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Progressive Orthogonal Wavelets: a Review

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Abstract. The concept of progressive wavelets, or wavelets supported on a half-line, has been introduced a long time ago by Björn Jawerth, for the purpose of analyzing moving images. Subsequently, it has fallen somewhat in between the fields of wavelet analysis and signal processing. Our paper is an attempt to unify these two strands of literature and review the current state of knowledge on this topic, with accent on splines based examples.

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§1. Introduction

Progressive wavelets. In this paper we review and reexamine some results of [9, 3, 4] on a class of wavelets proposed by Björn Jawerth.

Definition 1. *Progressive orthogonal wavelets* are real valued functions $\phi(x) \in L_2(R_+)$, with $\phi(x) = 0$ for $x < 0$ satisfying:

1. The L_2 orthogonality of integer translates $\phi(x-k), k \in \mathbb{Z}$.
2. A functional equation:

$$\sigma(x) = \phi(x) - \sum_{k=1}^{N-1} \alpha_k \phi(x-k) = \left(I - \sum_{k=1}^{N-1} \alpha_k \mathcal{T}^k\right)\phi(x), \tag{1}$$

where $\alpha_1, \alpha_2, ..., \alpha_{N-1}$ are real numbers, N is an integer, I denotes the identity operator, and $\mathcal{T}$ denotes the translation operator defined by

$$\mathcal{T} f(x) = f(x-1).$$

Note that

$$\begin{cases} \phi(x) = \sigma(x) & x \text{ in } [0,1], \\ \phi(x) = \sigma(x) + \alpha_1\sigma(x-1) & x \text{ in } [1,2], \\ \phi(x) = \sigma(x) + \alpha_1\sigma(x-1) + (\alpha_2+\alpha_1^2)\sigma(x-2) & x \text{ in } [2,3], \\ \phi(x) = \sigma(x) + \alpha_1\sigma(x-1) + (\alpha_2+\alpha_1^2)\sigma(x-2) + (\alpha_3 + 2\alpha_2\alpha_1 + \alpha_1^3)\sigma(x-3) & x \text{ in } [3,4], \\ \dots & \\ \phi(x) = \sigma(x) + \beta_1\sigma(x-1) + \beta_2\sigma(x-2) + \dots + \beta_n\sigma(x-n) & x \text{ in } [n, n+1]. \end{cases}$$

Explicit expressions for the coefficients β_n are given in Theorem 4. The fact that $\phi(x)$ is given by a combination involving more and more of the translates of σ to the right when x increases suggests the name "progressive function". Since $\phi(x)$ is completely determined by σ, we call σ an *updating function*. Such a function may be used to profit in designing wavelets via multiresolution analysis [3, 4].

Solving the transfer functional equation (1). Looking for solutions of (1) such that $\phi(x), \phi(x-1), ...$ are orthogonal on $L_2(\mathbb{R})$ is formally equivalent to a MA (moving averages) model of time series, from the point of view of "operator calculus" based on the operator $\mathcal{T}$. The formal solution of (1) is:

$$\phi(x) = (\alpha(\mathcal{T}))^{-1}\, \sigma(x), \text{ where } \alpha(z) = 1 - \sum_{k=1}^{N-1} \alpha_k z^k \tag{2}$$

will be called the **transfer function**. There are several ways to make sense out of this, as long as $\alpha(z) \neq 0$ on the unit circle $|z| = 1$, by the famous **Wiener Lemma** (see [17, 12, 14, 7]).

Definition 2. A Laurent series $\alpha(z) = \sum_k \alpha_k z^k$ is said to belong to the **Wiener class** W, if it has coefficient sequence α_k in $\ell_1(\mathbb{Z})$, and if $\alpha(z) \neq 0$, for all z on the unit circle $|z| = 1$ (see [1, p. 140]).

