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Nonconforming finite element approximation of an elliptic interface problem with NXFEM

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Abstract. We study a Nitsche's eXtended Finite Element Method for an elliptic interface problem, approximated by nonconforming finite elements on triangular meshes. In order to bound the consistency error, we add some stabilisation on the sides cut by the interface. We show stability of the discrete formulation and optimal a priori error estimates, which are robust with respect to the geometry and to the diffusion parameters. Finally, we present a numerical example illustrating the theoretical results.

Keywords: interface, NXFEM, elliptic problem, nonconforming elements, stabilisation.

AMS classification: 65N30, 65N12, 35J20.

§1. Introduction

Several finite element methods have been proposed in the last years in order to take into account discontinuities which are not necessarily aligned with the mesh. One of them is NXFEM (Nitsche's eXtended Finite Element Method), introduced by A. Hansbo and P. Hansbo in [6] and based on the use of Nitsche's method to treat the transmission conditions on the interface. It uses standard finite element spaces, which are enriched on the cells cut by the interface such that the degrees of freedom are doubled on these cells. Some recent developments of NXFEM concern its robustness with respect to the geometry (see for instance [2] or [1]), its a posteriori analysis or its application to different model problems, such as fluid flow or fluid-structure interaction.

At our knowledge, NXFEM has been used so far with continuous finite elements. Our goal is to extend it to the case of nonconforming elements on triangular meshes. Besides their small stencil, nonconforming elements present other advantages such as inf-sup stability for Stokes equations or robustness with respect to small parameters and locking.

For P^1-continuous elements, the degrees of freedom are associated to the nodes, which belong to only one of the sub-domains delimited by the interface. Meanwhile, the degrees of freedom of the Crouzeix-Raviart P^1-nonconforming elements [4] are associated to the edges, so those associated to the cut edges belong to two sub-domains simultaneously. Consequently, this leads to some difficulties in the estimation of the consistency error.

In this paper, we focus on an elliptic equation with discontinuous coefficients and we discretise it by P^1-nonconforming elements on triangular meshes which are not aligned with the interface. To tackle the previously mentioned difficulty, we propose to add, in addition to the usual stabilisation on the interface specific to NXFEM, some stabilisation on the cut edges which compensate the nonconformity error. The weights used for the definition of the means as well as the stabilisation parameters in each sub-domain are chosen in order to get robustness of the method with respect to the geometry and to the diffusion parameters. We show the well-posedness of the discrete problem as well as *a priori* error estimates.

Another approach, based on the modification of the nonconforming finite elements such that the new degrees of freedom are associated to only one sub-domain, is studied in [5]. No additional term is then necessary to ensure the stability, but the difficulty now lies in estimating the interpolation error in a robust way.

The outline of the paper is as follows. Section 2 contains the notation and the presentation of the original NXFEM method for conforming finite elements, whereas in Section 3 we introduce the extension to the nonconforming case. The aim of Section 4 is threefold: we are concerned with the stability of the formulation, the consistency error and the interpolation error. Numerical tests are presented in Section 5, confirming the optimal convergence rate predicted by the theoretical results.

The extension to a nonconforming finite element approximation of the Stokes equations in the presence of an interface can be found in [5].

§2. Original NXFEM with conforming finite elements

Let Ω a bounded domain of $\mathbb{R}^2$, with polygonal boundary $\partial\Omega$ and an internal smooth boundary Γ dividing Ω into open sets Ω^{in} and Ω^{ex}. We consider the same model problem as in [6]:

$$\begin{cases} -\mathrm{div}\,(\kappa\nabla u) &= f & \text{in } \Omega^{in}\cup\Omega^{ex}, \\ u &= 0 & \text{on } \partial\Omega, \\ [u] &= 0 & \text{on } \Gamma, \\ [\kappa\nabla u\cdot n] &= g & \text{on } \Gamma, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ and n is the unit normal to the interface Γ oriented from Ω^{in} towards Ω^{ex}. For the sake of simplicity, we suppose that κ is a piecewise constant coefficient, discontinuous across Γ and taking the values κ^{in} and κ^{ex} in the sub-domains Ω^{in} and Ω^{ex}.

