}0 &= E[((W_j,g_n,g_m)(t) - (W_j,g_n,g_m)(s) - ((t-s)Q(g_n),g_m)_U)\psi(\hat{u}_{M_j},\hat{u}_{L_j},B_{M_j},B_{L_j},W_j)]\\ &\to E[((W_\infty,g_n,g_m)(t) - (W_\infty,g_n,g_m)(s) - ((t-s)Q(g_n),g_m)_U\psi(u^1_\infty,u^2_\infty,B^1_\infty,B^2_\infty,W_\infty))]\end{aligned} \tag{25}$$

for $j \to \infty$, where

$$(W,g_n,g_m)(r) := (W(r),g_n)_U(W(r),g_m)_U$$

for $W(r) \in U$, $r \in [0,T]$, thus (24) holds true. □

Corollary 7. *For $i = 1,2$, the process*

$$M_i(t) := \int_0^t H(u^i_\infty)\,dW_\infty,\ t \in [0,T] \tag{26}$$

is a $\mathcal{F}_t^\infty$-martingale with quadratic variation process

$$\ll M_i \gg_t = \int_0^t (H(u^i_\infty) \circ Q^{1/2}) \circ (H(u^i_\infty) \circ Q^{1/2})^*\,ds$$

for all $t \in [0,T]$.

Remark 2. For $i = 1,2$, $u^i : \hat{\Omega} \times [0,T] \to L^2(D)$ is assumed to be a stochastic process. By definition of $(\mathcal{F}_t^\infty)$ we get that u^i is $\mathcal{F}_t^\infty$-adapted and by Lemma 3 it follows that u^i is square integrable. From [8], Remark 1.1., p.45 it follows that u^i is a.s. equal to a predictable process, thus the stochastic integral in (26) is well-defined.

Lemma 8. *For $i = 1,2$ we have the cross quadratic variation process*

$$\ll W_\infty, B^i_\infty, \gg_t = \int_0^t Q \circ H^*(u^i_\infty)\,ds. \tag{27}$$

Proof. Since, for any square-integrable and predictable $\phi \in L^2(\hat{\Omega} \times (0,T); HS(L^2(D))$ and any Q-Wiener process $W(t)$ we have

$$\begin{aligned}\int_0^t (\phi \circ Q^{1/2}) \circ (\phi \circ Q^{1/2})^*\,ds &= \ll \int_0^\cdot \phi\,dW \gg_t\\ &= \ll \int_0^\cdot \phi\,dW, \int_0^\cdot \phi\,dW \gg_t\\ &= \int_0^t \phi\,d \ll W, \int_0^\cdot \phi\,dW \gg_s,\end{aligned} \tag{28}$$

and from (28) it follows that

$$\ll W, \int_0^\cdot \phi\, dW \gg = \int_0^t Q^{1/2} \circ (\phi \circ Q^{1/2})^* \, ds = \int_0^t Q \circ \phi^* \, ds. \tag{29}$$

Therefore, for all $j \in \mathbb{N}$ and $t \in [0, T]$ we have

$$\ll W_j, B_{M_j} \gg_t = \int_0^t Q \circ H^*(\hat{u}_{M_j})\, ds \tag{30}$$

and

$$\ll W_j, B_{L_j} \gg_t = \int_0^t Q \circ H^*(\hat{u}_{L_j})\, ds. \tag{31}$$

We choose orthonormal bases (g_l) of U and (e_l) of $L^2(D)$, fix $t \in [0, T]$, $0 \le s \le t$, $\psi \in C_b(L^2(0, s; L^2(D))^2 \times C([0, T]; L^2(D))^2 \times C([0, s]; U))$ and $n, m \in \mathbb{N}$. Using the convergence results of Lemma 3 and Assumption 14 from (30) and (31) it follows that

$$\begin{aligned} 0 &= E[((B_{M_j}, W_j, e_n, g_m)(t) - (B_{M_j}, W_j, e_n, g_m)(s) - \int_s^t (Q \circ H^*(\hat{u}_{M_j})(e_n), g_m)_U \, dr)\psi_j] \\ &\to E[((B^1_\infty, W_\infty, e_n, g_m)(t) - (B^1_\infty, W_\infty, e_n, g_m)(s) - \int_s^t (Q \circ H^*(u^1_\infty)(e_n), g_m)_U \, dr)\psi_\infty] \end{aligned} \tag{32}$$