Alternatively, if we put $z = e^{-it}$ and $\tilde{\alpha}(t) = \alpha(z) = \alpha(e^{-it}) = \sum_k \alpha_k e^{-ikt}, t \in [0, 2\pi)$ we say that the periodic function $\tilde{\alpha}(t)$ is in the Wiener class W if its Fourier sequence α_k is in $\ell_1(\mathbb{Z})$, and if $\alpha(t) \neq 0$ for all $t \in [0, 2\pi)$ (see [7, p.179]). Note that if $\tilde{\alpha}(t) \in W$, then $\tilde{\alpha}(t)$ is continuous on $[0, 2\pi)$, since its Fourier series converges absolutely.

Lemma 1. *a)* ***Wiener's Lemma****. If $\alpha(z) \in W$, then there exists $\beta(z) = \frac{1}{\alpha(z)} = \sum_{k\in\mathbb{N}} \beta_k z^k \in W$, for some sequence $\{\beta_k\} \in \ell_1(\mathbb{Z})$.*

b) ***Sylvester's Lemma****. If $\alpha(z) = (1-\lambda_1 z)(1-\lambda_2 z)...(1-\lambda_n z) = 1 + \alpha_1 z + ... + \alpha_n z^n$, is a polynomial, then $\{\beta_k\}$ satisfy the recurrence*

$$\alpha(\mathcal{T})\beta_j = 0, j \geq 1, \beta_0 = 1, \beta_i = 0, \forall i < 0, \Leftrightarrow \sum_{k=0}^{n} \alpha_k \beta_{j-k} = 0, j \geq 1, \beta_0 = 1, \beta_i = 0, \forall i < 0,$$

where $\mathcal{T}(\beta_j) = \beta_{j-1}$ and $\alpha_0 = 1$. Moreover $\{\beta_j\}$ are given explicitly by the Newton divided differences:

$$\beta_j = (\lambda^{j+n-1})_{[\lambda_1,\lambda_2,...,\lambda_n]} := \sum_{i=1}^{n} \frac{\lambda_i^{j+n-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)} \tag{3}$$

Furthermore, the solution for the nonhomogeneous recurrence

$$\alpha(\mathcal{T})x_k = \epsilon_k, k \geq 1,$$

is $x_k = \sum_{j=0}^{k-1} \beta_j \epsilon_{k-j}$. β_k is called the "fundamental solution"of the operator $\alpha(\mathcal{T})$■

For $\alpha(z) \in W$, one may

1. Apply Fourier transform on both sides of (1) and solve for $\widehat{\phi}(\xi)$, yielding

$$\widehat{\phi}(\xi) = \frac{\widehat{\sigma}(\xi)}{\alpha(e^{-i\xi})} = \beta(e^{-i\xi})\widehat{\sigma}(\xi) = \left(\sum_{k=0}^{\infty} \beta_k e^{-ik\xi}\right)\widehat{\sigma}(\xi). \tag{4}$$

2. Or, with N finite, one may factor $\alpha(z)$ as in Lemma 1 b), and develop $\beta(z) = 1/\alpha(z)$ in a power series $\sum_{k=0}^{\infty} \beta_n z^n$, using partial fractions. The formal result

$$\phi(x) = \sum_{k=0}^{\infty} \beta_k \sigma(x-k) \tag{5}$$

converges for $\alpha(z) \in W$ by Wiener's Lemma.

Remark 1. A further step, already investigated in [4], is to consider analytic scaling symbols $s(z)$. In this paper we restrict ourselves to the easier polynomial case, since here we may exploit the power series expansion of reciprocals of polynomials due to Sylvester (3), and that makes it possible to establish a relation between the scaling symbol $f(z)$ of $\phi(x)$ and $\alpha(z)$ – see (15) below.

Refinable updating functions. In this paper we suppose that the updating function satisfies a **two scale refinement equation**

$$\frac{1}{2}\sigma(\frac{x}{2}) = \sum_{j=0}^{N} s_j \sigma(x-j) \Leftrightarrow \widehat{\sigma}(w) = s(e^{-iw/2})\widehat{\sigma}(w/2). \tag{6}$$

Following [5], we assume that the **polynomial symbol**

$$s(z) = \sum_{j=0}^{N} s_j z^j$$

has real coefficients, $s_0 > 0, s_N > 0$, all its roots are in the left half-plane $z : Re(z) \leq 0$, at least two of them in $z : Re(z) < 0$, and that $s(-1) = 0, s(1) = 1$. It follows then [5] that the refinement equation (6) has a unique solution satisfying $\int_{-\infty}^{\infty} \sigma(x)dx = 1$, and that $\sigma(x)$ is continuous, non-negative and has support in $[0, N]$.