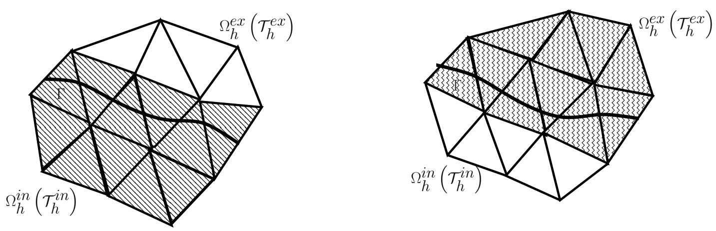


Figure 1: Underlying meshes for the domains Ω^{in} and Ω^{ex}.

Let $(\mathcal{T}_h)_h$ be a regular family of triangulations of Ω, each $\mathcal{T}_h$ consisting of triangles. We denote by $\mathcal{T}_h^{\Gamma} = \{T \in \mathcal{T}_h;\ T\cap\Gamma \neq \emptyset\}$ the set of cut cells and we also introduce $\mathcal{T}_h^{i} = \left\{T \in \mathcal{T}_h;\ T\cap\Omega^i \neq \emptyset\right\}$ for $i = in,\ ex$, see Figure 1. $\mathcal{S}_h$ denotes the set of sides of the triangulation $\mathcal{T}_h$, while $\mathcal{S}_h^{i,cut}$ denotes the set of cut sides contained in Ω^i and $\mathcal{S}_h^{nc}$ the set of uncut sides of $\mathcal{T}_h$. It is useful to introduce $\Gamma_T = T \cap \Gamma$ for any $T \in \mathcal{T}_h^\Gamma$. For a given side $S \in \mathcal{S}_h$, we fix once for all a unit normal n_S; if S is situated on the boundary $\partial\Omega$, then n_S coincides with the outward normal n_Ω.

For $x \in \Gamma$ and v a piecewise smooth function, we set

$$v^{in}(x) = \lim_{\varepsilon \longrightarrow 0} v(x - \varepsilon n), \qquad v^{ex}(x) = \lim_{\varepsilon \longrightarrow 0} v(x + \varepsilon n)$$

and we define its jump across Γ as well as the following weighted means by:

$$[v] = v^{in} - v^{ex}, \qquad \{v\} = \alpha^{ex} v^{ex} + \alpha^{in} v^{in}, \qquad \{v\}_* = \alpha^{in} v^{ex} + \alpha^{ex} v^{in},$$

where the weights satisfy $\alpha^{in} + \alpha^{ex} = 1$ and $0 < \alpha^{in}, \alpha^{ex} < 1$.

We next recall the NXFEM formulation of (1), introduced in [6] for the case of a piecewise linear, continuous finite approximation on a mesh of Ω which is not aligned with the interface Γ. The idea is to use standard finite element spaces but to double the degrees of freedom on all the cut cells (see Figure 1), and to treat the transmission conditions on Γ weakly, by means of Nitsche's method [7].

Let the finite dimensional spaces:

$$W_h^i = \{v \in H^1(\Omega_h^i);\ v|_T \in P^1,\ \forall T \in \mathcal{T}_h^i,\ v|_{\partial\Omega} = 0\}, \quad i = in, ex$$

and let the product space $W_h = W_h^{in} \times W_h^{ex}$. Let us introduce:

$$a_h(u_h, v_h) = \int_{\Omega^{in} \cup \Omega^{ex}} \kappa \nabla u_h \cdot \nabla v_h - \int_\Gamma \{\kappa \nabla u_h \cdot n\}[v_h] - \int_\Gamma \{\kappa \nabla v_h \cdot n\}[u_h] + \gamma \sum_{T \in \mathcal{T}_h^\Gamma} \int_{\Gamma_T} \gamma_T [u_h][v_h],$$