for $j \to \infty$ and

$$\begin{aligned} 0 &= E[((B_{L_j}, W_j, e_n, g_m)(t) - (B_{L_j}, W_j, e_n, g_m)(s) - \int_s^t (Q \circ H^*(\hat{u}_{L_j})(e_n), g_m)_U \, dr)\psi_j] \\ &\to E[((B^2_\infty, W_\infty, e_n, g_m)(t) - (B^2_\infty, W_\infty, e_n, g_m)(s) - \int_s^t (Q \circ H(u^2_\infty)(e_n), g_m)_U \, dr)\psi_\infty] \end{aligned} \tag{33}$$

for $j \to \infty$, where

$$\psi_j := \psi(\hat{u}_{M_j}, \hat{u}_{L_j}, B_{M_j}, B_{L_j}, W_j), \ \psi_\infty := \psi(u^1_\infty, u^2_\infty, B^1_\infty, B^2_\infty, W_\infty)$$

and

$$(B, W, e_n, g_m)(r) := (B(r), e_n)_2 (W(r), g_m)_U$$

for $r \in [0, T]$ and $W(r) \in U$, $B(r) \in L^2(D)$. □

Lemma 9. *For $i = 1, 2$ and all $t \in [0, T]$ we have*

$$\ll \int_0^\cdot H(u^i_\infty)\, dW_\infty - B^i_\infty \gg_t = 0, \tag{34}$$

Proof. For $i = 1, 2$ from Lemmas 4-6 and Corollary 7 it follows that

$$\begin{aligned} &\ll \int_0^\cdot H(u^i_\infty)\, dW_\infty - B^i_\infty \gg_t \\ &= \ll \int_0^\cdot H(u^i_\infty)\, dW_\infty \gg_t - 2 \ll \int_0^\cdot H(u^i_\infty)\, dW_\infty, B^i_\infty \gg_t + \ll B^i_\infty \gg_t \\ &= 2\int_0^t (H(u^i_\infty) \circ Q^{1/2}) \circ (H(u^i_\infty) \circ Q^{1/2})^* \, ds - 2\int_0^t H(u^i_\infty)\, d \ll W_\infty, B^i_\infty \gg_s \end{aligned} \tag{35}$$

where, according to Lemma 8

$$\int_0^t H(u^i_\infty)\, d \ll W_\infty, B^i_\infty \gg_s$$
$$= \int_0^t H(u^i_\infty) \circ Q^{1/2} \circ (Q^{1/2})^* \circ H^*(u^i_\infty)\, ds = \int_0^t (H(u^i_\infty) \circ Q^{1/2}) \circ (H(u^i_\infty) \circ Q^{1/2})^*\, ds \quad (36)$$

Now, (34) follows from (35) and (36). □

Corollary 10. *From Lemma 9 it follows that for $i = 1, 2$*

$$B^i_\infty(t) = \int_0^t H(u^i_\infty)\, dW_\infty,\ t \in [0, T].$$

§6. Conclusion

If, in addition, $G^i = |\nabla u^i_\infty|^{p-2}\nabla u^i_\infty$ in $L^{p'}(\hat{\Omega} \times Q_T)^d$ holds true for $i = 1, 2$, from Assumption (15) it then follows that u^1_∞ and u^2_∞ satisfy

$$u^i_\infty(t) = u_0 + \int_0^t \operatorname{div}(|\nabla u^i_\infty|^{p-2}\nabla u^i_\infty + F(u^i_\infty))\, ds + \int_0^t H(u^i_\infty)\, dW_\infty \quad (37)$$

and we have constructed two martingale solutions to (1) with respect to the same stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, (\mathcal{F}^\infty)_t, \hat{P})$ and the same $\mathcal{F}^\infty_t$-Wiener process $W_\infty(t)$. From Proposition 1 it follows that $u^1_\infty(t) = u^2_\infty(t)$ as elements of $L^2(D)$ a.s. $\hat{\Omega}$ for all $t \in [0, T]$. Recall that the joint subsequence (u_{M_j}, u_{L_j}) which was extracted from (u_N) on the original probability space $(\Omega, \mathcal{F}, P)$ converges in law on $L^2(0, T; L^2(D)) \times L^2(0, T; L^2(D))$ to a probability measure $\mu = (\mu_1, \mu_2)$ and by Skorokhod's theorem we have $\mu_1 = \mathcal{L}(u^1_\infty)$, $\mu_2 = \mathcal{L}(u^2_\infty)$. Therefore, the support of μ is contained in the diagonal of $L^2(0, T; L^2(D)) \times L^2(0, T; L^2(D))$ and from Lemma 2 it follows that (u_N) converges in probability.

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