Example 1. In the special case $s_j \geq 0, \sum_j s_j = 1$, the solution $\sigma(x)$ of the refinement equation (6) may be interpreted as the probability density function of an absolutely continuous random variable X satisfying

$$2X \stackrel{\mathcal{L}}{=} X + Y,$$

where X, Y are independent, and $P(Y = j) = s_j$ for $j = 0, ..., N$. The probability density of X satisfies the equation (6).

Example 2. In the important special case $s(z) = (\frac{z+1}{2})^N$, $\sigma(x)$ is the uniform B-spline B_N of degree $N-1$ with knots $0, ..., N$.

The autocorrelation function. For any $\sigma \in L_2(\mathbb{R})$, let us introduce the functions

$$[\sigma, \sigma](x) = \int_{\mathbb{R}} \sigma(x+y)\overline{\sigma(y)}dy, \qquad C_\sigma(z) = \sum_{n=-N+1}^{N-1} [\sigma, \sigma](n) z^n \tag{7}$$

called respectively autocorrelation function and autocorrelation symbol. By the Poisson summation formula [10, Thm.6.5.3], the autocorrelation function is related to σ by:

$$\sum_{n\in\mathbb{Z}} \left|\widehat{\sigma}(\xi + 2\pi n)\right|^2 = \sum_{n\in\mathbb{Z}} [\sigma, \sigma](n) e^{-in\xi} = C_\sigma(e^{-i\xi}). \tag{8}$$

The following orthogonality condition for translates of a scaling function can be found in many references – see for example [11, Thm 5].

Theorem 2.

a) *For any function $\phi \in L_2(\mathbb{R})$, $\{\phi(x-n) : n \in \mathbb{Z}\}$ is an orthonormal family if and only if*

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi n)|^2 = 1, \forall \xi \in \mathbb{R}^{\S}. \tag{9}$$

b) ***Smith-Barnwell condition***. *If ϕ satisfies a two-scale refinement equation $\phi(x) = 2\sum_k f_k \phi(2x-k) \iff \widehat{\phi}(2w) = f(e^{-iw})\widehat{\phi}(w)$, where $f(z) = \sum_{k\in Z} f_k z^k$, then (9) is equivalent to*

$$f(z)f(z^{-1}) + f(-z)f(-z^{-1}) = 1 \Longrightarrow |f(e^{-iw})|^2 + |f(-e^{-iw})|^2 = 1.$$

Corollary 3. *Let $\sigma(x)$ be an updating function in $L_2(\mathbb{R})$ and let ϕ be a progressive function in $L_2(\mathbb{R})$ satisfying (1) with N finite. Then, $\{\phi(x-n) : n \in \mathbb{Z}\}$ is an orthonormal sequence if and only if*

$$\sum_{n\in\mathbb{Z}} \left|\widehat{\sigma}(\xi + 2\pi n)\right|^2 = C_\sigma(e^{-i\xi}) = \left|\alpha(e^{-i\xi})\right|^2 \text{ for any } \xi \in \mathbb{R}.$$

Proof. By Theorem 2 the translates $\{\phi(x-n) : n \in \mathbb{Z}\}$ are orthonormal if and only if

$$\sum_{n\in\mathbb{Z}} \left|\widehat{\phi}(\xi + 2\pi n)\right|^2 = 1 \tag{10}$$

The corollary follows by equation 4 and the periodicity of $\alpha(e^{-i\xi})$:

$$\sum_{n\in\mathbb{Z}} \frac{|\widehat{\sigma}(\xi + 2\pi n)|^2}{|\alpha(e^{-i(\xi+2\pi n)})|^2} = \frac{\sum_{n\in\mathbb{Z}} |\widehat{\sigma}(\xi + 2\pi n)|^2}{|\alpha(e^{-i(\xi)})|^2} = 1 \tag{11}$$

§2. Progressive wavelets with polynomial transfer function

We collect now several results on progressive wavelets scattered in the literature.