$$l_h(v_h) = \int_\Omega f v_h + \int_\Gamma g \{v_h\}_*,$$

where $\gamma > 0$ is a stabilisation parameter and where the coefficients $(\alpha^{in}, \alpha^{ex}, \gamma_T)$ are defined as follows:

$$\alpha^{in} = \frac{\kappa^{ex}|T^{in}|}{\kappa^{ex}|T^{in}| + \kappa^{in}|T^{ex}|}, \qquad \alpha^{ex} = \frac{\kappa^{in}|T^{ex}|}{\kappa^{ex}|T^{in}| + \kappa^{in}|T^{ex}|}, \qquad \gamma_T = \frac{\kappa^{in}\kappa^{ex}|\Gamma_T|}{\kappa^{in}|T^{ex}| + \kappa^{ex}|T^{in}|}.$$

We use here above the expressions proposed in [2, 1], which ensure robustness of the method with respect to both the mesh-interface geometry and to the diffusion parameters, under standard assumptions on the interface.

Then the discrete problem reads:

$$u_h \in W_h, \qquad a_h(u_h, v_h) = l_h(v_h), \qquad \forall v_h \in W_h. \tag{2}$$

It is well-known cf. [6] that (2) is consistent and stable for γ sufficiently large, with respect to the norm:

$$\|v\|_h^2 = \sum_{i=in,ex} \|\kappa^{1/2} \nabla v\|_{0,\Omega^i}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} |\Gamma_T| \|\{\kappa \nabla v \cdot n\}\|_{0,\Gamma_T}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \gamma_T \|[v]\|_{0,\Gamma_T}^2. \tag{3}$$

One can improve the robustness with respect to the previous norm by replacing $|\Gamma_T|$ in front of $\{\kappa\nabla v \cdot n\}$ by the weight $\dfrac{|\Gamma_T|}{\gamma_T h_T}$, see [5]. This classically yields the *a priori* error estimate:

$$||u - u_h||_h \leq C \inf_{v_h \in W_h} ||u - v_h||_h.$$

In the case of a smooth solution $u = (u^{in}, u^{ex}) \in H^2(\Omega^{in}) \times H^2(\Omega^{ex})$, one retrieves an optimal convergence rate $O(h)$ by using the global interpolation operator $\mathcal{L}_h = \left(\mathcal{L}_h^{in}, \mathcal{L}_h^{ex}\right)$: $H^2(\Omega^{in}) \times H^2(\Omega^{ex}) \longrightarrow W_h^{in} \times W_h^{ex}$, defined as follows:

$$v|_{\Omega^i} \longrightarrow \mathcal{E}^i v|_\Omega \longrightarrow (\mathcal{L}_h^* \circ \mathcal{E}^i)v|_\Omega \longrightarrow (\mathcal{L}_h^* \circ \mathcal{E}^i)v|_{\Omega_h^i} =: \mathcal{L}_h^i v, \qquad i = in,\ ex. \tag{4}$$

Here above, $\mathcal{E}^i$ denotes a continuous extension operator from $H^2(\Omega^i)$ to $H^2(\Omega)$ and $\mathcal{L}_h^*$ is the Lagrange interpolation operator associated to the mesh $\mathcal{T}_h$ of Ω. See [6] for more details.

§3. Extension to nonconforming finite elements

In what follows, we are interested in the discretisation of the interface problem (1) by Crouzeix-Raviart nonconforming elements [4]. We recall that their degrees of freedom are given by $\frac{1}{|S|}\int_S v$ for any side $S \in \mathcal{S}_h$, such that the finite element space associated to the triangulation $\mathcal{T}_h^i$ (of sides $\mathcal{S}_h^i$) is defined by

$$V_h^i = \left\{\varphi \in L^2(\Omega_h^i);\ \varphi|_T \in P^1,\ \forall T \in \mathcal{T}_h^i,\ \int_S [\varphi]_S = 0,\ \forall S \in \mathcal{S}_h^i\right\}, \qquad i = in,\ ex,$$

where $[\cdot]_S$ denotes the jump across S; on a boundary side, the jump is equal to the trace.

