Monografías Matemáticas *García de Galdeano* **39**, 57–63 (2014)

CHARACTERIZATIONS AND TESTS FOR ALMOST M–MATRICES

Álvaro Barreras and Juan Manuel Peña

Abstract. The concept of almost nonsingular M–matrix is analyzed and characterized. Other related concepts are studied and some applications are given.

Keywords: M–matrices, Stieltjes matrices, diagonal dominance, negative determinant.
AMS classification: 15A23, 65F30, 65F05, 15B48.

§1. Introduction

Nonsingular M–matrices are very important in many applications: economics, dynamical systems, linear programming or numerical analysis, among other fields (cf. [4]). Besides, they can be characterized in many different ways. In fact, in [4], more than 50 different characterizations can be found. Recently, it has been shown that diagonally dominant M–matrices is one of the few classes of matrices for which one can find accurate algorithms; for instance, for computing the singular values (see [3], [6], [9]), or the smallest eigenvalue ([2]), or the matrix inverse (cf. [1]). By accurate algorithms we mean that they can be performed to high relative accuracy independently of the conditioning of the problem (see [5]).

In this paper we introduce the concept of an almost nonsingular M–matrix and other related concepts. We prove that they inherit many properties and characterizations of nonsingular M–matrices, with the natural adaptations.

In Section 2 we introduce the main concepts and we characterize almost nonsingular M–matrices in different ways. The characterization of Theorem 4 (*vi*) provides a practical test (of $O(n^3)$ elementary operations) to check if an $n \times n$ matrix is an almost nonsingular M–matrix. Section 3 analyzes some subclasses of almost nonsingular M–matrices adding either symmetric or diagonal dominant properties. As an application of this last subclass of matrices, we give a very simple test (of $O(n^2)$ elementary operations) to check if a given $n \times n$ matrix has negative determinant.

§2. Characterizations of almost M–matrices

Let $A = (a_{ij})_{1\le i,j\le n}$ be a real square matrix. Given $k, l \in \{1, 2, \dots, n\}$, let α, β be two increasing sequences of k and l positive integers respectively less than or equal to n. Then we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A containing rows numbered by α and columns numbered by β. For principal submatrices, we use the notation $A[\alpha] := A[\alpha|\alpha]$. A principal submatrix of A of the form $A[1, \dots, k]$ for $k \in \{1, \dots, n\}$ is called a leading principal submatrix. We also denote by $A(\alpha) := A[\alpha^c]$, where α^c is the increasing rearranged complement of α in $\{1, \dots, n\}$, that is, $\alpha^c = \{1, \dots, n\} \setminus \alpha$. A real matrix with nonpositive off–diagonal entries is called a *Z–matrix*. An *M–matrix* is a Z–matrix A such that it can be expressed as $A = sI - B$, with $B \ge 0$ and

$s \geq \rho(B)$ (where $\rho(B)$ is the spectral radius of B). Let us recall that, given a Z–matrix A, then A is a nonsingular M–matrix if and only if A^{-1} is nonnegative. There are many characterizations of nonsingular M–matrices (see for instance Theorem 2.3 of Chapter 6 of [4]). We now recall some of them in the following result, which collects some conditions of the statement and proof of Theorem 2.3 of Chapter 6 of [4].

Theorem 1. *Let A be a Z–matrix, then the following properties are equivalent:*

(i) A is a nonsingular M–matrix.

(ii) All leading principal minors of A are positive.

(iii) All principal minors of A are positive.

(iv) $A = LU$, with L a nonsingular lower triangular Z–matrix with positive diagonal and U a nonsingular upper triangular Z–matrix with positive diagonal.

We now introduce the main definition of the paper.

Definition 1. A nonsingular Z–matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is called an *almost nonsingular M–matrix* if A is not an M–matrix and $A[1,\ldots,n-1]$ is a nonsingular M–matrix.

The following theorem shows that almost nonsingular M–matrices have LDU decompositions such that L and U are M–matrices. Let us recall that an LDU decomposition of a square matrix A is a factorization $A = LDU$ where L is a lower triangular matrix with unit diagonal, D is a diagonal matrix with nonzero diagonal entries and U is an upper triangular matrix with unit diagonal. It is well–known that, for a nonsingular matrix, this decomposition is unique.

