Monografías Matemáticas *García de Galdeano* **40**, 21–27 (2016)

# INTERVALS OF STRUCTURED MATRICES

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**Abstract.** For some structured matrices, such as SBD matrices or nonsingular $M$–matrices, some results on intervals of these matrices are presented.

*Keywords:* Intervals of matrices, matrix ordering, bidiagonal decomposition.

*AMS classification:* 15A24, 15A45, 15B35.

## §1. Introduction

A matrix with all its principal minors positive is called a *P–matrix*. Interval results (i.e., results for intervals of matrices) for $P$–matrices and their subclasses have been obtained in the last decades. For instance, see [8] for $P$–matrices and the usual entrywise ordering. An important subclass of $P$–matrices with applications to many fields (such as Statistics, Approximation Theory, Combinatorics, Economy, or Computer Aided Geometric Design) is given by the nonsingular totally positive matrices. Let us recall that a matrix is called *totally positive* (TP) if all its minors are nonnegative. If they are all positive, then the matrix is called *strictly totally positive* (STP). Interval results for TP matrices with the checkerboard ordering can be seen in [7, 9, 1].

The class of SBD matrices is a subclass of $P$–matrices that contains nonsingular TP matrices as well as their inverses [3, 4]. In Section 2, we present this class of SBD matrices and some preliminary results. In Section 3, we provide some interval results for SBD matrices with an ordering generalizing checkerboard ordering. We also present the class of SSBD matrices, which contains STP matrices and their inverses. Some important properties of these matrices are derived. An interval result for SSBD matrices is also provided.

Finally, in Section 4 we consider another important subclass of $P$–matrices: the nonsingular $M$–matrices. After defining a partial ordering for these matrices, an interval result is included.

## §2. Preliminary results

Given $k \in \{1, 2, \dots, n\}$ let $Q_{k,n}$ be the set of increasing sequences of $k$ positive integers less than or equal to $n$. Given $\alpha = (\alpha_1, \dots, \alpha_k) \in Q_{k,n}$, we define its dispersion number by $d(\alpha) = \alpha_k - \alpha_1 - (k-1)$. Observe that if $d(\alpha) = 0$ then $\alpha$ consists of $k$ consecutive integers. If $\alpha, \beta \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of $A$ containing rows numbered by $\alpha$ and columns numbered by $\beta$. Let us recall that submatrices $A[\alpha] := A[\alpha|\alpha]$ are called principal submatrices. We also define a principal minor as the determinant of a principal submatrix, $\det A[\alpha]$. We also denote by $\alpha^c$ the increasingly rearranged $\{1, \dots, n\} \setminus \alpha$. Then, we denote $A(\alpha|\beta) := A[\alpha^c|\beta^c]$.

Let $k$ be a positive integer and let us consider a $k$–vector of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ with $\varepsilon_j \in \{\pm 1\}$ for all $j \leq k$, $\varepsilon$ is called a *signature*. Given a signature sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1})$, let us define a diagonal matrix $K_\varepsilon = \mathrm{diag}(k_1, \ldots, k_n)$ with $k_i$ satisfying

$$k_1 = 1, \qquad k_i \in \{-1, 1\}\ \forall\, i = 2, \ldots, n, \qquad k_i = \varepsilon_1 \cdots \varepsilon_{i-1}\ \forall\, i = 2, \ldots, n. \tag{1}$$

Observe that

$$k_i k_{i+1} = \varepsilon_i, \quad \forall i = 1, \ldots, n-1. \tag{2}$$

Let $A$ be a nonsingular $n \times n$ matrix. Suppose that we can write $A$ as a product of bidiagonal matrices

$$A = L^{(1)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(1)}, \tag{3}$$

where $D = \mathrm{diag}(d_1, \ldots, d_n)$, and, for $k = 1, \ldots, n-1$, $L^{(k)}$ and $U^{(k)}$ are unit diagonal lower and upper bidiagonal matrices respectively, with off-diagonal entries $l_i^{(k)} := (L^{(k)})_{i+1,i}$ and $u_i^{(k)} := (U^{(k)})_{i,i+1}$, $(i = 1, \ldots, n-1)$ satisfying

1. $d_i \neq 0$ for all $i$,
2. $l_i^{(k)} = u_i^{(k)} = 0$ for $i < n-k$,
3. $l_i^{(k)} = 0 \Rightarrow l_{i+s}^{(k-s)} = 0$ for $s = 1, \ldots, k-1$ and
   $u_i^{(k)} = 0 \Rightarrow u_{i+s}^{(k-s)} = 0$ for $s = 1, \ldots, k-1$.

