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BIFURCATIONS IN THE ATTITUDE DYNAMICS OF AN AXIAL-SYMMETRIC SATELLITE IN A CENTRAL FIELD

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Abstract. The global parametric evolution of the normalized Hamiltonian, in the problem of the attitude dynamics of an axial-symmetric satellite in a gravity field is obtained. The phase portrait is represented in a Mercator map. Pitch-fork bifurcations and degeneracies (a dense set of equilibria) are found.

Keywords: Attitude dynamics, rigid body motion, stability, bifurcations .

AMS classification: 70F15, 70F50, 34D20.

§1. Introduction

Attitude dynamics of a rigid body has been studied by the most famous scientists (see e.g. the classical Leimanis' textbook [13] which contains a detailed list of problems and authors involved with rigid body motion).

The rigid body in free rotation is the simplest case of the three integrable problems in the rigid-body dynamics, but besides still is complex enough since in its integration, elliptic functions appear. Because its relative simplicity, it has been used as benchmark of new set of variables, or numerical methods integrating rigid body rotations among other applications. It is also used in Astrodynamics, since a good knowledge of the attitude of spinning satellite is essential in designing a mission, and also in astronomical problems like the rotation of planets, satellites, asteroids, etc.

In order to maintain alive a spatial mission, it is essential the stability analysis of the rotational motion [14, 15, 16, 7, 6], since the choice of a set of wrong initial conditions could put the satellite in tumbling rotation, leading to a chaotic regime and ruining the mission.

Although there are substantial progresses in the understanding of small perturbations of integrable systems by means of KAM theory and Chaos indicators (e.g. [4]), in this work, we proceed in the following way. By means of averaging or normalization obtained by a Lie-Deprit transformation, the original non-integrable Hamiltonian is replaced by an integrable approximation. This allows us to know how the phase flow varies according to the initial conditions and what are the stability regions, through a qualitative analysis of the equations of the motion [11, 12].

In the present paper, we assume that the Earth possesses a spherically symmetry mass distribution, that the satellite is small compared to its distance from the mass center of the primary, and although for most of the paper we consider the satellite has three different moments of inertia, for the computations (Section 3) we will assume that the spacecraft has axial symmetry. Finally, let us mention that an extension of this communication has been published elsewhere [2].

§2. Hamiltonian of the Problem

Let us consider the problem of the rotational–translational motion of a rigid body (the satellite) attracted by the Newtonian gravity field of the Earth (a point mass). To describe the problem we will employ the polar–nodal variables $(r, \theta, \nu, R, \Theta, N)$.

For the attitude motion, we use the classical Serret–Andoyer variables (ℓ, g, h, L, G, H). For a detailed explanation on these variables and their canonicity, the reader is addressed to [8, 9]. They are well known in the context of the rigid body rotation, but since we relate then with the orbital ones, we briefly describe the chosen set of reference frames in order to obtain a simpler formulation. We consider the following frames centered at the center of mass of the satellite.

- A fixed space frame $O\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3$.
- The principal body frame of inertia $O\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$ moving rigidly with the satellite.
- The system $O\ell_0\mathbf{m}_0\mathbf{n}$, where $\mathbf{n}$ is the unit vector in the direction of the rotational angular momentum $\mathbf{G}$ and ℓ_0 is the ascending node of the plane perpendicular to the vector $\mathbf{G}$ and the space plane $O\mathbf{s}_1\mathbf{s}_2$.

The Serret–Andoyer variables (ℓ, g, h, L, G, H) are defined as usual: The longitude h of the ascending node ℓ_0 reckoned from the axis $\mathbf{s}_1$; the longitude g of the node ℓ_1 of the equatorial body plane $O\mathbf{b}_1\mathbf{b}_2$ on the plane perpendicular to the angular momentum reckoned from the axis ℓ_0; the longitude ℓ of the body axis $\mathbf{b}_1$ reckoned from the node ℓ_1.