Theorem 2. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a Z–matrix. The following conditions are equivalent:*

(i) *A is almost nonsingular M–matrix.*

(ii) *$A = LDU$, where L is a lower triangular nonsingular M–matrix and U is an upper triangular nonsingular M–matrix, both with unit diagonal, and $D = diag(d_i)_{i=1}^n$ with $d_i > 0$ for all $i < n$ and $d_n < 0$.*

Proof. (i) ⇒ (ii) Since $A[1,\ldots,n-1]$ is a nonsingular M–matrix, by Theorem 1 all its leading principal minors are positive and, since A is nonsingular and is not an M–matrix, $\det A < 0$ again by Theorem 1. Since A is also nonsingular, all its leading principal minors are nonzero and then it is well–known that $A = LDU$, where L (resp. U) is lower (resp. upper) triangular with unit diagonal and $D = diag(d_1,\ldots,d_n)$ is a diagonal matrix with $d_1 = a_{11}$ (> 0) and $d_i = \det A[1,\ldots,i]/\det A[1,\ldots,i-1]$ (> 0) for each $i = 2,\ldots,n-1$. In addition, we have the following LDU decomposition of $A[1,\ldots,n-1]$:

$$A[1,\ldots,n-1] = L[1,\ldots,n-1]D[1,\ldots,n-1]U[1,\ldots,n-1]. \tag{2.1}$$

Let us prove, by induction on j, that $l_{nj} \leq 0$ for $j = 1,\ldots,n-1$. We have that $a_{n1} = l_{n1}d_1$ and, since $a_{n1} \leq 0$ and $d_1 > 0$ by hypothesis, we conclude that $l_{n1} \leq 0$. Suppose now that $l_{nk} \leq 0$ for all $k \leq j-1$. We know that

$$0 \geq a_{nj} = \sum_{k=1}^{j} l_{nk}d_k u_{kj} = l_{nj}d_j u_{jj} + \sum_{k=1}^{j-1} l_{nk}d_k u_{kj} = l_{nj}d_j + \sum_{k=1}^{j-1} l_{nk}d_k u_{kj}.$$

Taking into account that, by hypothesis, $l_{nk}, u_{kj} \leq 0$ for $k = 1, \ldots, j-1$ and $d_j > 0$ for $j = 1, \ldots, n-1$, we can derive that $l_{nj} \leq 0$ for all $j \leq n-1$. Analogously, we can prove that $u_{jn} \leq 0$ for all $j = 1, \ldots, n-1$. By (2.1), Theorem 1 and the uniqueness of the LDU decomposition, we can deduce that $L[1, \ldots, n-1]$ and $U[1, \ldots, n-1]$ are Z–matrices. Thus, L and U are triangular Z–matrices with unit diagonal and so, by Theorem 1, L and U are nonsingular M–matrices. Finally, let us observe that

$$d_n = \frac{\det A}{\det A[1, \ldots, n-1]} < 0,$$

and so, (ii) follows.

$(ii) \Rightarrow (i)$ By hypothesis, A is a Z–matrix, and so $A[1, \ldots, n-1]$ is a Z–matrix. The leading principal minors $\det A[1, \ldots, k]$ of A are $d_1 \cdots d_k > 0$, $k = 1, \ldots, n-1$. Then, by Theorem 1 $(ii) \Rightarrow (i)$, $A[1, \ldots, n-1]$ is a nonsingular M–matrix. Finally, $\det A = d_1 \cdots d_n < 0$ and (i) follows from Theorem 1. □

We can extend the previous theorem to a larger class of matrices. We say that A is a *generalized almost nonsingular M–matrix* if there exists a permutation matrix P such that PAP^T is an almost nonsingular M–matrix.

Theorem 3. *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a Z–matrix. The following conditions are equivalent:*

- (i) *A is a generalized almost nonsingular M–matrix.*
- (ii) *There exists a permutation matrix P such that $PAP^T = LDU$, where L (resp. U) is a lower (resp. upper) triangular nonsingular M–matrix with unit diagonal and $D = diag(d_i)_{i=1}^n$, with $d_i > 0$ for all $i < n$ and $d_n < 0$.*

Proof. It is only necessary to apply Theorem 2 to the almost nonsingular M–matrix PAP^T. □

Let us recall that a *P–matrix* is a matrix with all its principal minors positive. If a nonsingular matrix A is not a P–matrix but $A[1, \ldots, n-1]$ is a P–matrix, then we say that A is an *almost P–matrix*.