Theorem 4. *Let $\sigma(x) \in L_2(\mathbb{R})$ with support $[0, N]$ for a positive integer N. Assume that $[\sigma, \sigma](N-1) \neq 0$, where $[\sigma, \sigma]$ denotes the autocorrelation of σ defined in (7), and let $C(z) = C_\sigma(z)$ denote the autocorrelation symbol. Then,*

[§] As a corollary, the translates of any function $\phi^\perp(x)$ obtained by

$$\widehat{\phi^\perp}(w) = \frac{\widehat{\phi}(w)}{\sqrt{\sum_{k\in\mathbb{Z}} |\widehat{\phi}(w + 2\pi k)|^2}}$$

provide an orthonormal base of the space V_0 generated by $\{\phi(x-k), k \in \mathbb{Z}\}$. Finally, applying a "mirror filter" $(-1)^k p_{1-k}$ yields a wavelet function $\psi(x) = \sum_k (-1)^k p_{1-k} \phi^\perp(2x-k)$ whose scaled translates enjoy also orthogonality between the spaces $V_n, n \in \mathbb{Z}$ at different scales.

1. *If* $C(z) = \sum_{n\in\mathbb{Z}} \left|\widehat{\sigma}(\xi + 2\pi n)\right|^2 > 0$, *for* $|z| = 1$, *then there exists a unique* ***canonical spectral factor*** $\alpha(z) \in W$ *satisfying*

$$\alpha(z)\alpha(z^{-1}) = C(z) \Longrightarrow |\alpha(z)|^2 = C(z), z = e^{-i\xi},$$

and having no roots in the interior of the unit circle $|z| = 1$. *If* $z_1, ...z_{N-1}$ *are the roots of* $C(z)$ *outside the unit disc,* $|z_k| > 1$, $k = 1, ..., N-1$, *then*

$$\alpha(z) = \sqrt{A}\prod_{k=1}^{N-1}(z - z_k) = \alpha_0 + \alpha_1 z + ... + \alpha_{N-1}z^{N-1}$$

where $A = (-1)^{N-1}[\sigma, \sigma](N-1)/\prod_{k=1}^{N-1} \bar{z}_k > 0$
Furthermore, $\alpha_k = 0$ *for all* $k \geq N$ *in (1) and* $\max\{k : \alpha_k \neq 0\} = N - 1$.

2. *As a consequence, the coefficients* β_k *of the reciprocal series satisfy the "Yule-Walker recursion"* [§]

$$\beta_j = \alpha_1\beta_{j-1} + \alpha_2\beta_{j-2} + ... + \alpha_{N-2}\beta_{j-N+2} + \alpha_{N-1}\beta_{j-N+1}, \beta_0 = 1, \beta_i = 0, \forall i < 0. \quad (12)$$

Moreover the $\{\beta_j\}$ *are given explicitly in terms of the roots of* $\alpha(z)$ *by* (3), *and decay exponentially.*

3. *The* $\{\beta_n\}$ *are given explicitly in terms of the coefficients of* $\alpha(z)$ *by*

$$\beta_n = \sum_{k_1+2k_2+3k_3+...+mk_m=n} \begin{pmatrix} k_1 + k_2 + \cdots + k_m \\ k_1, k_2, ..., k_m \end{pmatrix} \alpha_1^{k_1}\alpha_2^{k_2}\alpha_3^{k_3}\cdots\alpha_m^{k_m}, \quad (13)$$

where $k_i \geq 0$ *for* $0 \leq i \leq N-1$, $m = n$ *for* $0 \leq n \leq N-1$, $m = N-1$ *for* $n > N-1$, *and*

$$\begin{pmatrix} k_1 + k_2 + \cdots + k_m \\ k_1, k_2, ..., k_m \end{pmatrix} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!k_2!\cdots k_m!}$$

are the multinomial coefficients.