The conjugate moments are: $G = ||\mathbf{G}||$, the norm of the rotation angular momentum vector; H, the projection of this vector on the space axis $\mathbf{s}_3$, $(H = G\cos\epsilon)$; L, the projection of $\mathbf{G}$ on the body axis b_3, $(L = G\cos\sigma)$. For details, see e.g. [8, 9, 10].

The Hamiltonian function of the problem, considering only terms up to the third power of the inverse of the distance, and after some simplifications (see [3]) may be written as

$$\mathcal{H} = \mathcal{H}_K + \mathcal{H}_E + \mathcal{H}_C \tag{2.1}$$

where

$$\begin{aligned}
\mathcal{H}_K &= \frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{\mu}{r}, \\
\mathcal{H}_E &= \left(\frac{\sin^2\ell}{2I_1} + \frac{\cos^2\ell}{2I_2}\right)(G^2 - L^2) + \frac{1}{2I_3}L^2, \\
\mathcal{H}_C &= -\frac{\mu}{2r^3}\Big[(I_1 - I_2)(1 - 3\alpha^2) + (I_3 - I_2)(1 - 3\gamma^2)\Big],
\end{aligned}$$

being I_1, I_2, I_3 the principal moments of inertia of the satellite (which at this stage we assume $I_1 \leq I_2 \leq I_3$), and (α, β, γ) the unit vector in the radial direction in the body frame.

The two nodes, the orbital ν and the rotational one h, are on the same plane $(O\mathbf{s}_1\mathbf{s}_2)$, and in the development of the potential function, they appear only as the combination $(h - \nu)$. Besides, in the problem of motion given by the Hamiltonian (2.1), the total angular momentum vector **c**, is an integral of the motion, and this allows us to choose the space frame in such a way that the axis $\mathbf{s}_3$ coincides with this vector. (For more details, see [1, 5]).

With this election, and $\mathbf{\Theta}$ and $\mathbf{G}$ standing for the orbital and rotational angular moments respectively, in the space frame we have

$$\mathbf{\Theta} + \mathbf{G} = \mathbf{c} = (0, 0, c) \quad with \quad c = constant, \tag{2.2}$$

and the nodes satisfy the relation $h - \nu = \pi$. The angles h and ν being cyclic, the problem has four degrees of freedom in the variables $(r, \theta, \ell, g, R, \Theta, L, G)$.

Let us assume that the Hamiltonian (2.1) may be decomposed as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1,$$

where the zero order term is formed by the Keplerian and Eulerian parts $\mathcal{H}_K$ and $\mathcal{H}_E$, whereas the coupled terms ($\mathcal{H}_C$) are of order one.

We are interested in the study of the behavior of the phase flow for different values of the initial conditions. In order to do that, we shall make a simplification by averaging the Hamiltonian (2.1) over the Keplerian mean anomaly.

After performing automatically this average and dropping those terms which do not contain the variables, we obtain:

$$\begin{aligned} \mathcal{H}^* = & \left(\frac{\sin^2 \ell}{2I_1} + \frac{\cos^2 \ell}{2I_2}\right)(G^2 - L^2) + \frac{1}{2I_3} L^2 \\ & - \frac{\mu n}{2\Theta q}\left[(I_1 - I_2)(1 - 3\alpha^{*2}) + (I_3 - I_2)(1 - 3\gamma^{*2})\right] \end{aligned} \tag{2.3}$$

where n is the orbital mean motion, q is the semilatus rectus, and

$$\begin{aligned} \alpha^{*2} &= A_0 + A_1 \cos g + A_2 \cos 2g + A_3 \cos 2\ell + A_4 \cos(g + 2\ell) \\ &\quad + A_5 \cos(g - 2\ell) + A_6 \cos(2g + 2\ell) + A_7 \cos(2g - 2\ell), \\ \gamma^{*2} &= G_0 + G_1 \cos g + G_2 \cos 2g, \end{aligned} \tag{2.4}$$

with the coefficients A_i and G_i expressions depending on the moments Θ, N, L, G, H (for mor details see [1]).