As in the conforming case, we introduce the product space $V_h = V_h^{in} \times V_h^{ex}$ and we define a global interpolation operator $\mathcal{I}_h = (\mathcal{I}_h^{in}, \mathcal{I}_h^{ex})$ following the previous approach (4). This ensures $\int_S(\mathcal{I}_h^i v - v) = 0$ on any side $S \in \mathcal{S}_h^i$, but this property clearly does not hold on the segments of cut sides:

$$\int_S \mathcal{I}_h^i v \neq \int_S v, \qquad \forall S \in \mathcal{S}_h^{i,cut} \qquad i = in,\ ex.$$

Consequently, we cannot estimate the consistency error on the cut sides:

$$\sum_{i=in,\,ex}\ \sum_{S \in \mathcal{S}_h^{i,cut}} \int_S \kappa^i \nabla u \cdot n_S [v_h]_S.$$

To overcome this difficulty, we propose in this paper to add some stabilisation terms in order to balance the previous consistency error.

Remark 1. Another solution, which consists in modifying the basis functions on the cut cells such that the corresponding interpolation operator satisfies $\int_S(\mathcal{I}_h^i v - v) = 0$ on any cut side $S \in \mathcal{S}_h^{i,cut}$, was proposed in [5].

For this purpose, we first define the jump and weighted mean of a piecewise smooth, discontinuous function v on a segment $S \in \mathcal{S}_h^{i,cut}$ of a cut side. For an interior segment S, we denote by T^l and T^r the two neighbour cells such that the normal n_S is oriented from T^l towards T^r. We denote then $v|_{T^l} = v^l$, $v|_{T^r} = v^r$ and we set

$$[v]_S = v^l - v^r, \qquad \{v\}_S = \beta^l v^l + \beta^r v^r,$$

where the weights β^l, β^r satisfy $\beta^l + \beta^r = 1$ and $0 < \beta^l, \beta^r < 1$. The same definition of the jump is employed on a side completely contained in Ω^i. If S is situated on $\partial\Omega$, then the jump and the mean coincide with the trace. We refer to Figure 2 for additional notation.

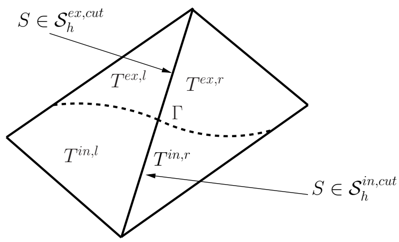


Figure 2: Notations related to adjacent cut cells.

We next introduce the bilinear forms:

$$A_h(u_h, v_h) = -\sum_{i=in,ex} \sum_{S \in \mathcal{S}_h^{i,cut}} \int_S \{\kappa \nabla u_h \cdot n_S\}_S [v_h]_S + \{\kappa \nabla v_h \cdot n_S\}_S [u_h]_S,$$

$$J_h^i(u_h, v_h) = \sum_{S \in \mathcal{S}_h^{i,cut}} \int_S \delta_S^i \, [u_h]_S \, [v_h]_S \, ,$$

$$J_h(u_h, v_h) = \sum_{i=in,ex} \delta^i \, J_h^i(u_h, v_h),$$

where $\delta^i > 0$ $(i = in, ex)$ are stabilisation parameters. These terms are similar to those of Nitsche's method, except that they are written on the segments of cut sides and that the coefficients δ_S^i as well as the weights β^l, β^r are different. We take:

$$\beta^l = \frac{|T^{i,l}|}{|T^{i,l}| + |T^{i,r}|}, \qquad \beta^r = \frac{|T^{i,r}|}{|T^{i,l}| + |T^{i,r}|}, \qquad \delta_S^i = \frac{\kappa^i |S|}{\left|T^{l,i}\right| + \left|T^{r,i}\right|} \tag{5}$$

and we consider the discrete problem:

$$u_h \in V_h, \quad (a_h + A_h + J_h)(u_h, v_h) = l_h(v_h), \quad \forall v_h \in V_h. \tag{6}$$

§4. Stability and error estimates

In what follows, we are interested in the well-posedness of (6) and in error bounds.