In the following theorem we prove that, for Z–matrices, the concepts of almost P–matrix and almost nonsingular M–matrix are equivalent. We also provide more equivalent properties of this class of matrices. In particular, (v) characterizes almost nonsingular M–matrices in terms of their leading principal minors and (vi) through Gaussian elimination.

Theorem 4. *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a Z–matrix. The following statements are equivalent:*

- (i) *A is almost nonsingular M–matrix.*
- (ii) $\det A < 0$ *and* $A[1, \ldots, n-1]$ *is a nonsingular M–matrix.*
- (iii) *A is a nonsingular matrix with an odd number of negative eigenvalues, and all eigenvalues of* $A[1, \ldots, n-1]$ *has positive real part.*
- (iv) *A is an almost P–matrix.*
- (v) $\det A < 0$ *and* $\det A[1, \ldots, k] > 0$ *for all $k < n$.*
- (vi) *Gaussian elimination of A can be performed without row exchanges and the pivots d_i satisfy $d_i > 0$ for $i = 1, \ldots, n-1$ and $d_n < 0$.*

Proof. $(i) \Leftrightarrow (ii)$ This equivalence can be derived using Theorem 1.

$(ii) \Leftrightarrow (iii)$ It is well known (cf. Theorem 2.5.3 of [8]) that the Z–matrix $A[1,\ldots,n-1]$ is a nonsingular M–matrix if and only if $A[1,\ldots,n-1]$ has all its eigenvalues with positive real part. Furthermore, since $\det A = \prod_{i=1}^{n} \lambda_i$, with $\lambda_1,\ldots,\lambda_n$ the eigenvalues of A, we conclude that $\det A < 0$ if and only if A has an odd number of negative eigenvalues.

$(ii) \Rightarrow (iv)$ By Theorem 1 the submatrix $A[1,\ldots,n-1]$ is a nonsingular M–matrix if and only if $\det A[\alpha] > 0$ for all α such that $n \notin \alpha$, that is, $A[1,\ldots,n-1]$ is a P–matrix. So, if (ii) holds, then A is an almost P–matrix.

$(iv) \Rightarrow (ii)$ Since we have seen in the previous paragraph that $A[1,\ldots,n-1]$ is a nonsingular M–matrix if and only if $A[1,\ldots,n-1]$ is a P–matrix, it remains to prove that, if a nonsingular Z–matrix A is not a P–matrix and $A[1,\ldots,n-1]$ is a nonsingular P–matrix, then $\det A < 0$. Otherwise, $\det A > 0$ and, by Theorem 1, A is a nonsingular M–matrix because it has all its leading principal minors positive. Then, again by Theorem 1, all principal minors of A positive, contradicting the fact that A is not a P–matrix.

$(ii) \Leftrightarrow (v)$ It can be derived applying Theorem 1 to the submatrix $A[1,\ldots,n-1]$.

$(v) \Leftrightarrow (vi)$ Take into account that Gaussian elimination can be performed without row exchanges if and only if all $n-1$ first leading principal minors are nonzero and that, in this case, the pivots are given by $d_1 = a_{11}$ and $d_i = \det A[1,\ldots,i]/\det A[1,\ldots,i-1]$ for $i = 2,\ldots,n$. □

Observe that condition (vi) provides a test of $O(n^3)$ elementary operations to check if an $n \times n$ Z–matrix is an almost nonsingular M–matrix.

§3. Some subclasses of almost nonsingular M–matrices

This section considers two classes of almost nonsingular M–matrices and includes an application of the second class.

Let us recall that a symmetric nonsingular M–matrix is called a *Stieltjes matrix* (see [4]). Recall that a real symmetric matrix is a positive definite matrix if and only if all its leading principal minors are positive. Then a Z–matrix is Stieltjes if and only if it is positive definite. If A is a nonsingular symmetric Z–matrix such that $A[1,\ldots,n-1]$ is a Stieltjes matrix and A is not a Stieltjes matrix, then we say that A is an *almost Stieltjes matrix*. Clearly a matrix is almost Stieltjes if and only if it is a symmetric almost nonsingular M–matrix. The following result characterizes almost Stieltjes matrices.