§3. Qualitative analysis of the phase flow in the axisymmetric case.

For the sake of simplicity, we shall consider the case in which the satellite has axial symmetry of inertia, i.e., $I_1 = I_2$. Under this hypothesis, the variable ℓ becomes cyclic, and the Hamiltonian is reduced to:

$$\begin{aligned} \mathcal{H}^* &= \frac{1}{2I_1}(G^2 - L^2) + \frac{1}{2I_3} L^2 \\ &\quad - \frac{\mu n}{2\Theta a(1 - e^2)}(I_3 - I_1)\left[1 - 3\left(G_0 + G_1 \cos g + G_2 \cos 2g\right)\right] \end{aligned}$$

which exclusively depends on g and G and where a is the semimajor axis and e the orbital eccentricity of the orbit. The averaged problem is of one-degree of freedom and, therefore, it is integrable.

The zero-order of the above Hamiltonian corresponds to the rotation of an axis-symmetric rigid body, and it is well known that its motion consists of rotations about the axis of symmetry, thus, we will take into consideration only the perturbation.

The behavior of the flow corresponding to the perturbed part is the same as that given by

$$\mathcal{K} = G_0 + G_1 \cos g + G_2 \cos 2g, \tag{3.1}$$

where G_0, G_1, G_2 are

$$\begin{aligned} G_0 &= \tfrac{1}{2}\sin^2\epsilon\cos^2\sigma + \tfrac{1}{4}(1+\cos^2\epsilon)\sin^2\sigma, \\ G_1 &= \sin\epsilon\cos\epsilon\sin\sigma\cos\sigma, \\ G_2 &= \tfrac{1}{4}(\cos^2\epsilon\sin^2\sigma - \sin^2\epsilon), \end{aligned}$$

which result after dropping summand constant terms and time-scaling the problem in order to get rid of the constant factor.

Having in mind that the inclination angles σ and ϵ are given in terms of the moments L, G, H by

$$\cos\sigma = \frac{L}{G}, \qquad \cos\epsilon = \frac{H}{G}, \qquad \text{and} \qquad \cos\sigma = (L/H)\cos\epsilon,$$

we choose $p = L/H$ as a parameter to make a qualitative analysis of the phase flow.

Making the change

$$\eta = \cos\epsilon, \quad \text{where} \quad |\eta| \le \widehat{\eta} = \min\{1, 1/p\},$$

the Hamiltonian becomes simpler

$$\begin{aligned} \mathcal{K} = {} & \tfrac{1}{2}p^2\eta^2(1-\eta^2) + \tfrac{1}{4}(1+\eta^2)(1-p^2\eta^2) \\ & + p\eta^2\sqrt{1-p^2\eta^2}\sqrt{1-\eta^2}\cos g + \tfrac{1}{4}(2\eta^2 - 1 - p^2\eta^4)\cos 2g, \end{aligned} \tag{3.2}$$

with the coordinates $(g, \eta) \in [0, 2\pi) \times [-\widehat{\eta}, \widehat{\eta}]$.

Let us note that the Hamiltonian (3.2) enjoys some symmetries. Indeed, it is symmetric respect to the axis $\eta = 0$ and the line $g = \pi$. So, we may restrict our analysis to the region $(g, \eta) \in [0, \pi] \times [0, \widehat{\eta}]$. Besides, since $\mathcal{K}(g, \eta; p) = \mathcal{K}(g \pm \pi, \eta; -p)$, we reduce our analysis to $p \ge 0$.