4.1. Stability

The discrete space V_h is equipped with the norm

$$[[v]]^2 = \|v\|_h^2 + J_h^{in}(v,v) + J_h^{ex}(v,v)\,.$$

Lemma 1. *For all* $S \in \mathcal{S}_h^{i,cut}$ *(*$i = in,\ ex$*), one has:*

$$\|\{\kappa\nabla u_h \cdot n_s\}_S\|_{0,S}^2 \le \delta_S^i \left(\|\kappa^{1/2}\nabla u_h\|_{0,T^{i,l}}^2 + \|\kappa^{1/2}\nabla u_h\|_{0,T^{i,r}}^2\right).$$

Proof. Using that $\kappa\nabla u_h$ is piecewise constant, $0 < \beta^l, \beta^r < 1$ and $\beta^l + \beta^r = 1$, it follows by Cauchy-Schwarz inequality that

$$\int_S \{\kappa\nabla u_h \cdot n_s\}_S^2 \le \beta^l \int_S |\kappa\nabla u_h|_{T^{i,l}}^2 + \beta^r \int_S |\kappa\nabla u_h|_{T^{i,r}}^2 = \frac{\beta^l|S|}{|T^{i,l}|}\int_{T^{i,l}} |\kappa\nabla u_h|^2 + \frac{\beta^r|S|}{|T^{i,r}|}\int_{T^{i,r}} |\kappa\nabla u_h|^2.$$

Thanks to the expressions (5), we get that

$$\int_S \{\kappa\nabla u_h \cdot n_s\}_S^2 \le \frac{|S|}{|T^{i,l}| + |T^{i,r}|}\int_{T^{i,l}\cup T^{i,r}} |\kappa\nabla u_h|^2 = \delta_S^i \int_{T^{i,l}\cup T^{i,r}} |\kappa^{1/2}\nabla u_h|^2,$$

which proves the lemma. □

This result immediately implies the uniform continuity of the bilinear form $A_h(\cdot,\cdot)$. For γ, δ^{in} and δ^{ex} sufficiently large, we deduce in a standard way (by using Young's inequality) the uniform stability of the approximation method:

$$(a_h + A_h + J_h)(v_h, v_h) \ge C[[v_h]]^2, \quad \forall v_h \in V_h.$$

Therefore, problem (6) is well-posed.

In order to estimate the *a priori* error, we use Strang's lemma:

$$[[u - u_h]] \le C\left(\inf_{v_h\in V_h}[[u - v_h]] + \sup_{v_h\in V_h}\frac{(a_h + A_h + J_h)(u - u_h, v_h)}{[[v_h]]}\right).$$

The first term represents the interpolation error and the second one, the consistency error due to the nonconformity of the approximation.

In what follows, we bound each term under a standard regularity assumption.

4.2. Consistency error

Lemma 2. *Assume* $(u^{in}, u^{ex}) \in H^2(\Omega^{in}) \times H^2(\Omega^{ex})$*. Then one has that*

$$(a_h + A_h + J_h)(u, v_h) - l_h(v_h) = \sum_{S\in\mathcal{S}_h^{nc}} \int_S \kappa\nabla u \cdot n_S\,[v_h]_S, \qquad \forall v_h \in V_h.$$

Proof. Note first that for $(u^{in}, u^{ex}) \in H^2(\Omega^{in}) \times H^2(\Omega^{ex})$, the trace of ∇u^i on a given side is well-defined and moreover, $[u]_S = 0$ for any segment $S \in \mathcal{S}_h^{i,cut}$ (as well as on any side completely contained in Ω^i). This implies $J_h(u, v_h) = 0$ for any $v_h \in V_h$.