Then we denote (3) by $\mathcal{BD}(A)$, a bidiagonal decomposition of $A$.

Let us consider a class of matrices with a signed bidiagonal decomposition presented in [4].

**Definition 1.** Given a signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$ and a nonsingular $n \times n$ matrix $A$, we say that $A$ is an *SBD matrix with signature* $\varepsilon$ if there exists a $\mathcal{BD}(A)$ such that

1. $d_i > 0$ for all $i$,
2. $l_i^{(k)}\varepsilon_i \geq 0$, $u_i^{(k)}\varepsilon_i \geq 0$ for $1 \leq k \leq n-1$ and $n-k \leq i \leq n-1$.

We say that $A$ is an SBD matrix if it is SBD for some signature $\varepsilon$.

By Proposition 4.1 of [4], all the principal minors of SBD matrices are positive and then SBD matrices form a subclass of $P$–matrices.

It is possible to characterize SBD matrices in terms of nonsingular TP matrices, as the following result shows (see Theorem 3.1 of [4]).

**Theorem 1.** *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a nonsingular matrix and let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$ be a signature sequence. Then A is SBD with signature $\varepsilon$ if and only if $K_\varepsilon A K_\varepsilon = |A| := (|a_{ij}|)_{1 \leq i,j \leq n}$ is TP, where $K_\varepsilon$ is a diagonal matrix satisfying (1).*

Observe that the class of SBD matrices contains nonsingular TP matrices as well as their inverses. In fact, by Corollary 3.3 of [4], $A^{-1}$ is TP if and only if $A$ is SBD with signature $(-1, \ldots, -1)$. A detailed study of SBD matrices, with many properties, accurate computations and relation with other classes of important matrices, can be found in [3] and [4].

Let us recall that $<, \leq$ is used to denote the usual entrywise partial ordering on matrices; that is, given $A = (a_{ij})_{1 \leq i,j \leq n}$ and $B = (b_{ij})_{1 \leq i,j \leq n}$ we say that $A < B$ (resp., $A \leq B$) if $a_{ij} < b_{ij}$ (resp., $a_{ij} \leq b_{ij}$) for all $i, j \in \{1, \ldots, n\}$. We also write $0 < A$ (resp., $0 \leq A$) if $0 < a_{ij}$ (resp., $0 \leq a_{ij}$) for all $i, j \in \{1, \ldots, n\}$. As we can see in Example 3.2.1 of [6], if we have $0 < A < B$

two TP matrices, then not all matrices $C$ such that $A \leq C \leq B$ need to be TP. A similar comment can be applied to SBD matrices. In fact, let us consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 5 \\ 5 & 13 \end{pmatrix}.$$

Observe that $A$ and $B$ are TP and so SBD. We have that

$$C = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$$

satisfies $A < C < B$ but $C$ is not TP and, by Theorem 1, it is not SBD either.

In order to obtain intervals of TP matrices we need to define a different matrix ordering. Let us recall the *checkerboard ordering* (see [1], [6]). Let us denote the matrices $J_n = \mathrm{diag}(1, -1, \ldots, (-1)^{n-1})$ and $A^* := J_n A J_n$. Then we have that $A \leq^* B$ if and only if $A^* \leq B^*$.

In [1] we can find the following result for intervals of nonsingular TP matrices.