3.1. Analytical study

Equilibria are the solutions of the system

$$\frac{dg}{dt} = \frac{\partial\mathcal{K}}{\partial G} = \frac{\partial\mathcal{K}}{\partial\eta}\frac{\partial\eta}{\partial G} = \frac{-H}{G^2}\left(\frac{\partial G_0}{\partial\eta} + \frac{\partial G_1}{\partial\eta}\cos g + \frac{\partial G_2}{\partial\eta}\cos 2g\right) = 0, \tag{3.3}$$

$$\frac{dG}{dt} = -\frac{\partial\mathcal{K}}{\partial g} = (G_1 + 4G_2\cos g)\sin g = 0. \tag{3.4}$$

After some algebra, we have

$$\begin{aligned}
&\frac{\partial G_0}{\partial \eta} + \frac{\partial G_1}{\partial \eta}\cos g + \frac{\partial G_2}{\partial \eta}\cos 2g = -\frac{1}{2}\eta(6\eta^2 p^2 - 1 - p^2) \\
&\qquad + \frac{\eta p(2 - 3(1+p^2)\eta^2 + 4p^2\eta^4)}{\sqrt{1-\eta^2}\sqrt{1-p^2\eta^2}}\cos g + \eta(1 - p^2\eta^2)\cos 2g = 0. \\
&(G_1 + 4G_2\cos g)\sin g = \\
&\qquad \left(p\eta^2\sqrt{1-\eta^2}\sqrt{1-p^2\eta^2} + (-1 + 2\eta^2 - p^2\eta^4)\cos g\right)\sin g = 0.
\end{aligned} \tag{3.5}$$

This system is complex enough to solve it, but some particular solutions can be obtained.

• If $\eta = 0$ the first equation of the system 3.5 is always null. Besides, the second equation reduces to $\sin 2g = 0$, which is satisfied for $g = k\pi/2$ ($k \in \mathbb{Z}$). Since our analysis is reduced to the interval $g \in [0, \pi]$, the equilibria are the points $E_0 = (0, 0)$, $E_1 = (0, \pi/2)$, and $E_2 = (0, \pi)$.

• If $\eta \neq 0$, we have two cases:

a) The second equation of 3.5 is satisfied for $g = k\pi$ ($k \in \mathbb{Z}$). Then, the first equation of system 3.5 becomes

$$\frac{\partial G_0}{\partial \eta} \pm \frac{\partial G_1}{\partial \eta} + \frac{\partial G_2}{\partial \eta} = 0,$$

depending on either $g = 0$ (+) or $g = \pi$ (−), or explicitly,

$$\eta\left(\frac{1}{2}(3 + p^2 - 8p^2\eta^2) \pm \frac{p(2 - 3(1+p^2)\eta^2 + 4p^2\eta^4)}{\sqrt{1-\eta^2}\sqrt{1-p^2\eta^2}}\right) = 0.$$

If $p \neq 1$, isolated equilibrium points can appear if $\eta \in [-\widehat{\eta}, \widehat{\eta}]$.

When $p = 1$ and $g = \pi$, the expression $\partial(G_0 - G_1 + G_2)/\partial\eta \equiv 0$ for whatever value of $\eta \in [-1, 1]$. Thus, for $p = 1$, the segment $D_\pi = (\pi, \eta)$ is made of equilibria; hence, we are in presence of a degeneracy.

b) Assume now that $\sin g \neq 0$, then the second equation 3.4 holds when

$$G_1 + 4G_2\cos g = 0 \quad \text{or} \quad \cos g = -G_1/(4G_2). \tag{3.6}$$

b.1) Case $p = 1$.

Then the above equation (from the second equation of Eq. 3.5) becomes $\eta^2 - (1 - \eta^2)\cos g = 0$, that is,

$$\eta = \sqrt{\frac{\cos g}{1 + \cos g}}. \tag{3.7}$$

By replacing this value of η into the first equation of the system 3.5, we see that it is always satisfied for whatever value of g, hence, we meet another degeneracy (D_0) for $p = 1$. Every point on the curve 3.7 for $p = 1$ is an equilibrium of the system.

b.2) Case $p \neq 1$;