An integration by part yields, for all $v_h \in V_h$:

$$a_h(u, v_h) - l_h(v_h) = \sum_{S \in \mathcal{S}_h^{nc}} \int_S \kappa \nabla u \cdot n_S \, [v_h]_S + \sum_{i=in,\,ex} \sum_{S \in \mathcal{S}_h^{i,cut}} \int_S \kappa \nabla u \cdot n_S \, [v_h]_S \,.$$

Noting that $[\kappa \nabla u \cdot n_S]_S = 0$ for any $S \in \mathcal{S}_h^{i,cut}$ and therefore,

$$A_h(u, v_h) = -\sum_{i=in,ex} \sum_{S \in \mathcal{S}_h^{i,cut}} \int_S \{\kappa \nabla u \cdot n_S\}_S \, [v_h]_S = -\sum_{i=in,ex} \sum_{S \in \mathcal{S}_h^{i,cut}} \int_S \kappa \nabla u \cdot n_S \, [v_h]_S \,,$$

we immediately get the announced result. □

Then the consistency error estimate is immediate.

Lemma 3. *Assume* $(u^{in}, u^{ex}) \in H^2(\Omega^{in}) \times H^2(\Omega^{ex})$*. There exists a constant* $C > 0$ *independent of the discretisation and of the interface, such that*

$$\sup_{v_h \in V_h} \frac{(a_h + A_h + J_h)(u - u_h, v_h)}{[[v_h]]} \leq C\, h\, |\kappa^{1/2} u|_{2,\Omega^{ex} \cup \Omega^{in}} \,.$$

Proof. We use the discrete problem (6) and the previous Lemma to write, for any $v_h \in V_h$:

$$(a_h + A_h + J_h)\,(u - u_h, v_h) = \sum_{S \in \mathcal{S}_h^{nc}} \int_S \kappa \nabla u \cdot n_S \, [v_h]_S = \sum_{S \in \mathcal{S}_h^{nc}} \int_S \kappa \nabla (u - \mathcal{I}_h u) \cdot n_S \, [v_h - C_S]_S \,,$$

where $\mathcal{I}_h$ is the usual Crouzeix-Raviart interpolation operator; C_S denotes any constant on S. The last equality holds true because $[C_S]_S = 0$, $\int_S [v_h]_S = 0$ on any non-cut side S and $\kappa \nabla \mathcal{I}_h u \cdot n_S$ is constant on S. The rest of the proof is classical, cf. for instance [3]. □

4.3. Interpolation error

We assume here, for the sake of simplicity, that Γ_T is a segment for all $T \in \mathcal{T}_h^\Gamma$ and that:

$$\exists C < 1 \;\text{ s.t. }\; |S| \leq C |\tilde{S}|, \qquad \forall S \in \mathcal{S}_h^{i,cut} \quad (i = in,\, ex), \tag{7}$$

where $\tilde{S}$ is the whole side containing the cut segment S; the general case can be found in [5].

Lemma 4. *Assume* $(u^{in}, u^{ex}) \in H^2(\Omega^{in}) \times H^2(\Omega^{ex})$ *and (7). There exists a constant* $C > 0$ *independent of the discretisation such that*

$$[[u - \mathcal{I}_h u]] \leq C\, h\, |\kappa^{1/2} u|_{2,\Omega^{ex} \cup \Omega^{in}} \,.$$

Proof. The norm $||u - \mathcal{I}_h u||_h$ can be bounded similarly to the case of conforming finite elements, see [6] for the proof. We only focus here on the additional terms

$$J_h^i(u - \mathcal{I}_h^i u, u - \mathcal{I}_h^i u) = \sum_{S \in \mathcal{S}_h^{i,cut}} \delta_S^i \, ||[u - \mathcal{I}_h^i u]_S||_{0,S}^2 \,, \qquad i = in,\, ex. \tag{8}$$