Theorem 5. *Let A be an $n\times n$ symmetric Z–matrix. The following statements are equivalent:*

(i) *A is an almost Stieltjes matrix.*

(ii) $\det A < 0$ *and* $A[1,\ldots,n-1]$ *is an Stieltjes matrix.*

(iii) *$A = LDL^T$, where L is a lower triangular M–matrix with unit diagonal and $D = diag(d_i)_{i=1}^{n}$ with $d_i > 0$ for all $i < n$ and $d_n < 0$.*

(iv) *A has $n-1$ positive eigenvalues and 1 negative eigenvalue and* $A[1,\ldots,n-1]$ *has positive eigenvalues.*

(v) *A is an almost P–matrix.*

Proof. $(i) \Leftrightarrow (ii)$ This equivalence is consequence of Theorem 4.

$(ii) \Rightarrow (iii)$ Observe that a matrix A satisfying (ii) is, by Theorem 4, an almost nonsingular M–matrix. Then we know that the LDU decomposition of A satisfies Theorem 2 (ii). Since A is symmetric, $U = L^T$ and the LDU factorization of A is $A = LDL^T$, and (iii) follows.

$(iii) \Rightarrow (iv)$ If $A = LDL^T$, then the matrices A and D are congruent and so, by the Sylvester's law of inertia (cf. Theorem 4.5.8 of [7], [10]) they have the same number of positive (resp. negative) eigenvalues. In addition, (2.1) holds and, again by Sylvester's law of inertia all eigenvalues of $A[1, \dots, n-1]$ are positive.

$(iv) \Rightarrow (ii)$ By (iv), $\det A < 0$. Since all eigenvalues of $A[1, \dots, n-1]$ are positive, this submatrix is positive definite and so an Stieltjes matrix.

$(ii) \Leftrightarrow (v)$ It is a consequence of the equivalence of (ii) and (iv) of Theorem 4. □

We say that a matrix A is a *generalized almost Stieltjes matrix* if there exists a permutation matrix P such that PAP^T is an almost Stieltjes matrix. Analogously to Theorem 3, we can derive from Theorem 5 the following result.

Theorem 6. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a symmetric Z–matrix. The following conditions are equivalent:*

(i) *A is a generalized almost Stieltjes matrix.*

(ii) *There exists a permutation matrix P such that $PAP^T = LDL^T$, where L is a lower triangular nonsingular M–matrix with unit diagonal and $D = diag(d_i)_{i=1}^n$, with $d_i > 0$ for all $i < n$ and $d_n < 0$.*

We now recall some notations related to Gaussian elimination. Given a square matrix $A = (a_{ij})_{1 \le i,j \le n}$ such that Gaussian elimination can be performed without row exchanges, Gaussian elimination consists of a succession of at most $n-1$ major steps resulting in a sequence of matrices:

$$A = A^{(1)} \longrightarrow A^{(2)} \longrightarrow \cdots \longrightarrow A^{(n)} = U,$$

where $A^{(t)} = (a_{ij}^{(t)})_{1 \le i,j \le n}$ has zeros below its main diagonal in the first $t-1$ columns and U is upper triangular with the pivots on its main diagonal. In order to obtain $A^{(t+1)}$ from $A^{(t)}$ we produce zeros in column t below the *pivot* $a_{tt}^{(t)}$ $(\neq 0)$ by subtracting multiples of row t from the rows beneath it.

A matrix $A = (a_{ij})_{1 \le i,j \le n}$ is strictly diagonally dominant (*SDD*) if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for each $i = 1, \dots, n$. If a nonsingular matrix A is not SDD but $A[1, \dots, n-1]$ is SDD then we say that A is an *almost SDD matrix*. Finally, a matrix A is a *generalized almost SDD matrix* if there exists a permutation matrix P such that PAP^T is an almost SDD matrix.

By the Levy-Desplanques Theorem (cf. Corollary 5.6.17 of [7]) an SDD matrix is nonsingular. If, in addition, all diagonal entries are positive then it is well–known that A has positive determinant. In fact, using the Gershgorin circles, it can be deduced that all eigenvalues have positive real part, and so the determinant is positive. The following result shows a sufficient condition for negative determinant, which corresponds to a class of generalized almost SDD matrices.

Theorem 7. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a Z–matrix and let $A(k)$ be the principal submatrix of A removing row and column k. If $a_{tt} < 0$, $a_{kk} > 0$ for all $k \neq t$ and $A(k)$ is an SDD matrix, then* $\det A < 0$.