If we replace 3.6 into equation 3.3, the possible equilibria will result by solving the equation

$$\frac{\partial G_0}{\partial \eta} - \frac{\partial G_1}{\partial \eta}\frac{G_1}{4G_2} + \frac{\partial G_2}{\partial \eta}\left(\frac{G_1^2}{8G_2^2} - 1\right) = 0,$$

which in terms of η, is

$$\frac{\eta(1-p^2)(-1+4\eta^2-4(1-p^2)\eta^4-4p^2\eta^6+p^4\eta^8)}{2(1-2\eta^2+p^2\eta^4)^2} = 0,$$

or equivalently,

$$-1+4\eta^2-4(1-p^2)\eta^4-4p^2\eta^6+p^4\eta^8 = 0, \tag{3.8}$$

since the cases $\eta = 0$ and $p = 1$ have been already studied.

For each value of the parameter p we have to solve the above equation 3.8, to obtain η and with it we get the angle g from Eq. 3.6.

The above equation can be put as

$$-1+4\xi-4(1-p^2)\xi^2-4p^2\xi^3+p^4\xi^4 = 0, \quad with \quad \xi = \eta^2, \quad \eta \leq \widehat{\eta}.$$

easier to solve numerically. Note that, by means of Descartes rule of signs, it follows that this equation always has a negative root, hence there is at least one root and at most three positive roots. There are two positive roots if the resultant of the polynomial vanishes, which happens for $p_d = 1.2033783313$; there are three positive roots for $p < p_d$ and there is only one positive root when $p > p_d$. But the root ξ_0 must be in the interval $\xi_0 \in [0, \widehat{\eta}^2]$, with $\widehat{\eta} = \min\{1/p, 1\}$, and this condition is satisfied for $p \leq 2.64575$. That is to say, this equilibrium (let us call it S_2) only exist for $0 < p \leq 2.64575$. In addition, for $0 < p < 1$ another equilibrium exists (we name it S_1).

3.2. Stability and bifurcations

The linear stability of the equilibria is determined by the characteristic equation of the differential system.

After a time scaling $t \longmapsto \tau$ given by the relation

$$d\tau = \frac{\partial \eta}{\partial G}\, dt,$$

the variational equations of the motion in an equilibrium point are

$$\frac{d\,\delta g}{d\,\tau} = A\,\delta\eta + B\,\delta g, \qquad \frac{d\,\delta\eta}{d\,\tau} = -C\,\delta\eta - D\,\delta g,$$

or in matricial form

$$\frac{d}{d\,\tau}\begin{pmatrix}\delta g\\ \delta\eta\end{pmatrix} = \begin{pmatrix} B & A\\ -D & -C\end{pmatrix}\begin{pmatrix}\delta g\\ \delta\eta\end{pmatrix}$$

where

$$A = \frac{\partial^2 \mathcal{K}}{\partial \eta^2},\; B = \frac{\partial^2 \mathcal{K}}{\partial \eta \partial g} = C,\; D = \frac{\partial^2 \mathcal{K}}{\partial g^2},$$

evaluated at the equilibrium.

The associated characteristic polynomial equation is $\lambda^2 + (AD - B^2) = 0$ and, therefore, the equilibrium is unstable when $AD - B^2 < 0$.

Let us consider some of the equilibria just obtained.

For the point $(\pi, 0)$, the characteristic polynomial becomes

$$\lambda^2 + (AD - B^2) = \lambda^2 + \tfrac{1}{2}(3 - 4p + p^2),$$

thus, this point will be unstable when $3 - 4p + p^2 < 0$, that is, when $p \in (1, 3)$. Bifurcations (that is, change of stability) occur in the extrema of this interval, that is to say, for $p = 1$ and for $p = 3$. In fact, these two values are the bifurcation values of p as portrayed in the phase space graphics (see Fig. 1).

For the point $(\pi/2, 0)$, the characteristic polynomial becomes

$$\lambda^2 + (AD - B^2) = \lambda^2 + \frac{1}{2}(1 - p^2),$$

hence, this point is stable for $p \in [0, 1)$ and unstable for $p > 1$. The bifurcation occurs at $p = 1$.