Let $S \in \mathcal{S}_h^{i,cut}$ and let $\tilde{S} \in \mathcal{S}_h^i$ the whole side containing the segment S and T^l, T^r the neighbour cells. The trace inequality gives

$$\begin{aligned}
\|[u - \mathcal{I}_h^i u]_S\|_{0,S} \leq & \|[\mathcal{E}^i u - \mathcal{I}_h \circ \mathcal{E}^i u]_S\|_{0,\tilde{S}} \\
\leq & c|\tilde{S}|^{1/2}\Big(\frac{1}{h_{T^r}}\|\mathcal{E}^i u - \mathcal{I}_h \circ \mathcal{E}^i u\|_{0,T^r} + |\mathcal{E}^i u - \mathcal{I}_h \circ \mathcal{E}^i u|_{1,T^r} \\
& + \frac{1}{h_{T^l}}\|\mathcal{E}^i u - \mathcal{I}_h \circ \mathcal{E}^i u\|_{0,T^l} + |\mathcal{E}^i u - \mathcal{I}_h \circ \mathcal{E}^i u|_{1,T^l}\Big).
\end{aligned}$$

The optimal approximation properties of the Crouzeix-Raviart interpolation operator $\mathcal{I}_h$ yield:

$$\|[u - \mathcal{I}_h^i u]_S\|_{0,S} \leq c|\tilde{S}|^{1/2}\, h\, |\mathcal{E}^i u|_{2,T^r \cup T^l}.$$

Using next the expression of δ_S^i, we deduce that

$$(\delta_S^i)^{1/2}\|[u - \mathcal{I}_h^i u]_S\|_{0,S} \leq c \max\{\sqrt{\frac{|S||\tilde{S}|}{|T^{i,l}|}}, \sqrt{\frac{|S||\tilde{S}|}{|T^{i,r}|}}\}\, h\, (\kappa^i)^{1/2}|\mathcal{E}^i u|_{2,T^l \cup T^r}.$$

The hypothesis (7) implies that $|S|h_{T^l} \leq c|T^{i,l}|$, $|S|h_{T^r} \leq c|T^{i,r}$ and since $|\tilde{S}| \leq \max\{h_{T^l}, h_{T^r}\}$, it follows that

$$(\delta_S^i)^{1/2}\|[u - \mathcal{I}_h^i u]_S\|_{0,S} \leq C \max\{h_{T^l}, h_{T^r}\}(\kappa^i)^{1/2}|\mathcal{E}^i u|_{2,T^l \cup T^r}.$$

Finally, thanks to the continuity of the extension operator $\mathcal{E}^i : H^2(\Omega^i) \longrightarrow H^2(\Omega)$, we get:

$$J_h^i(u - \mathcal{I}_h^i u, u - \mathcal{I}_h^i u)^{1/2} \leq C\, h\, |\kappa^{1/2} u|_{2,\Omega^i}, \qquad i = in,\ ex$$

and we obtain the desired estimate by summing upon i. □

§5. Numerical experiments

To illustrate the previous results, we consider the same test-case as in [6], where it was discretised by NXFEM with conforming finite elements. The exact solution on $\Omega =]-1, 1[\times]-1, 1[$ is given by:

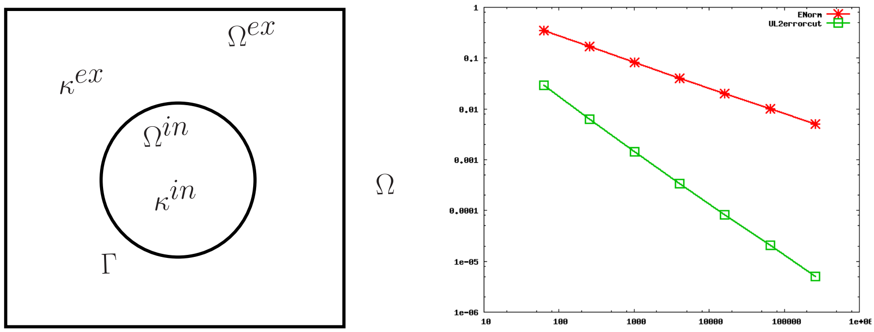
$$u(x, y) = \frac{r^2}{\kappa^{in}} \quad \text{if } r \leq r_0, \qquad u(x, y) = \frac{r^2}{\kappa^{ex}} - \frac{r_0^2}{\kappa^{ex}} + \frac{r_0^2}{\kappa^{in}} \quad \text{if } r > r_0,$$

