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A BIVARIATE HOMOGENEOUS STOCHASTIC VASICEK DIFFUSION PROCESS

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Abstract. In this paper, we extend Vasicek's homogeneous univariate stochastic diffusion process(cf. [18] and [12]) to the bivariate case, in the same way as has been described for the bivariate Gompertz and Gamma processes (cf. [8] and [11]). We first obtain the analytical expression of the process to resolve the stochastic differential equation that characterises the process in question, its probabilistic distribution and the conditional and unconditional trend functions of the process. Then, using matrix differential calculus, we study the problem of estimating the parameters present in the drift vector and in the diffusion matrix, by maximum likelihood with discrete sampling.

Keywords: Homogeneous Vasicek model, Trend functions, Likelihood estimation in diffusion process, Matrix differential calculus..

AMS classification: 60J60, 62M05.

§1. Introduction

Stochastic diffusion processes (SDP), which are described by stochastic differential equations (SDE) or Kolmogorov partial differential equations, arise naturally in a variety of applications. In addition to the traditional uses in biology and economics, these processes have become indispensable in fields such as stochastic economy and finance, news technology, cell growth, radiotherapy, chemotherapy, energy consumption and the emissions of CO_2 and greenhouse gases.

The statistical inference in SDP is a crucial but non-trivial task, especially when the process is observed continuously. In most cases (with continuous sampling), the estimation of parameters in these models, in particular, are based on methods for approximating the likelihood function, the likelihood estimators or on alternatives such as non parametric methods. An extensive review of this theory for the general case can be consulted in Prakasa-Rao [15], Bibby et al. [1] and Fan [2], and in particular diffusion cases, in Ferrante et al. [3] for the Gompertz process, in Giovanis et al. [5] in the case of the logistic model and in Gutiérrez et al. [9] for the inverse CIR model, among others. Furthermore, many studies have been conducted in the context of inference by discrete sampling, including those related to particular processes such as the homogeneous and non homogeneous lognormal processes, by Tintner et al. [17], the Gompertz diffusion process, in both the homogeneous and non homogeneous cases, by Nafidi [14], Gutiérrez et al. [10] and Rupsys et al. [16] and the Vasicek non homogeneous process, by Gutiérrez et al. [12].

Various useful extensions of the ordinary univariate SDP have also been proposed, such as non homogeneous processes and multivariate cases. However, very few univariate SDP have been extended to the multivariate case; in this respect, we can cite the non homogeneous lognormal process with exogenous factors proposed by Gutiérrez et al. [6], or the extension to the lognormal process with a vector of exogenous factors, proposed by Gutiérrez et al. [7]. In addition, we have the homogeneous Gompertz process studied by Gutiérrez et al. [8], the multivariate Gompertz process with delay considered by Fran et al. [4] and finally the Gamma diffusion process defined by Gutiérrez et al. [11].

The one-dimensional homogeneous SDP defined by Vasicek [18] has been extremely influential in the field of stochastic economics and finance. Various researchers have applied it to obtain interest rate models and short rate models, taking into account diverse extensions of the process, such as extending their linear drift to the non-linear function, for both the homogeneous and the non homogeneous cases, the non homogeneous version with a multifactorial drift function. More details of these extensions can be found in Gutiérrez et al. [12], who also studied a new non homogeneous extension of this process, introducing exogenous factors into the drift process, in a linear way. This approach was applied to data on the emissions of CO_2, with respect to the growth in GDP and the total electricity consumption in Morocco as exogenous factors.

The present study extends the one dimensional Vasicek diffusion process to the multivariate case; specifically a bivariate case. The rest of this paper is organised as follows: in the second section, the proposed model is defined in terms of the Ito stochastic differential equation. We then determine the analytical expression of the process to resolve the SDE that characterises the process in question. We obtain the probability transition density function (ptdf) of the proposed processes, together with the probabilistic distribution, the marginal moments, the conditional and non conditional trend functions and the covariance function. The third section presents an integrated study of the estimation of the drift and diffusion parameters. The first of these are estimated by maximum likelihood methods, using discrete sampling and matrix differential calculus.