Proof. Let us consider the permutation matrix P that permutes the rows t and n of A and let $B = (b_{ij})_{1\le i,j\le n} := PAP^T$. Recall that, by Theorem 2.5.3 of [8], an SDD Z–matrix is a nonsingular M–matrix. Thus, we have that the matrix $B[1,\dots,n-1]$ is a nonsingular M–matrix. Since the (n,n) entry (a principal submatrix) of B is negative by hypothesis, we conclude from Theorem 1 that B is not an M–matrix.

Since the matrix $B[1,\dots,n-1]$ is a Z–matrix with positive leading principal minors, then it is easy to check that we can perform the Gaussian elimination without row exchanges until obtaining an upper triangular matrix U with the first $n-1$ diagonal entries $d_1,\dots,d_{n-1}$ positive and nonpositive off–diagonal entries (in fact, $B[1,\dots,n-1]$ is a nonsingular M–matrix by Theorem 1 and it is well–known that Gaussian elimination preserves this property). Let us see that at each step of Gaussian elimination, the (n,n) entry decreases and let us denote by $d_n := b_{nn}^{(n)}$ the (n,n) entry of $B^{(n)} = U$. It is sufficient to prove it at the first step $B^{(1)} \longrightarrow B^{(2)}$ (analogously, it can be proved at any step). The (n,n) entry of $B = B^{(1)}$ is updated as

$$b_{nn}^{(2)} = b_{nn}^{(1)} - \frac{b_{n1}^{(1)}}{b_{11}^{(1)}} b_{1n}^{(1)},$$

where $b_{n1}, b_{1n} \le 0$ and $b_{11} > 0$. Thus $b_{nn}^{(2)} \le b_{nn}$ (< 0) and, continuing Gaussian elimination, we can prove that $d_n = b_{nn}^{(n)} \le b_{nn}^{(n-1)} \le \dots \le b_{nn}^{(2)} \le b_{nn} < 0$. Finally, we have that $\det A = d_1 \cdots d_n < 0$. □

Observe that checking if a given $n \times n$ Z–matrix satisfies the hypothesis of Theorem 7 requires $O(n^2)$ elementary operations.

An illustrative example of the criterion of negative determinant of Theorem 7 is given by the following matrix

$$A = \begin{pmatrix} 8 & -2 & 0 & -1 \\ -2 & 7 & -1 & -8 \\ -2 & -5 & 8 & -7 \\ 0 & -7 & -1 & -2 \end{pmatrix}.$$

It has negative determinant by Theorem 7. A direct computation shows that $\det A = -5586$.

Acknowledgements

Research Partially Supported by the Spanish Research Grant MTM2012-31544, by Gobierno de Aragón and Fondo Social Europeo.

References

[1] Alfa, A. S., Xue, J., and Ye, Q. Entrywise perturbation theory for diagonally dominant M-matrices with applications. *Numer. Math. 90* (1999), 401–414.

[2] Alfa, A. S., Xue, J., and Ye, Q. Accurate computation of the smallest eigenvalue of a diagonally dominant M-matrix. *Math. Comp. 71* (2001), 217–236.

[3] Barreras, A., and Peña, J. M. Accurate and efficient LDU decomposition of diagonally dominant M-matrices. *Electronic J. Linear Algebra 24* (2012), 153–167.

[4] BERMAN, A., AND PLEMMONS, R. J. *Nonnegative matrices in the mathematical sciences*. Classics in Applied Mathematics, 9, SIAM, Philadelphia, 1994.

[5] DEMMEL, J., GU, M., EISENSTAT, S., SLAPNICAR, I., VESELIC, K., AND DRMAC, K. Computing the singular value decomposition with high relative accuracy. *Linear Algebra Appl. 299* (1999), 21–80.

[6] DEMMEL, J., AND KOEV, P. Accurate SVDs of weakly diagonally dominant M-matrices. *Numer. Math. 98* (2004), 99–104.

[7] HORN, R. A., AND JOHNSON, C. R. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.

[8] HORN, R. A., AND JOHNSON, C. R. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1994.

[9] PEÑA, J. M. LDU decompositions with L and U well conditioned. *Electronic Transactions of Numerical Analysis 18* (2004), 198–208.

[10] SYLVESTER, J. J. A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares. *Phil. Mag. 4* (1852), 142.

Álvaro Barreras and Juan Manuel Peña
Departamento de Matemática Aplicada/IUMA
Universidad de Zaragoza
50009 Zaragoza, Spain
`albarrer@unizar.es` and `jmpena@unizar.es`