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ON THE MINIMAL DEGREE OF LOGARITHMIC VECTOR FIELDS OF LINE ARRANGEMENTS

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Abstract. Let $\mathcal{A}$ be a real line arrangement and $\mathcal{D}(\mathcal{A})$ the module of $\mathcal{A}$–derivations view as the set of polynomial vector fields which possess $\mathcal{A}$ as an invariant set. We first characterize polynomial vector fields having an infinite number of invariant lines. Then we prove that the minimal degree of polynomial vector fields fixing only a finite set of lines in $\mathcal{D}(\mathcal{A})$ is not determined by the combinatorics of $\mathcal{A}$.

§1. Introduction

Let $\mathcal{A} = \{L_1, \ldots, L_n\}$ be a *real line arrangement* in $\mathbb{R}^2$. Denote by $|\mathcal{A}| = n$ the number of lines of the arrangement and by $\operatorname{Sing}\mathcal{A}$ the set of singular points of $\mathcal{A}$, i.e. the intersection points between lines. We define by $L(\mathcal{A}) = \{\emptyset \neq L_i \cap L_j \mid L_i, L_j \in \mathcal{A}\} \cup \mathcal{A}$ the *intersection poset* of $\mathcal{A}$ partially ordered by reverse inclusion of the subsets, which codifies the combinatorial data of $\mathcal{A}$.

The influence of combinatorics of line arrangements over the properties of its realizations on different ambient spaces (as $\mathbb{R}^2$, $\mathbb{C}^2$, $\mathbb{F}_p^2$ and its projectives) was largely studied, e.g. [1], [5], [7]. A classical object is the *module of $\mathcal{A}$–derivations* of a line arrangement $\mathcal{A}$, denoted by $\mathcal{D}(\mathcal{A})$ or the *module of logarithmic* 1*–forms* $\Omega^1(\mathcal{A})$ (see [6]).

In this note, we study the relation between $\mathcal{A}$, the poset $L(\mathcal{A})$, and the module $\mathcal{D}(\mathcal{A})$. Our approach is first to give a *dynamical* interpretation of $\mathcal{D}(\mathcal{A})$ as the set of polynomial vector fields owning $\mathcal{A}$ as invariant set. Using this point of view, we are able to characterize the geometry of the set $\mathcal{D}(\mathcal{A})$ (Proposition 1 and 2). We introduce the notion of maximal line arrangement for a given polynomial vector field, to characterize those having an infinite number of invariant lines (Theorem 5). Finally, we prove that the minimal degree $d_f(\mathcal{A})$ of the elements in $\mathcal{D}(\mathcal{A})$ fixing only a finite set of lines is not determined by the combinatorics of a line arrangement $\mathcal{A}$ (Theorem 7). Related results on polynomial vector fields and lines arrangements can be found from a different point of view in [2] and [8].

§2. Line arrangements and vector fields

For a line $L \in \mathcal{A}$, consider $\alpha_L : \mathbb{R}^2 \to \mathbb{R}$ an associated affine form such that $L = \ker \alpha_L$. The *defining polynomial* of $\mathcal{A}$ is given by $Q(\mathcal{A}) = \prod_{L\in\mathcal{A}} \alpha_L$. Let $\operatorname{Der}_{\mathbb{R}}(\mathbb{R}[x, y])$ be the algebra of $\mathbb{R}$-*derivations* of $\mathbb{R}[x, y]$.

Definition 1. Let $\mathcal{A}$ be a line arrangement and $Q = Q(\mathcal{A})$ its defining polynomial. The *module of $\mathcal{A}$–derivations* is the $\mathbb{R}[x,y]$-module defined by $\mathcal{D}(\mathcal{A}) = \{\chi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}[x,y]) \mid \chi Q \in \mathcal{I}_Q\}$, where $\mathcal{I}_Q$ is the ideal generated by Q.

Remark 1. From the definition, it is easy to deduce the following characterization by lines $\mathcal{D}(\mathcal{A}) = \bigcap_{L\in\mathcal{A}}\{\chi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}[x,y]) \mid \chi\alpha_L \in \mathcal{I}_{\alpha_L}\}$, where $\mathcal{I}_{\alpha_L}$ is the ideal generated by α_L.

As $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}[x,y])$ coincide with *polynomials vector fields* on the plane, we obtain a dynamical interpretation of $\mathcal{D}(\mathcal{A})$: the elements of $\mathcal{D}(\mathcal{A})$ correspond to the polynomial vector fields admitting $\mathcal{A}$ as invariant set. We use this point of view in the following.