4. *Let now* $\phi(x)$ *be defined by*

$$\phi(x) = \sum_{n\geq 0} \beta_n \sigma(x - n), \quad (14)$$

where the convergence is uniform, since β_n *decay exponentially. Then,* ϕ *satisfies the functional equation (1) and* $\{\phi(x-k) : k \in \mathbb{Z}\}$ *is an orthonormal sequence.*

5. *If* $\sigma(x)$ *is refinable with symbol* $s(z)$, *then so is the function* $\phi(x)$, *with refinement equation*

$$\phi(x) = 2\sum_k f_k \phi(2x - k),$$

[§] This recursion is used a lot in time series, for example when inverting the transfer function, in order to switch from an autoregressive to a moving average model, or viceversa. Note that the famous Fibonacci recursion induced by $\alpha(z) = 1 - z - z^2$, is not "acceptable/stationary" due to one root smaller $z = 0.618034 \notin [-1, 1]$ which induces non-exponential decay. Before using this model, this root must be replaced by its reciprocal.

and scaling symbol $f(z)$ given by

$$f(z) = s(z)\frac{\alpha(z)}{\alpha(z^2)} \Leftrightarrow F(z) := f(z)f(z^{-1}) = s(z)s(z^{-1})\frac{C(z)}{C(z^2)} = S(z)\frac{C(z)}{C(z^2)}. \tag{15}$$

Proof.

1. This follows by the well-known Fejér-Riesz lemma (see e.g. [8, p. 235], [13, p. 330]), since $C(z) \in W$ by (8) (i.e. $C(e^{-in\xi}) > 0, \forall \xi \in \mathbb{R}$). Furthermore, The leading term of $C(z)$ is $[\sigma, \sigma](N-1)z^{N-1}$. Comparing coefficients with the (possibly infinite) Laurent series $\alpha(z)\,\alpha(z^{-1})$, we conclude that $\max\{k : \alpha_k \neq 0\} = N-1$.
2. This widely used result (for example in signal processing and time series) may be easily checked.
3. For a proof, see for example [4].
4. It is easy to check that (14) satisfies (1). Taking then Fourier transform, we find that

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi n)|^2 = \frac{\sum_{n\in\mathbb{Z}} \left|\widehat{\sigma}(\xi + 2\pi n)\right|^2}{\left|\alpha(e^{-i\xi})\right|^2} = 1.$$

and hence the translates $\phi(x-k)$ are orthonormal by Corollary 3.

5. Following [9, (43)], let us look for a refinement relation $\widehat{\phi}(2w) = f(e^{-iw})\widehat{\phi}(w)$. Using $\widehat{\phi}(w) = \frac{\widehat{\sigma}(w)}{\alpha(e^{-iw})}$ and $\widehat{\sigma}(2w) = s(e^{-iw})\widehat{\sigma}(w)$ we find

$$\widehat{\phi}(2w) = \frac{\widehat{\sigma}(2w)}{\alpha(e^{-2iw})} = \frac{s(e^{-iw})}{\alpha(e^{-2iw})}\widehat{\sigma}(w) = \frac{s(e^{-iw})\alpha(e^{-iw})}{\alpha(e^{-2iw})}\widehat{\phi}(w).$$

Putting $z = e^{-iw}$ yields (15)□

§3. Linear B-spline progressive wavelets

Recall that the linear B-spline/"roof" function

$$\Lambda(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases},$$

satisfies the two-scale relation

$$\Lambda(x) = \frac{1}{2}\Lambda(2x) + \Lambda(2x-1) + \frac{1}{2}\Lambda(2x-2), \tag{16}$$

with scaling symbol

$$s(z) = (\frac{1+z}{2})^2.$$

For $\nu \geq 0$, let V_R^ν be the closure in L_2 of the subspace spanned by the functions $\Lambda_{\nu,k}$ for $k \geq 0$, and let W_R^ν be the orthogonal complement of V_R^ν in $V_R^{\nu+1}$. We want to construct an orthonormal basis of V_R^0 of the form $\{\phi(x-k) : k \geq 0\}$, with a progressive function ϕ defined by $\phi(x) = C\Lambda(x) + \alpha\phi(x-1)$, with constants α and $C = C_\alpha$ determined so that the set of translates $\{\phi(x-k)\}$ is orthonormal.