§2. The bivariate Vasicek stochastic diffusion process

2.1. The model and its analytical expression

The bivariate Vasicek diffusion process can be defined by the bidimensional stochastic process $\{x(t) = (x_1(t), x_2(t))'; t \in [t_0, T]; t_0 \geq 0\}$ that satisfies the following Ito SDE:

$$dx(t) = (a - \beta x(t))dt + B^{1/2}dw(t) \quad ; \quad x(t_0) = x_{t_0} \tag{1}$$

where $\{w(t); t \in [t_0, T]\}$ is a 2-dimensional standard Wiener process, x_{t_0} is a fixed vector belonging to $(0, \infty)^2$, $a = (a_1, a_2)'$ and $B = \left(b_{ij}\right)_{i,j}$ is a 2×2 symmetric non negative definite matrix. The parameters a_1, a_2, β and $b_{i,j}$ for $1 \leq i, j \leq 2$ are real and must be estimated.

By applying the Ito formula to the transform

$$y(t) = e^{\beta t} x(t) = e^{\beta t} (x_1(t), x_2(t))'$$

we obtain the following SDE:

$$dy(t) = e^{\beta t}\left(a - \frac{b}{2}\right)dt + e^{\beta t}B^{1/2}dw(t) \quad , \quad y(t_0) = e^{\beta t_0}x_{t_0}$$

where we denote by $b = diag(B) = (b_{11}, b_{22})'$
Then, by integrating, we have

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} e^{\beta\theta}d\theta(a - \frac{b}{2}) + B^{1/2}\int_{t_0}^{t} e^{\beta\theta}dw(\theta) \\ &= y(t_0) + \frac{e^{\beta t} - e^{\beta t_0}}{\beta}\left(a - \frac{b}{2}\right) + B^{1/2}\int_{t_0}^{t} e^{\beta\theta}dw(\theta) \end{aligned}$$

from which we can deduce the solution of our original SDE Eq.(1) namely

$$x(t) = e^{-\beta(t-t_0)}x_{t_0} + \frac{1 - e^{-\beta(t-t_0)}}{\beta}(a - \frac{b}{2}) + B^{1/2}\int_{t_0}^{t} e^{-\beta(t-\theta)}dw(\theta)$$

2.2. The pdf and the moments of the model

The random vector $\int_s^t e^{\beta\theta}dw(\theta)$ has a bivariate normal distribution $\mathcal{N}_2\left(0, \int_s^t e^{2\beta\theta}d\theta I_2\right)$ (where I_2 denotes the 2×2 identity matrix). It can then be deduced that $x(t) \mid x(s) = x_s \sim \mathcal{N}_2\left(\mu(s,t,x_s), \Sigma(s,t)\right)$ which has a bivariate normal distribution with

$$\mu(s,t,x) = e^{-\beta(t-s)}x + \frac{1 - e^{-\beta(t-s)}}{\beta}(a - b/2)$$

$$\Sigma(s,t) = \frac{1}{2\beta}(1 - e^{-2\beta(t-s)})B$$

The transition density function of this process is then expressed as $f(y,t \mid x,s)$ (for $y = (y_1, y_2)'$ and $x = (x_1, x_2)'$) has the form

$$f(y,t \mid x,s) = (2\pi)^{-1} \mid \Sigma(s,t) \mid^{-\frac{1}{2}} \exp\left\{-\frac{Q}{2}\right\} \qquad (2)$$

where $\mid B \mid$ is the determinant of the matrix B, and Q is a quadratic form given by

$$Q = (y - \mu(s,t,x))' \left(\Sigma(s,t)\right)^{-1} (y - \mu(s,t,x))$$

The marginal conditional trend functions of the processes, for $i = 1, 2$, are:

$$\mathbb{E}\left(x_i(t) \mid x_i(s) = x_{s,i}\right) = e^{-\beta(t-s)}x_{s,i} + \frac{1 - e^{-\beta(t-s)}}{\beta}\left(a_i - \frac{b_{ii}}{2}\right) \qquad (3)$$