The required condition for derivations in $\mathcal{D}(\mathcal{A})$ is equivalent to the definition of algebraic invariant set in complex dynamical systems: a complex algebraic curve $C = \{f = 0\}$ is invariant by a polynomial vector field χ if there exists $K \in \mathbb{C}[x,y]$ such that $\chi f = Kf$ (see [3]). Since real line arrangements are defined by products of real affine forms, this condition also holds geometrically.

§3. Structure theorems

Let $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}_d[x,y]) = \{\chi = P\partial_x + Q\partial_y \mid \deg P, \deg Q \leq d\}$ and $F_d(\mathcal{D}(\mathcal{A})) = \mathcal{D}(\mathcal{A}) \cap \mathrm{Der}_{\mathbb{R}}(\mathbb{R}_d[x,y])$, defining an ascending filtration of $\mathcal{D}(\mathcal{A})$ by degree. We denote by $\mathcal{D}_d(\mathcal{A}) = F_d(\mathcal{D}(\mathcal{A})) \setminus F_{d-1}(\mathcal{D}(\mathcal{A}))$ the set of *polynomial vector fields of degree d fixing $\mathcal{A}$*. Consider $C(d)$ the $\mathbb{R}$-linear space of coefficients of a pair of polynomials of degree less than d. We have $C(d) = \mathbb{R}^{(d+1)(d+2)/2} \oplus \mathbb{R}^{(d+1)(d+2)/2} \simeq \mathbb{R}^{(d+1)(d+2)}$. Using the classical properties of an ideal, we have:

Proposition 1 (Structure of polynomial vector fields). *Let $\mathcal{A}$ be a line arrangement. For each $d \in \mathbb{N}$, the set $F_d(\mathcal{D}(\mathcal{A}))$ is a vector sub-space of the set of coefficients $C(d)$.*

Proposition 2 (Structure of fixed line arrangements). *Let χ be a polynomial vector field. The set $\mathcal{F}_n(\chi)$ of arrangements with n lines fixed by χ is an algebraic sub-variety of $\left(\mathbb{RP}^2\right)^n$ (view as the set of the coefficients $\alpha_i, \beta_i, \gamma_i$ defining the lines of $\mathcal{A}$).*

This last Proposition can be deduced from the following:

Proposition 3 (Invariant line)**.** *Let L be a line of $\mathbb{R}^2$ defined by the equation $\alpha x + \beta y + \gamma = 0$, and let $\chi = P(x,y)\partial_x + Q(x,y)\partial_y$ be a polynomial vector field on $\mathbb{R}^2$. The line L is invariant for χ if and only if: (i) $\beta = 0$ and $P(-\gamma/\alpha, y) = 0$, (ii) $\beta \neq 0$ and $\alpha P(\beta y, -\alpha y - \gamma/\beta) + \beta Q(\beta y, -\alpha y - \gamma/\beta) = 0$.*

§4. Polynomial vectors fields admitting a finite/infinite number of invariant lines

In order to characterize efficiently line arrangements as invariant sets of a vector field, we distinguish them according to finiteness requirements over the set of its invariant lines.

4.1. Finiteness of families of fixed lines

The first step is to obtain conditions on the finiteness of the family of invariant lines. This leads us to the notion of maximal line arrangement fixed by a polynomial vector field.

Definition 2. Let χ be a polynomial vector field in the plane. We said that a line arrangement $\mathcal{A}$ is *maximal fixed by* χ if any line $L \in \mathbb{R}^2$ invariant by χ belongs to $\mathcal{A}$.

Trivial examples of polynomial vector fields in the plane which do not possess a maximal line arrangement are: the null vector field, the "central" vector field $\chi_c = x\partial_x + y\partial_y$ or the "parallel" vector field $\chi_p = (x+1)\partial_y$. In Theorem 5 we prove that derivations which do not admit a maximal fixed arrangement are essentially of these kind.

Definition 3. We said that χ fixes *only a finite* (resp. *an infinite*) set of lines if there exists (resp. does not exist) a maximal arrangement fixed by χ.

Let $\mathcal{A}$ be a line arrangement, we consider the partition $\mathcal{D}(\mathcal{A}) = \mathcal{D}^\infty(\mathcal{A}) \sqcup \mathcal{D}^f(\mathcal{A})$ where $\mathcal{D}^f(\mathcal{A})$ (resp. $\mathcal{D}^\infty(\mathcal{A})$) is the subset of elements in $\mathcal{D}(\mathcal{A})$ fixing only a finite (resp. infinite) set of lines. We define $\mathcal{D}^\infty_d(\mathcal{A}) = \mathcal{D}_d(\mathcal{A}) \cap \mathcal{D}^\infty(\mathcal{A})$ and $\mathcal{D}^f_d(\mathcal{A}) = \mathcal{D}_d(\mathcal{A}) \cap \mathcal{D}^f(\mathcal{A})$, for $d \in \mathbb{N}$.