**Theorem 2.** *Let $A, B, Z \in \mathbb{R}^{n\times n}$ with $A \leq^* Z \leq^* B$. If $A, B$ are nonsingular TP matrices, then $Z$ is nonsingular TP.*

Let us observe that, as in the previous example, this ordering does not guarantee that given two SBD matrices $A, B$ such that $A \leq^* B$, then all matrices $C$ such that $A \leq^* C \leq^* B$ are SBD. Let us consider

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -5 \\ -5 & 13 \end{pmatrix}.$$

Observe that, by Theorem 1, $A$ and $B$ are SBD. Then

$$C = \begin{pmatrix} 3 & -4 \\ -4 & 5 \end{pmatrix}$$

satisfies $A \leq^* C \leq^* B$ but $C$ is not SBD again by Theorem 1.

Let us define a new matrix ordering that we can use to study intervals of SDB matrices. Given $A, B$ two $n \times n$ SBD matrices with the same signature $\varepsilon$, we consider a matrix $K_\varepsilon = \mathrm{diag}(k_1, \ldots, k_n)$ satisfying (1). Let us denote $A^{\dagger_\varepsilon} := (J_n K_\varepsilon) A (J_n K_\varepsilon)$. Then we consider the following matrix ordering: $A \leq^{\dagger_\varepsilon} B$ if and only if $A^{\dagger_\varepsilon} \leq B^{\dagger_\varepsilon}$.

## §3. Intervals of SBD and SSBD matrices

The following result extends Theorem 2 to the class of SBD matrices and uses the new ordering to determine the interval of matrices.

**Theorem 3.** *Let $A, B, Z \in \mathbb{R}^{n\times n}$ with $A \leq^{\dagger_\varepsilon} Z \leq^{\dagger_\varepsilon} B$. If $A$ and $B$ are SBD matrices with the same signature $\varepsilon$, then $Z$ is SBD with signature $\varepsilon$.*

*Proof.* Let us consider the matrix $K_\varepsilon$ asociated with $A$ and $B$, defined as in Theorem 1. Observe that, since $K_\varepsilon$ and $J_n$ are diagonal matrices, we have that $K_\varepsilon J_n = J_n K_\varepsilon$ and then, by hypothesis, we have that $J_nK_\varepsilon AK_\varepsilon J_n \le J_nK_\varepsilon ZK_\varepsilon J_n \le J_nK_\varepsilon BK_\varepsilon J_n$. By Theorem 1, we know that $K_\varepsilon AK_\varepsilon$ and $K_\varepsilon BK_\varepsilon$ are nonsingular TP matrices. Thus, we can apply Theorem 2 (using checkerboard ordering) for TP matrices and we conclude that the matrix $K_\varepsilon ZK_\varepsilon$ is nonsingular TP. Then, again by Theorem 1, $Z$ is SBD with signature $\varepsilon$. □

Finally, let us consider the inverse of a TP matrix, which is a particular case of SBD matrix with signature $\varepsilon = (-1, \ldots, -1)$. Since $\le^{\dagger_\varepsilon}$ coincides for this $\varepsilon$ with the usual ordering $\le$ because $K_\varepsilon = J_n$, we deduce the following corollary.

**Corollary 4.** *Let $A, B, Z \in \mathbb{R}^n$ with $A \le Z \le B$. If $A^{-1}, B^{-1}$ are TP, then $Z^{-1}$ is TP.*

Let us now study intervals of another class of matrices: strictly SBD matrices. An $n \times n$ matrix $A$ is said to be strictly SBD, or SSBD, with signature $\varepsilon$ if there exists a matrix $K_\varepsilon = \mathrm{diag}(k_1, \ldots, k_n)$ satisfying (1) such that $K_\varepsilon AK_\varepsilon$ is STP. We say that A is SSBD if it is SSBD for some signature $\varepsilon$.

Now, let us recall formula (1.32) of [2],

$$\det(J_nA^{-1}J_n)[\alpha \mid \beta] = \frac{\det A(\beta \mid \alpha)}{\det A}, \qquad \text{for } \alpha, \beta \in Q_{k_n}. \tag{4}$$

Let us also recall that, by Theorem 3.3 of [2], an $n \times n$ matrix $A$ is TP if and only if $J_nA^{-1}J_n$ is TP. The following lemma extends this result to STP matrices and it is a consequence of formula (4).