Theorem 5. *a) The functional equation*

$$\phi(x) = C\Lambda(x) + \alpha\phi(x-1), \tag{17}$$

admits a unique solution $\phi = \phi_\alpha \in L_2(\mathbf{R})$ *when* $|\alpha| < 1$. *Moreover*

$$\phi(x) = C\sum_{k\geq 0} \alpha^k \Lambda(x-k). \tag{18}$$

b) The function ϕ *satisfies the scaling equation*

$$\phi(x) = 2\sum_{k\geq 0} f_k \phi(2x-k),$$

where the filter coefficients f_k *are given by*

$$f_0 = \tfrac{1}{4}, \quad f_1 = \tfrac{1}{2} - \tfrac{\alpha}{4} \quad f_2 = \tfrac{1}{4}(1-\alpha),$$

$$f_{2k+1} = f_{2k+2} = \tfrac{1-\alpha}{4}\alpha^k, \quad \textit{for} \quad k \geq 1$$

Proof a) The equation (17) can be written as

$$(I - \alpha\mathcal{T})\phi = C\Lambda,$$

and for $|\alpha| < 1$, the unique solution ϕ is given by the Neumann series (18).

b) Write

$$\phi(x) = (1-\alpha\mathcal{T})^{-1}\Lambda(x) = (1-\alpha\mathcal{T})^{-1}D_2(2s(\mathcal{T}))\Lambda(x) = (1-\alpha T)^{-1}D_2(2s(\mathcal{T}))(1-\alpha\mathcal{T})\phi(x), \tag{19}$$

where D_2 is the dilation operator by 2, and $s(z) = (\frac{1+z}{2})^2$. Using $\mathcal{T}^k D_2 = D_2\mathcal{T}^{2k}$, we find that

$$\begin{aligned}
\phi(x) &= \frac{1}{4}(\sum_{k\geq 0} \alpha^k\mathcal{T}^k) D_2(1+\mathcal{T})^2(1-\alpha\mathcal{T})\phi(x) \\
&= \frac{1}{4}D_2(\sum_{k\geq 0}\alpha^k\mathcal{T}^{2k})(1+2\mathcal{T}+\mathcal{T}^2)(1-\alpha\mathcal{T})\phi(x) \\
&= \frac{1}{4}D_2(\sum_{k\geq 0}\alpha^k\mathcal{T}^{2k})(1+\mathcal{T}(2-\alpha)+\mathcal{T}^2(1-2\alpha)-\alpha\mathcal{T}^3)\phi(x) \\
&= \frac{1}{4}D_2(I+\mathcal{T}(2-\alpha)+\sum_{k\geq 1}\mathcal{T}^{2k}(\alpha^k+\alpha^{k-1}(1-2\alpha))+\mathcal{T}\sum_{k\geq 1}\mathcal{T}^{2k}(\alpha^k(2-\alpha)-\alpha^{k-1}\alpha))\phi(x) \\
&= \frac{1}{4}D_2(I+\mathcal{T}(2-\alpha)+\sum_{k\geq 1}\mathcal{T}^{2k}(\alpha^{k-1}(1-\alpha))+\mathcal{T}\sum_{k\geq 1}\mathcal{T}^{2k}(\alpha^k(1-\alpha)))\phi(x)
\end{aligned}$$

□

Theorem 6. *The family $\{\phi_\alpha(x-k) : k \geq 0\}$ is an orthonormal basis for V_R^0 if and only if $\alpha = -2+\sqrt{3}$ and $C^2 = C_\alpha^2 = -6\alpha = \frac{3}{2}(1+\alpha^2)$.*

Proof: It will be enough to determine α so that

$$<\phi_\alpha(x), \phi_\alpha(x-1)>= 0$$

$$<\phi_\alpha(x), \phi_\alpha(x)>= 1.$$

By using that $<\Lambda(x), \Lambda(x)>= \frac{2}{3}$ and $<\Lambda(x), \Lambda(x-1)>= \frac{1}{6}$, these equations lead to a system of equations in C and α

$$C^2\frac{1}{6}+\alpha = 0$$

$$C^2\frac{2}{3}+C^2\alpha\frac{1}{6} = 1$$

with solution $\alpha = -2+\sqrt{3} \in (-1,1)$ and $C^2 = C_\alpha^2 = -6\alpha$.