Definition 4. A vector field χ is said to be: (i) *central* if there is a point $(x_0, y_0) \in \mathbb{R}^2$ such that all the vectors $(x - x_0, y - y_0)$ and $\big(P(x,y), Q(x,y)\big)$ are collinear, (ii) *parallel* if there is a vector v such that for all $(x,y) \in \mathbb{R}^2$, the vectors $\big(P(x,y), Q(x,y)\big)$ and v are collinear.

Let $m(\mathcal{A})$ be the maximal multiplicity of singular points of a line arrangement $\mathcal{A}$, and let $p(\mathcal{A})$ be the maximal number of parallel lines of $\mathcal{A}$.

Theorem 4. *If* $d < \max(m(\mathcal{A}) - 1, p(\mathcal{A}))$ *then* $\mathcal{D}_d(\mathcal{A}) = \mathcal{D}^\infty_d(\mathcal{A})$.

Proof. We decompose this proof in two cases.

First, suppose $d + 1 < m(\mathcal{A})$. Up to an affine deformation, we may assume that the singular point P of multiplicity $d+2$ of $\mathcal{A}$ is the origin, and that no one of these lines are vertical (i.e. $x = 0$). Let $y = \alpha_i x$ be the $d+2$ lines passing by point P. Proposition 3 implies that for all $i \in \{1, \cdots, d+2\}$ we have

$$\alpha_i P(y, -\alpha_i y) + Q(y, -\alpha_i y) = \sum_{n=0}^{d} \Big(\sum_{j=0}^{n} (\alpha_i a_{n-j,j} + b_{n-j,j})(-\alpha_i)^j \Big) y^n = 0,$$

which is equivalent to the system of $(d+2)(d+1)$ equations defined, for all $n \in \{0, \cdots, d\}$ and $i \in \{1, \cdots, d+1\}$, by $Eq_{(n,j)} : \sum_{j=0}^{n} (\alpha_i a_{n-j,j} + b_{n-j,j})(-\alpha_i)^j = 0$. We regroup them in $d+1$ systems S_n formed by the $d+2$ equations (indexed by i). These equations are polynomial of degree $n+1$ in α_i. We denote by c_k the coefficient of α^k, that is $c_0 = b_{n,0}$, $c_n = a_{0,n}$ and $c_k = a_{k,n-k} - b_{k-1,n-k+1}$ for $k \in \{1, n-1\}$. If we restrict the system S_n to their $n+2$ first equations, then we remark that the square system in c_k obtained is in fact a Vandermonde system. Since all the α_i are distinct then the system admits a unique solution $c_k = 0$. This implies that $a_{0,n} = 0$, $b_{n,0} = 0$ and $a_{k,d-k} = b_{k-1,d-k+1}$ for $k \in \{1, d\}$. Thus we have $yP(x,y) = xQ(x,y)$, which is a central vector field.

In a second case, assume that $d < p(\mathcal{A})$ thus $\mathcal{A}$ has at least $d+1$ parallel lines. Then, without lost of generality, we may assume that these lines are vertical. By Proposition 3, for any fixed y, $P(x,y) = 0$ for $d+1$ distinct values of x, since P is a polynomial of degree smaller than d. Then $P(x,y) = 0$ and χ fixes all the vertical lines. □

4.2. Structure of $\mathcal{D}^\infty(\mathcal{A})$

In this subsection, we give a characterization of polynomial vector fields fixing an infinity of lines, that is:

Theorem 5. *Let χ be a polynomial vector field fixing an infinity of lines, then χ is null, central or parallel.*

The proof is based on the following lemma, about the number of singular points in an arrangement with a countable infinity of lines.

Lemma 6. *Let $\mathcal{A}_\infty = \{L_1, L_2, L_3, \ldots\}$ be an infinite collection of different lines in the plane, then we have $\#\operatorname{Sing}(\mathcal{A}_\infty) \in \{0, 1, \infty\}$.*