**Lemma 5.** *Let $A$ be an $n \times n$ nonsingular matrix. Then $A$ is STP if and only if $J_nA^{-1}J_n$ is STP.*

The following result and Corollary 7 extend Theorem 3.1 and Corollary 3.3 of [4], valid for SBD matrices, to SSBD matrices. The class of SSBD matrices is closed for the inversion of matrices, as the following result shows.

**Proposition 6.** *Let $A$ be an $n \times n$ nonsingular matrix. Then $A$ is SSBD with signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$ if and only if $A^{-1}$ is SSBD with signature $-\varepsilon = (-\varepsilon_1, \ldots, -\varepsilon_{n-1})$.*

*Proof.* Recall that, by definition of SSBD matrices, $A$ is SSBD with signature $\varepsilon$ if and only if $K_\varepsilon AK_\varepsilon$ $(= |A|)$ is STP, where $K_\varepsilon = \mathrm{diag}(k_1, \ldots, k_n)$ is a matrix satisfying (1). By Lemma 5, $|A|$ is STP if and only if $J_n|A|^{-1}J_n$ is STP. Observe that $(K_\varepsilon AK_\varepsilon)^{-1} = |A|^{-1} = K_\varepsilon A^{-1}K_\varepsilon$ and so, $A^{-1} = K_\varepsilon|A|^{-1}K_\varepsilon$. Thus, we have $\hat{K}_\varepsilon A^{-1}\hat{K}_\varepsilon = J_n|A|^{-1}J_n = |A^{-1}|$, where $\hat{K}_\varepsilon := K_\varepsilon J_n = J_nK_\varepsilon = \mathrm{diag}(k_1, -k_2, \ldots, (-1)^{n-1}k_n)$ and then, by Theorem 1, $A^{-1}$ is SSBD with signature $-\varepsilon$ if and only if $J_n|A|^{-1}J_n$ is STP. □

Taking into account that $A$ is STP if and only if it is SSBD with signature $(1, \ldots, 1)$ and Proposition 6, we can deduce that inverses of STP matrices are SSBD matrices.

**Corollary 7.** *Let $A$ be a nonsingular matrix. If $A^{-1}$ is STP, then $A$ is SSBD with signature $(-1, \ldots, -1)$.*

By Theorem 4.3 of [11] we know that STP matrices are characterized by a (unique) bidiagonal decomposition $\mathcal{BD}(A)$ such that

1. $d_i > 0$ for all $i \le n$,
2. $l_i^{(k)} > 0$, $u_i^{(k)} > 0$, $k = 1, \ldots, n-1$, $i = n-k, \ldots, n-1$.

The following result characterizes SSBD matrices in terms of its bidiagonal decomposition.

**Proposition 8.** *Let $A$ be an $n \times n$ nonsingular matrix. Then $A$ is SSBD with signature $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1})$ if and only if there exists a $\mathcal{BD}(A)$ such that*

1. $d_i > 0$ *for all* $i \leq n$,

2′. $\varepsilon_i l_i^{(k)} > 0$, $\varepsilon_i u_i^{(k)} > 0$, $k = 1, \dots, n-1$, $i = n-k, \dots, n-1$.

*Proof.* $A$ is SSBD with signature $\varepsilon$ if and only if $K_\varepsilon A K_\varepsilon$ is STP, where $K_\varepsilon = \operatorname{diag}(k_1, \dots, k_n)$ satisfies (1). As we have seen above, we have that $K_\varepsilon A K_\varepsilon$ is STP if and only if it has a bidiagonal decomposition $K_\varepsilon A K_\varepsilon = \hat{L}^{(1)} \cdots \hat{L}^{(n-1)} \hat{D} \hat{U}^{(n-1)} \cdots \hat{U}^{(1)}$, with $\hat{D}$ and $\hat{L}^{(k)}$, $\hat{U}^{(k)}$, for all $k \leq n-1$, satisfying conditions 1 and 2 above.