By assuming the initial condition $\mathrm{P}(x(t) = x_{t_0}) = 1$, and using Eq.(3), the marginal (unconditional) trend functions for $i = 1, 2$ are:

$$\mathbb{E}\left(x_i(t)\right) = e^{-\beta(t-s)}x_{t_0,i} + \frac{1-e^{-\beta(t-t_0)}}{\beta}\left(a_i - \frac{b_{ii}}{2}\right)$$

the marginal variance function of the process, for $i = 1, 2$ is:

$$\mathrm{Var}\left(x_i(t)\right) = \frac{1-e^{-2\beta(t-t_0)}}{2\beta}b_{ii}$$

and the covariance function at the same moment is

$$\mathrm{Cov}\left(x_1(t), x_2(t)\right) = \frac{1}{2\beta}(1-e^{-2\beta(t-t_0)})b_{12}$$

§3. Likelihood parameters estimation

The parameters β, a and B are estimated by the maximum likelihood method based on discrete sampling. To construct the likelihood function associated with the process, we consider the discrete sampling of the process $\{x(t_1) = x_{t_1}; x(t_2) = x_{t_2}; \ldots, x(t_n) = x_{t_n}\}$ at times $t_1, t_2; \ldots; t_n$ (with $t_i - t_{i-1} = 1$ for $i = 2, \ldots, n$), in which each $x(t_\alpha)$ represents the bidimensional vector $x(t_\alpha) = (x_1(t_\alpha), x_2(t_\alpha))'$. For the sake of simplicity, we denote this as $x_{t_\alpha} = x_\alpha$, with the initial condition $\mathrm{P}[x(t_1) = x_1] = 1$. By applying the Markov property and making use of Eq.(2), the likelihood function associated with the sample considered, of size $(n-1)$, is given by:

$$\mathbb{L}(x_1, \ldots, x_n; \beta; \gamma; B) = (2\pi)^{-(n-1)}\nu_\beta{}^{-(n-1)} \mid B \mid^{-\frac{(n-1)}{2}} \prod_{\alpha=2}^{n} \exp\left\{-\frac{1}{2}\left[x_\alpha - e^{-\beta}x_{\alpha-1} - (1-e^{-\beta})\frac{\gamma}{\beta}\right]' \right.$$
$$\left. \nu_\beta^{-2}B^{-1}\left[x_\alpha - e^{-\beta}x_{\alpha-1}) - (1-e^{-\beta})\frac{\gamma}{\beta}\right]\right\}$$

where $\nu_\beta^2 = \dfrac{1-e^{-2\beta}}{2\beta}$ and $\gamma = a - \dfrac{b}{2}$

With the following change of variable $v_1 = \log(x_1)$ and for $\alpha = 2, \ldots, n$:

$$\mathbf{v}_\alpha(\beta) \equiv \mathbf{v}_\alpha = \nu_\beta{}^{-1}\left(x_\alpha - e^{-\beta}x_{\alpha-1}\right)$$

If we denote by $\xi_\beta = \nu_\beta^{-1}\dfrac{(1-e^{-\beta})}{\beta}$, then in terms of $\mathbf{v}_\alpha$, the likelihood function gives us:

$$\mathbb{L}_{\mathbf{v}_1,\ldots,\mathbf{v}_n}(\beta; \gamma; B) \equiv \mathbb{L} = (2\pi)^{-(n-1)}\nu_\beta^{-(n-1)} \mid B \mid^{-\frac{(n-1)}{2}}$$
$$\exp\left\{-\frac{1}{2}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)\right\}$$

The differential of the log-likelihood function is

$$
\begin{aligned}
d\log(\mathbb{L}) = {} & -(n-1)\nu_\beta^{-1}\frac{\partial \nu_\beta}{\partial \beta}d\beta - \frac{n-1}{2}\mathrm{tr}\left(B^{-1}dB\right) \\
& -\frac{1}{2}\sum_{\alpha=2}^{n}\left[-\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}(dB)B^{-1}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)\right] \\
& +\xi_\beta\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}(d\gamma) \\
& -\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\left(\frac{\partial \mathbf{v}_\alpha}{\partial \beta} - \frac{\partial \xi_\beta}{\partial \beta}\gamma\right)d\beta
\end{aligned}
$$