Remark 2. We may verify that the symbol $f(z)$ satisfies the Smith–Barnwell condition $f(z)f(z^{-1})+f(-z)f(-z^{-1}) = 1$ when $\alpha = -2+\sqrt{3}$.

§4. B-spline progressive wavelets

B-splines. The mth order cardinal B-splines $N_m(x)$ are the densities of a sum of m independent $U[0,1]$ r.v.'s. They are defined recursively by convolution:

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_{\mathbb{R}} N_{m-1}(x-t)N_1(t)dt, m \geq 2 \tag{20}$$

where N_1 is the characteristic function of the unit interval $[0,1)$. The corresponding Fourier transforms are

$$\int_{-\infty}^{\infty} e^{-iwt}N_m(t) = (\frac{1-e^{-iw}}{iw})^m = (e^{iw/2}\frac{sin(w/2)}{iw/2})^m.$$

N_m is supported on $[0,m]$, and is symmetric with respect to the center of its support $\frac{m}{2}$, i.e.

$$N_m\left(\frac{m}{2}+x\right) = N_m\left(\frac{m}{2}-x\right).$$

The two scale symbol of N_m is $s_m(z) = (\frac{1+z}{2})^m$ [2, (3.4.6)], and the corresponding two scale relation is [2, (3.4.7)]

$$\frac{1}{2}N_m(x/2) = 2^{-m}\sum_{j=0}^{m}\binom{m}{j}N_m(x-j) = s_m(\mathcal{T})N_m(x).$$

The autocorrelation of the cardinal B-splines is

$$[N_m, N_m](x) = \int_{\mathbb{R}} N_m(x+y)\overline{N_m(y)}dy = N_{2m}(m+x), m \geq 1,$$

and the autocorrelation symbol (8) is:

$$\sum_{n\in\mathbb{Z}} \left|\widehat{N}_m(\xi+2\pi n)\right|^2 = \sum_{n\in\mathbb{Z}}[N_m, N_m](n) = \sum_{n\in\mathbb{Z}} N_{2m}(m+n)e^{-in\xi} = \sum_{n=-m+1}^{m-1} N_{2m}(m+n)e^{-in\xi} = E_{2m}(e^{-i\xi}). \tag{21}$$

Here

$$E_m(z) = \sum_{j=-m+1}^{m-1} N_m(m+j)z^j,$$

is the well-known Euler-Frobenius Laurent polynomial [15].

The monic polynomials $\tilde{E}_m(z) = m!E_m(z)$ satisfy the recursion [15, (2.1.9)]

$$\begin{aligned}
&\tilde{E}_m(z) = (1+mz)\tilde{E}_{m-1}(z) + z(1-z)\tilde{E}'_{m-1}(z), \tilde{E}_0(z) = 1 \Longrightarrow\\
&\tilde{E}_1(z) = z+1,\\
&\tilde{E}_2(z) = z^2+4z+1,\\
&\tilde{E}_3(z) = z^3+11z^2+11z+1,\\
&\tilde{E}_4(z) = z^4+26z^3+66z^2+26z+1,\\
&\tilde{E}_5(z) = E_5(z) = z^5+57z^4+302z^3+302z^2+57z+1,\\
&\tilde{E}_6(z) = z^6+120z^5+1191z^4+2416z^3+1191z^2+120z+1.
\end{aligned}$$