Proof of Theorem 5. Let $P(x, y)$ and $Q(x, y)$ be such that $\chi = P\partial_x + Q\partial_y$. We define $\mathcal{A}_\infty = \{L_1, L_2, L_3, \ldots\}$ the set (or a subset) of the lines fixed by χ, and we denoted by α_i the equation of L_i. Up to now, we assume that we are not in the first case (i.e. $(P, Q) \neq (0, 0)$). The vector field χ fixes only a finite number of lines of $\mathcal{A}_\infty$ point by point. Indeed, L_i is fixed point by point by χ if and only if $\alpha_i \mid P$ and $\alpha_i \mid Q$. Since P and Q are polynomials then they have finite degree, thus only a finite number of α_i can divide them. Assume that these lines are $L_1, \ldots, L_k$. Denote by $\chi' = P'\partial_x + Q'\partial_y$ the derivation of components $P' = P/(\alpha_1 \cdots \alpha_k)$ and $Q' = Q/(\alpha_1 \cdots \alpha_k)$. It is clear that χ and χ' are collinear vector fields. Thus, if χ' is central (resp. parallel) then χ is central (resp. parallel). By construction, the set of points fixed by χ' (i.e. the common zeros of P' and Q') contain the intersection points of $\mathcal{A}'_\infty = \mathcal{A} \setminus \{L_1, \ldots, L_k\}$. By Lemma 6 we have 3 possible cases: (i) $\#\operatorname{Sing}(\mathcal{A}'_\infty) = 0$, then all the lines of $\mathcal{A}'_\infty$ are parallel. By the second part of the proof of Theorem 4, χ' is a parallel vector field. (ii) $\#\operatorname{Sing}(\mathcal{A}'_\infty) = 1$, then all the lines of $\mathcal{A}'_\infty$ are concurrent. By the first part of the proof of Theorem 4, χ' is a central vector field. (iii) $\#\operatorname{Sing}(\mathcal{A}'_\infty) = \infty$, then the polynomial P' and Q' have an infinity of zero. Which is impossible since P' and Q' are not simultaneously null. □

§5. Minimal degree and combinatorics

Definition 5. We denote by $d_f(\mathcal{A})$ the minimal integer d such that $\mathcal{D}^f_d(\mathcal{A})$ is not empty.

The study of the number $d_f(\mathcal{A})$ is related with the study of the Terao's conjecture in real space, which asks about the influence of combinatorics on the module of derivations of an arrangement when this one is *free*. In [4] we prove that, in general:

Theorem 7. *The minimal degree $d_f(\mathcal{A})$ is not determined by $L(\mathcal{A})$.*

The proof is composed of two parts. First, we give a purely combinatoric bound for which module of derivations is composed, up to a certain degree, only by derivations fixing a finite family of lines.

Theorem 8. *Let $\mathcal{A}$ be an arrangement. For all $0 < d < \min(|\mathcal{A}| - m(\mathcal{A}) + 1, |\mathcal{A}| - p(\mathcal{A}))$, the sets $\mathcal{D}_d(A)$ and $\mathcal{D}^f_d(A)$ are equal.*

Since $\mathcal{D}^\infty(\mathcal{A})$ and $\mathcal{D}^f(\mathcal{A})$ forms a disjoint partition of $\mathcal{D}(\mathcal{A})$, we conclude the following result from Theorem 4 and 8.

Corollary 9. *Let $\mathcal{A}$ be an arrangement, $\nu_\infty = \max(m(\mathcal{A}) - 1, p(\mathcal{A}))$ and $\nu_f = \min(|\mathcal{A}| - m(\mathcal{A}) + 1, |\mathcal{A}| - p(\mathcal{A})$. If $0 < d < \min(\nu_\infty, \nu_f)$ then $\mathcal{D}_d(\mathcal{A}) = \emptyset$.*

Then, we present two explicit counterexamples of line arrangements. As a first pair, we consider the configurations $(9_3)_1$ and $(9_3)_2$ realized in [9], called the *Pappus* and *non-Pappus* arrangements and denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Both arrangements have the same weak combinatorics (*i.e.* they share the same number of singularities for each multiplicity). We know that $\mathcal{D}_4(\mathcal{P}_1) \neq 0$ and $\mathcal{D}_4(\mathcal{P}_2) = 0$, but the previous theorem implies that $\mathcal{D}_4(\mathcal{P}_i) = \mathcal{D}_4^f(\mathcal{P}_i)$ (for $i = 1, 2$) and thus $d_f(\mathcal{P}_1) \neq d_f(\mathcal{P}_2)$. The second pair corresponds to Ziegler's arrangement $\mathcal{Z}_1$ and a small deformation $\mathcal{Z}_2$, with same strong combinatorics, *i.e.* $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$. In its paper [10], Ziegler proves that $\mathcal{D}_5(\mathcal{Z}_1) \neq 0$ and $\mathcal{D}_5(\mathcal{Z}_2) = 0$, but the previous theorem implies that $\mathcal{D}_5(\mathcal{Z}_i) = \mathcal{D}_5^f(\mathcal{Z}_i)$ (for $i = 1, 2$) and thus $d_f(\mathcal{Z}_1) \neq d_f(\mathcal{Z}_2)$.

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