By applying trace properties, the last differential can be written as:

$$
\begin{aligned}
d\log(\mathbb{L}) = {} & \frac{1}{2}\mathrm{tr}\left\{\sum_{\alpha=2}^{n}\left[B^{-1}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' - I_2\right]B^{-1}dB\right\} \\
& +\xi_\beta\mathrm{tr}\left\{\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}(d\gamma)\right\} \\
& +\left\{\nu_\beta^{-1}\frac{\partial \nu_\beta}{\partial \beta}\left[-(n-1) + \sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\mathbf{v}_\alpha\right]\right. \\
& \left. -\nu_\beta^{-1}e^{-\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}x_{\alpha-1} + \frac{\partial \xi_\beta}{\partial \beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\gamma\right\}d\beta
\end{aligned}
$$

From the relations $tr(AB) = Vec'(A')Vec(B)$ and $dVec(A) = Vec(dA)$, where Vec is the matrix vectorisation (see Magnus et al. [13]), we obtain

$$
\begin{aligned}
d\log(\mathbb{L}) = {} & Vec'\left\{\sum_{\alpha=2}^{n}\left[B^{-1}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' - I_2\right]B^{-1}\right\}dVec(B) \\
& +\xi_\beta Vec'\left\{B^{-1}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)\right\}dVec(\gamma) \\
& +\left\{\nu_\beta^{-1}\frac{\partial \nu_\beta}{\partial \beta}\left[-(n-1) + \sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\mathbf{v}_\alpha\right]\right. \\
& \left. -\nu_\beta^{-1}e^{-\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}x_{\alpha-1} + \frac{\partial \xi_\beta}{\partial \beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right)' B^{-1}\gamma\right\}d\beta
\end{aligned}
$$

and by the likelihood principle, the estimators of B and γ are obtained from the following equations:

$$
B^{-1}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha - \xi_\beta\gamma\right) = 0
$$

$$\sum_{\alpha=2}^{n}\left[B^{-1}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'-I_2\right]B^{-1}=0$$

with respect to the estimator of β, we have

$$\nu_\beta^{-1}\frac{\partial\nu_\beta}{\partial\beta}\left[-(n-1)+\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\mathbf{v}_\alpha\right]-\nu_\beta^{-1}e^{-\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}x_{\alpha-1}+\frac{\partial\xi_\beta}{\partial\beta}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_\beta\gamma\right)'B^{-1}\gamma=0$$

Then after various operations (not shown), the likelihood estimators $\hat{\gamma}$, $\hat{B}$ and $\hat{\beta}$ for the parameters are found to be:

$$\hat{\gamma}=\frac{1}{(n-1)\xi_{\hat{\beta}}}\sum_{\alpha=2}^{n}\mathbf{v}_\alpha$$

$$\hat{B}=\frac{1}{n-1}\sum_{\alpha=2}^{n}\left(\mathbf{v}_\alpha-\xi_{\hat{\beta}}\hat{\gamma}\right)\left(\mathbf{v}_\alpha-\xi_{\hat{\beta}}\hat{\gamma}\right)'$$

$$\hat{\beta}=\log\left\{\frac{\left(\sum_{\alpha=2}^{n}x'_{\alpha-1}\right)\left(\sum_{\alpha=2}^{n}x_{\alpha-1}\right)-(n-1)\sum_{\alpha=2}^{n}x'_{\alpha-1}x_{\alpha-1}}{\left(\sum_{\alpha=2}^{n}x'_{\alpha-1}\right)\left(\sum_{\alpha=2}^{n}x_{\alpha}\right)-(n-1)\sum_{\alpha=2}^{n}x'_{\alpha-1}x_{\alpha}}\right\}$$

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