Observe that we have

$$\begin{aligned} A &= K_\varepsilon \hat{L}^{(1)} \cdots \hat{L}^{(n-1)} \hat{D} \hat{U}^{(n-1)} \cdots \hat{U}^{(1)} K_\varepsilon \\ &= (K_\varepsilon \hat{L}^{(1)} K_\varepsilon) \cdots (K_\varepsilon \hat{L}^{(n-1)} K_\varepsilon)(K_\varepsilon \hat{D} K_\varepsilon)(K_\varepsilon \hat{U}^{(n-1)} K_\varepsilon) \cdots (K_\varepsilon \hat{U}^{(1)} K_\varepsilon). \end{aligned}$$

If we denote, for all $k = 1, \dots, n-1$, $L^{(k)} := K_\varepsilon \hat{L}^{(k)} K_\varepsilon$, $U^{(k)} := K_\varepsilon \hat{U}^{(k)} K_\varepsilon$ and $D := K_\varepsilon \hat{D} K_\varepsilon = \hat{D}$, it is easy to check that condition 2 (for $\hat{L}^{(k)}$, $\hat{U}^{(k)}$ and $K_\varepsilon A K_\varepsilon$) and condition 2′ (for $L^{(k)}$, $U^{(k)}$ and $A$) are equivalent because $k_i k_{i+1} = \varepsilon_i$ for all $i = 1, \dots, n-1$ by (2). Thus, the result follows. □

The following lemma provides a formula for the sign of the minors of certain matrices.

**Lemma 9.** *Let $A$ be an $n \times n$ matrix, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1})$ a signature and $K_\varepsilon = \operatorname{diag}(k_1, \dots, k_n)$ a matrix satisfying (1). Then*

$$\det(K_\varepsilon A K_\varepsilon)[\alpha \mid \beta] = \left( \prod_{i=1}^{k} \prod_{j=\min(\alpha_i,\beta_i)}^{\max(\alpha_i-1,\beta_i-1)} \varepsilon_j \right) \det A[\alpha \mid \beta] \tag{5}$$

*for all $\alpha, \beta \in Q_{k,n}$ and $k \leq n$.*

*Proof.* Let us consider the matrix $K_\varepsilon = \operatorname{diag}(k_1, \dots, k_n)$ and the sequences $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_k)$. Observe that we have

$$\begin{aligned} \det(K_\varepsilon A K_\varepsilon)[\alpha \mid \beta] &= (k_{\alpha_1} \cdots k_{\alpha_k})(k_{\beta_1} \cdots k_{\beta_k}) \det A[\alpha \mid \beta] \\ &= (k_{\alpha_1} k_{\beta_1} \cdots k_{\alpha_k} k_{\beta_k}) \det A[\alpha \mid \beta]. \end{aligned} \tag{6}$$

Since $K_\varepsilon = \operatorname{diag}(k_1, \dots, k_n)$ satisfies (1), we can write, for any $i \in \{1, \dots, k\}$,

$$\begin{aligned} \varepsilon_{\min(\alpha_i,\beta_i)} \cdots \varepsilon_{\max(\alpha_i-1,\beta_i-1)} &= (k_{\min(\alpha_i,\beta_i)} k_{\min(\alpha_i,\beta_i)+1})(k_{\min(\alpha_i,\beta_i)+1} k_{\min(\alpha_i,\beta_i)+2}) \\ &\quad \cdots (k_{\max(\alpha_i-1,\beta_i-1)} k_{\max(\alpha_i,\beta_i)}) \\ &= k_{\min(\alpha_i,\beta_i)} k_{\max(\alpha_i,\beta_i)} = k_{\alpha_i} k_{\beta_i}. \end{aligned}$$

Taking into account the previous formula and (6), we can derive that

$$\det(K_\varepsilon A K_\varepsilon)[\alpha \mid \beta] = \prod_{i=1}^{k} \varepsilon_{\min(\alpha_i,\beta_i)} \cdots \varepsilon_{\max(\alpha_i-1,\beta_i-1)},$$

and so, formula (5) holds. □

The following result characterizes SSBD matrices in terms of a reduced number of minors, extending Theorem 4.1 of [10] for STP matrices to the class of SSBD matrices. The proof of this proposition is a consequence of Theorem 4.1 of [10] and Lemma 9.