For the origin, the discriminant $\Delta = (AD - B^2) = (3 + 4p + p^2)/2$ is positive for $p > 0$, which means that is always stable in the studied interval and, therefore, the origin does not bifurcate.

The sign of the discriminant may be used to determine the stability of whatever equilibrium, and this is how we found the stability of the different cases shown in Table 1.

	(g, η)	Existence	Stable
E_0	(0,0)	always	$p \geq 0$
E_1	$(\pi/2, 0)$	always	$(0, 1)$
E_2	$(\pi, 0)$	always	$(0, 1) \cup (3, \infty)$
M_1	$(0, \eta_0)$	$p \neq 1$	$(0, 1)$
M_2	(π, η_π)	$1 < p \leq 3$	(1,3)
D_π	(π, η)	$p = 1$	Degeneracy
D_0	(g_η, η)	$p = 1$	Degeneracy
S_1	(g_1, η_1)	$(0, 1)$	Never
S_2	(g_2, η_2)	$0 < p \leq 2.64575$	$(1, 2.64575)$

Table 1: Equilibria, their stability and degeneracies of the phase flow

§4. Graphical analysis.

To show how the phase flow evolves with the parameter p, we shall use portraits of the phase flow for different values of the parameter. The equilibria corresponding to these chosen values of the parameter are given in Table 2.

In the plots of Fig. 1, we present the phase portrait on the Mercator chart (g, η) for several values of the parameter p. In fact, and due to the symmetries previously mentioned, the phase portrait is a cylinder, and because of the symmetry about the line $g = \pi$, we only need to consider the interval $0 \leq g \leq \pi$.

Let us start with $p = 3.1$; the phase flow is represented in upper left corner of Figure 1.

On the horizontal axis, $\eta = 0$, we can see three equilibria E_0, E_1 and E_2 at $g = 0, \pi/2, \pi$ respectively. Point $E_1 = (\pi/2, 0)$ is unstable, whereas the other two $E_0 = (0, 0)$ and $E_2 = (\pi, 0)$ are stable. Besides, the point $M_1 = (0, 0.290433)$ is an unstable equilibrium.

For $p = 3$ there is a pitch-fork bifurcation at the point $(\pi, 0)$. Indeed, as we can observe in Fig. 1, for $p \geq 3$ it is stable; however, for $p < 3$, this point is unstable and there is one new stable point, M_2, (Fig. 1, first row to the right) on the vertical axis $\eta = \pi$, inside the homoclinic orbit that springs out from the point $(\pi, 0)$.

Let us analyze the phase flow after the pitchfork bifurcation, that is, for $p < 3$; the flow is similar until we reach the value $p = 2.64575$ in which a new equilibrium point appears, S_2. For smaller values of p and $p > 1$, the flow is qualitatively the same; we can see it in Fig. 1 b), c) and d) for $p = 1.9$, $p = 1.5$ and $p = 1.1$ respectively. We see six equilibria, three stable E_0, M_2 and S_2, and other three unstable, namely, the points E_1, E_2 and M_1. Again, the numerical values of these points for those parameters are in Table 2.

p	M_1	M_2	S_1	S_2
3.1	(0, 0.290433)	—	—	—
1.9	(0, 0.439634)	$(\pi, 0.326986)$	—	(1.32596, 0.426428)
1.5	(0, 0.530549)	$(\pi, 0.486062)$	—	(1.26394, 0.461031)
1.1	(0, 0.664703)	$(\pi, 0.789791)$	—	(1.20922, 0.504821)
0.9	(0, 0.752870)	—	(1.18297, 0.531349)	(2.49967, 0.956788)
0.7	(0, 0.848647)	—	(1.15665, 0.561792)	(2.34531, 0.892025)

Table 2: Equilibria for several values of the parameter p.