with $r = \sqrt{x^2 + y^2}$ and $r_0 = 3/4$, see Figure 3(a). The diffusion coefficient is highly discontinuous across Γ: $\kappa^{in} = 1$ and $\kappa^{ex} = 10^3$. The values of the stabilisation parameters are $\gamma = 10$, $\delta^{in} = \delta^{ex} = 100$. The non-homogeneous Dirichlet boundary condition induces some trivial modifications of the method; it is treated in the code by Nitsche's method.

Table 1 shows the computed errors and orders of convergence under mesh refinement, with N representing the number of cells. We obtain the expected rates, that is $O(h)$ in the energy norm $[[\cdot]]$ and $O(h^2)$ in the L^2 norm. In Figure 3(b), we have represented these convergence rates in a *log-log* scale.

N	energy norm	ratio	L^2-norm	ratio
64	3.45e-01	—-	2.83e-02	—-
256	1.68e-01	2.05	6.27e-03	4.52
1024	8.03e-02	2.09	1.41e-03	4.45
4096	3.95e-02	2.03	3.38e-04	4.17
16384	1.97e-02	2.01	8.21e-05	4.11
65536	9.82e-03	2.00	2.02e-05	4.06

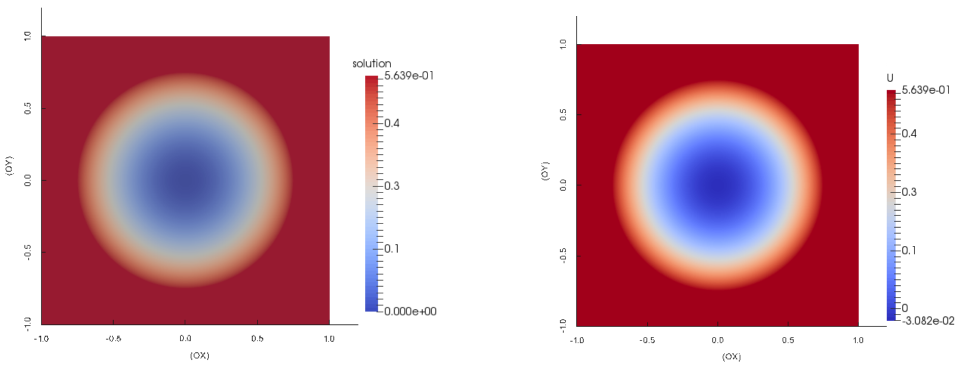
Table 1: Convergence in energy and L^2 norms with respect to mesh refinement.



(a) Domain of the test-case.

(b) L^2-norm and energy norm convergence rates.

Figure 3: Computational domain and convergence rates in *log-log* scale.



(a) Exact solution.

(b) Computed solution (N = 65536).

Figure 4: Comparison between exact and computed solutions.

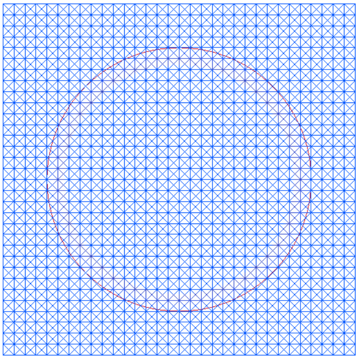


Figure 5: Mesh (1024 elements).

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