The symmetry of $\tilde{E}_{2m}(z) = z^{-m}E_{2m}(z) = \sum_{k=-m}^{m} a_k z^k$ is useful for performing the Riesz-Fejer factorization of $E_{2m}(z)$, providing a degree reduction from $2m$ to m – see [6, Sec. 2.3:Roots method]. Indeed, put $z+z^{-1} = w$ [§] . Using

$$z^k + z^{-k} = (z+z^{-1})^k - \sum_{j=1}^{k-1}\binom{k}{j} z^{k-2j}$$

yields

$$(z+z^{-1})^2 = w^2-2,\ (z+z^{-1})^3 = w^3-3w$$

and thus

$$\tilde{E}_2(z) = w+4,\ \tilde{E}_4(z) = w^2+26w+64,\ \tilde{E}_6(z) = w^3+120w^2+1188w+2176$$

(the first allows to recover the linear spline case by solving $z+z^{-1} = -4$).

For more properties and results about B-splines one can refer to [15, 16]. See also [9, Table II] for a list of "valid scaling symbols" $F(z)$ associated to B-splines.

Example 3. Consider now the quadratic B-spline $N_3(x)$. The corresponding progressive function ϕ satisfies the functional equation

$$\phi(x) + \alpha_1\phi(x-1) + \alpha_2\phi(x-2) = CN_3(x) \tag{22}$$

[§] An alternative to express $cos(kx), k = 1, ..., m$ as functions of $cos(x)^k, k = 1, ..., m$, and ultimately switch to $y = cos(x) \in [-1, 1]$.

where $C = C_\alpha = \sqrt{\frac{20}{11}(1+\alpha_1^2+\alpha_1^2)}$.

Solving $w^2 + 26w + 64 = 0$ wields $w_{1,2} = -13 \pm \sqrt{105}$, and each of the w_i roots yields a unique "reciprocal root" in the unit circle

$$z_1 = \frac{2}{-13 - \sqrt{105} - \sqrt{270 + 26\sqrt{105}}}, z_2 = \frac{1}{2}\left(-13 + \sqrt{105} + \sqrt{270 - 26\sqrt{105}}\right).$$

The transfer function is $(1-zz_1)(1-zz_2) = 1+\alpha_1 z+\alpha_2 z^2$ where $\alpha_1 = 13-\sqrt{135+4\sqrt{30}}, \alpha_2 = 2\left(8+\sqrt{30}\right) - \sqrt{375+64\sqrt{30}}$.

The general results of Theorem 4 are again confirmed in this case.

Theorem 7. *Let ϕ be the progressive function that satisfies the equation*

$$\phi(x) = CN_3(x) + \alpha_1\phi(x-1) + \alpha_2\phi(x-2)$$

and let

$$\phi(x) = C\sum_{k\geq 0}\beta_k N_3(x-k), \tag{23}$$

where the coefficients β_k are given inductively by $\beta_0 = 1$, $\beta_1 = \alpha_1$ and for $k \geq 2$, $\beta_k = \alpha_1\beta_{k-1} + \alpha_2\beta_{k-2}$. Then the function ϕ satisfies the scaling equation $\phi(x) = 2\sum_{k\geq 0} f_k\phi(2x-k)$, with filter coefficients given by

$$\begin{aligned}
&2f_0 = 1 \\
&2f_{2k-1} = -\alpha_2\beta_{k-3} + (1-3\alpha_1-3\alpha_2)\beta_{k-2} + (3-\alpha_1)\beta_{k-1} \\
&\vdots \\
&2f_{2k} = \beta_k + (-\alpha_1 - 3a_2)\beta_{k-2} + (3-3\alpha_1-\alpha_2)\beta_{k-1}.
\end{aligned}$$

In particular $2f_0 = 1, \quad 2f_1 = 3-\alpha_1 \quad 2f_2 = 3-2\alpha_1-\alpha_2 \ldots$

Remark 3. We may check again using Mathematica for example that (15) holds:

$$f(z) = s(z)\frac{\alpha(z)}{\alpha(z^2)} = (\frac{1+z}{2})^3\frac{(1-\alpha_1 z-\alpha_2 z^2)}{(1-\alpha_1 z^2-\alpha_2 z^4)}.$$

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