At $p = 1$ there are two degeneracies as we already proved in the above section. The first one (D_π) is the line $g = \pi$, which is a dense set of equilibria; indeed, as $p \to 1$, the homoclinic orbit emanating at the point $(\pi, 0)$, narrows until it collapses into the straight line $g = \pi$, which is made of equilibria (Fig. 1, e). As soon as $p < 1$, the degeneracy breaks out and only two equilibrium points remain, the stable $E_2 = (\pi, 0)$ and another one unstable, M_2.

Simultaneously, another degeneracy happens (D_0); the heteroclinic orbit connecting unstable points on the axis $g = 0$ coalesces as $p \to 1$ with the homoclinic orbit emanating from $E_1 = (\pi/2, 0)$ and the resulting curve is made of equilibria. As soon as $p < 1$, this degeneracy breaks out, and there are three new equilibria, one unstable S_1 and two stable E_1 and M_1 (Fig. 1 f) and g)). In sum, for $0 \leq p < 1$, in the phase rectangle $(g, \eta) \in [0, \pi] \times [0, 1/2)$, there are six equilibria (Fig. 1, last row), four stable (E_0, E_1, E_2 and M_1) and two unstable (S_1 and S_2).

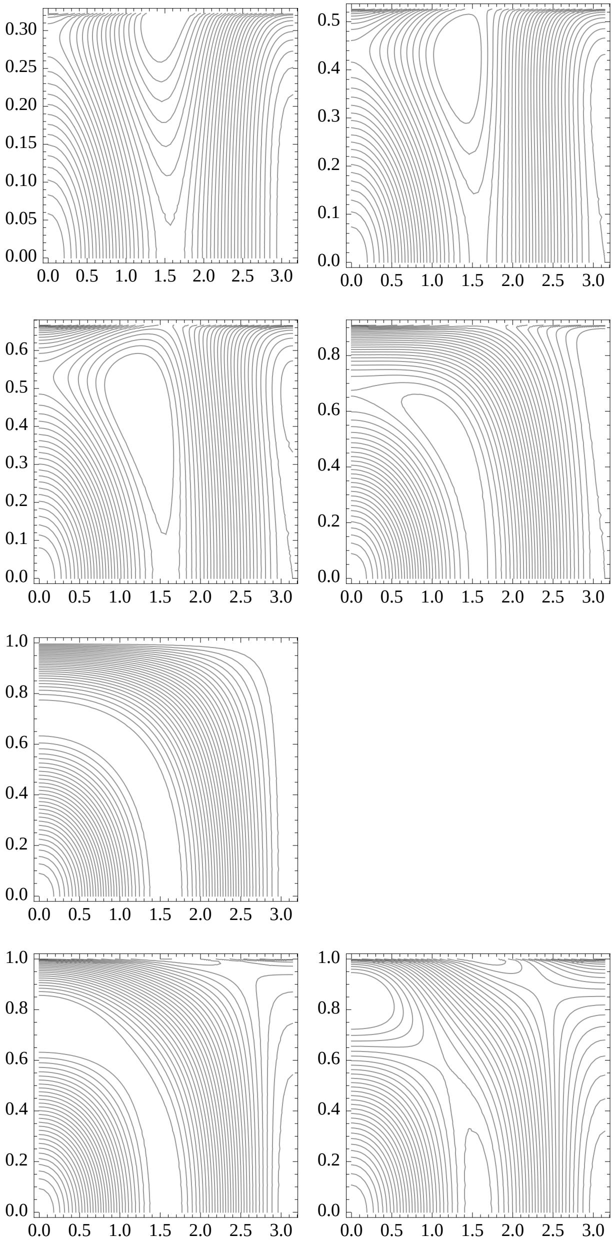

Figure 1: Evolution of the fhase flow through one pitch-fork bifurcation at $p = 3.0$ and two simultaneous degeneracies at $p = 1$ (third row). The values of the parameter p from left to right and from top to bottom are a)3.1, b)1.9, c)1.5, d)1.1, e)1.0, f)0.9 and g)0.7 .

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