**Proposition 10.** *Let $A$ be an $n \times n$ matrix. Then $A$ is SSBD with signature $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1})$ if and only if for every pair of sequences $\alpha = (\alpha_1, \dots, \alpha_k), \beta = (\beta_1, \dots, \beta_k) \in Q_{k,n}$ $(k \leq n)$ with $d(\alpha) = d(\beta) = 0$ such that either $\alpha_1 = 1$ or $\beta_1 = 1$, we have*

$$\left( \prod_{i=1}^{k} \prod_{j=\min(\alpha_i,\beta_i)}^{\max(\alpha_i-1,\beta_i-1)} \varepsilon_j \right) \det A[\alpha \mid \beta] > 0. \tag{7}$$

Intervals of strictly totally positive matrices were considered in Theorem 3.6 of [13] (see also [7]).

**Theorem 11.** *Let $A, B, Z \in \mathbb{R}^{n \times n}$ with $A \leq^* Z \leq^* B$. If $A$ and $B$ are strictly TP matrices, then $Z$ is a strictly TP matrix.*

The following result extends Theorem 11 to the class of SSBD matrices. The proof of this result is analogous to that of Theorem 3 and it uses the definition of SSBD matrices and Theorem 11.

**Theorem 12.** *Let $A, B, Z \in \mathbb{R}^{n \times n}$ with $A \leq^{\dagger_\varepsilon} Z \leq^{\dagger_\varepsilon} B$. If $A$ and $B$ are SSBD with the same signature $\varepsilon$, then $Z$ is SSBD with signature $\varepsilon$.*

## §4. Intervals of M–matrices

Intervals of other classes of matrices have also been considered. For instance, in [8] intervals of $P$–matrices and related matrices were considered. In this section we study intervals of a different class of matrices: the well-known nonsingular $M$–matrices, which appear in many applications in fields such as economy, linear programming, dynamical systems or biology (see [5]). Nonsingular $M$–matrices also form a subclass of $P$–matrices. Let us recall that the set of $n$–square matrices with nonpositive off–diagonal entries is denoted by $Z_n := \{A = (a_{ij})_{1 \leq i,j \leq n} \mid a_{ij} \leq 0 \text{ if } i \neq j\}$. A nonsingular matrix $A \in Z_n$ is an *M–matrix* if $A^{-1}$ has nonnegative entries, $A^{-1} \geq 0$ (see, for instance, Theorem 2.5.3 of [12]). An $n \times n$ matrix is a $Z$–matrix if $A \in Z_n$.

The following result deals with intervals of $M$–matrices.

**Theorem 13.** *Let $A, B, Z \in \mathbb{R}^{n \times n}$ with $J_n A^{-1} J_n \leq J_n Z^{-1} J_n \leq J_n B^{-1} J_n$ and $Z$ a nonsingular $Z$–matrix. If $A$ and $B$ are nonsingular $M$–matrices, then $Z$ is a nonsingular $M$–matrix.*

*Proof.* Let us denote $J_n A^{-1} J_n = (\hat{a}_{ij})_{1 \leq i,j \leq n}$, $J_n B^{-1} J_n = (\hat{b}_{ij})_{1 \leq i,j \leq n}$, $J_n Z^{-1} J_n = (\hat{z}_{ij})_{1 \leq i,j \leq n}$.

Since $A = (a_{ij})_{1 \leq i,j \leq n}, B = (b_{ij})_{1 \leq i,j \leq n}$ are $M$–matrices, we know that $A^{-1}, B^{-1} \geq 0$. Thus, it can be checked that $(-1)^{i+j} \hat{a}_{ij}, (-1)^{i+j} \hat{b}_{ij} \geq 0$, for all $i, j \in \{1, \dots, n\}$.

Observe that, since $J_n A^{-1} J_n \leq J_n Z^{-1} J_n \leq J_n B^{-1} J_n$ by hypothesis, we have that $(-1)^{i+j} \hat{z}_{ij} \geq 0$ for all $i, j \in \{1, \dots, n\}$. Thus, $0 \leq Z^{-1}$.

□

## Acknowledgements

Research Partially Supported by the Spanish Research Grant MTM2012-31544, by Gobierno de Aragón and Fondo Social Europeo.

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