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FRACTALITY TESTS FOR THE REFERENCE INDEX OF THE SPANISH STOCK MARKET (IBEX 35)

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Abstract. IBEX 35 is the official index of the Spanish computerized trading system, and it is a good indicator of the trends and evolution of the stock exchange. One of the main goals of our work is to inquire into the mathematical structure of the reference index IBEX. In particular, we want to elucidate if there are indications pointing at a fractal structure of the daily close IBEX values during the period corresponding to years 2000-2011.

Another objective of the study is the determination of useful parameters in order to characterize the stock recordings and their different patterns. We have computed the fractal dimension of the IBEX daily records over periods of twelve months and we have compared it with three different simulated time series (a uniform random variable, a Gaussian random motion and a sampled pure sinusoidal function).

Furthermore our results show that the IBEX daily records admit a model of fractional Brownian function with Hurst parameter H in the range 0.4-0.6. Consequently, the IBEX series is close to an ordinary Brownian motion ($H = 0.5$).

Keywords: Stock indices, IBEX 35, fractals, fractional Brownian function.

AMS classification: 28A80, 65D05, 65D10.

§1. Introduction

The stock indices reflect the time evolution of the stock prices quoted in the market. The assets composing the index follow criteria of election related to the trading volume and the market capitalization. Given that there are several types of assets (share, derivatives), different kinds of indices can be obtained, although the most used are referred to shares.

The indices try to reflect the behaviour of values quoted in the market as a unity. The importance of these numbers rests on their simplicity for handling the information. With a single number, the index summarizes a stock working day. They are not theoretical parameters, since they are based on the real behaviour of the market, and they offer a general view in order to take decisions.

IBEX 35 is the official index of the Spanish computerized trading system and is calculated, published and diffused by *Sociedad de Bolsas*. It is an index weighted by capitalization, composed by (temporarily) 36 companies quoted in the market of the four Spanish stock exchanges. Despite the reduced number of companies involved, IBEX represents a wide percentage of the total trading volume and the full capitalization of the Spanish market. Consequently, it is a good indicator of the trends and evolution of the stock exchange.

One of the main goals of our work is to inquire into the mathematical structure of the reference index IBEX of the Spanish stock market.

In particular, we want to elucidate if there are indications pointing at a fractal structure of the daily close IBEX values. For it, we have used both mathematical and statistical procedures coming from the fractal methodology and the approximation theory. Another objective of the study is the determination of useful parameters in order to characterize the stock recordings and their different patterns. Our main purpose is the conciliation between the apparently random behaviour of the signal and its trend curves. The theoretical framework is based on deterministic as well as random fractal function models.

1.1. Fractal dimensions

We have considered the daily close values of IBEX during the period corresponding to years 2000-2011. We computed the fractal dimension of the recordings in periods of twelve months. We considered in the first place, the daily data of the year 2000. Later on we removed the data of January 2000, being substituted by the values of January 2001, completing a new period of twelve months, and so on. The overlapping between consecutive segments was then eleven months. The fractal dimension of the interval was assigned to the last month recorded.

After the computation of the dimensions by means of fractal interpolation, we performed a smoothing filter of these numbers, by means of a moving average of third order. In this way, we obtained a weighted fractal dimension assigned to each month (the last one) of the period. In order to compare the IBEX values with different signals we obtained three different simulated recordings with the same number of samples:

1. A uniform random variable.
2. A Gaussian random variable.
3. A sampled pure sinusoidal variable.

We performed the same procedures for the generated signals: computation of overlapped fractal dimensions and smoothing in periods with the same length.

1.1.1. Fractal interpolation functions

In this Subsection we describe shortly the mathematical foundations of the fractal interpolation functions.

Let K be a complete metric space with respect to the distance $d(x, y)$, for $x, y \in K$.

Let $\mathcal{H}$ be the set of all complete not empty subsets of K. $\mathcal{H}$ is a complete metric space with the Hausdorff distance [3].

Let $w_n : K \to K$, for $n = 1, 2, \ldots, N$, be a set of continuous mappings. Then, the set $\{K, w_n; n = 1, \ldots, N\}$ is an Iterated Function System (IFS) on K.

Define the mapping $W : \mathcal{H} \to \mathcal{H}$ by

$$W(A) = \bigcup_{n=1}^{N} w_n(A) \quad \forall A \in \mathcal{H}.$$

Any set $G \in \mathcal{H}$ such that

$$G = W(G)$$

is an invariant set of the IFS (G is a fixed point of W). Furthermore, G is an attractor if

$$G = \lim_{m\to\infty} W^{(m)}(A) \qquad \forall A \in \mathcal{H}.$$

Here $W^{(m)}$ represents the composition of W with itself m times. The convergence of this limit is taken in the sense of the Hausdorff metric between sets [3].

Let us consider now a real compact interval $I = [a, b]$, a partition of I

$$\Delta : a = t_0 < t_1 < ... < t_N = b,$$

and corresponding ordinate values $(x_n)_{n=0}^{N}$.

Let K be defined as $K = I \times \mathbb{R}$, and the mappings $w_n : K \to K$ as:

$$w_n(t, x) = (L_n(t), F_n(t, x))$$

for $n = 1, 2, \ldots, N$, where

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t, x) = \alpha_n x + q_n(t), \end{cases} \tag{1.1}$$

with

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0}, \qquad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0} \tag{1.2}$$

and $q_n(t) = q_{n1} t + q_{n0}$, such that

$$q_{n1} = \frac{x_n - x_{n-1}}{t_N - t_0} - \alpha_n \frac{x_N - x_0}{t_N - t_0} \tag{1.3}$$

$$q_{n0} = \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} - \alpha_n \frac{t_N x_0 - t_0 x_N}{t_N - t_0}. \tag{1.4}$$

The Iterated Function System $\{K, w_n; n = 1, \ldots, N\}$ admits an attractor G, which is the graph of a continuous function $f : I \to \mathbb{R}$ interpolating the data $\{(t_n, x_n)_{n=0}^{N}\}$.

The parameters α_n are the vertical scaling factors of the IFS and must satisfy the inequality $|\alpha_n| < 1$ for any $n = 1, 2, \ldots, N$ [1]. The maps w_n transform a line segment r parallel to the y-axis into a line segment parallel to the y-axis. The ratio of the length of $w_n(r)$ to the length of r is the modulus of the vertical scaling factor $|\alpha_n|$.

1.1.2. Computation of fractal dimensions

The first step is the reconstruction of the signal by means of fractal interpolation functions, computing the parameters of the IFS associated to the data according to the fit proposed in the previous paragraph [7]. The computation of the fractal dimension is then performed by the use of the following equation:

$$\sum_{n=1}^{N} |\alpha_n|\, a_n^{D-1} = 1 \tag{1.5}$$

where α_n is the vertical scaling factor in the transformation w_n. If the nodes are equidistant, then $a_n = 1/N$ and the previous expression can be simplified as:

$$D = 1 + \frac{\ln\left(\sum_{n=1}^{N} |\alpha_n|\right)}{\ln N}. \tag{1.6}$$

This formula for the dimension is valid in the case $\sum_{n=1}^{N} |\alpha_n| > 1$. Otherwise, the fractal dimension is one [1]. This parameter lies between 1 and 2.

The computed values for all the records are displayed in the Figure 1 and Figure 2. The lower values (near 1) correspond to the sinusoidal function. The upper values (near 2) are the dimensions of the random variables. The intermediate numbers correspond to the IBEX data.

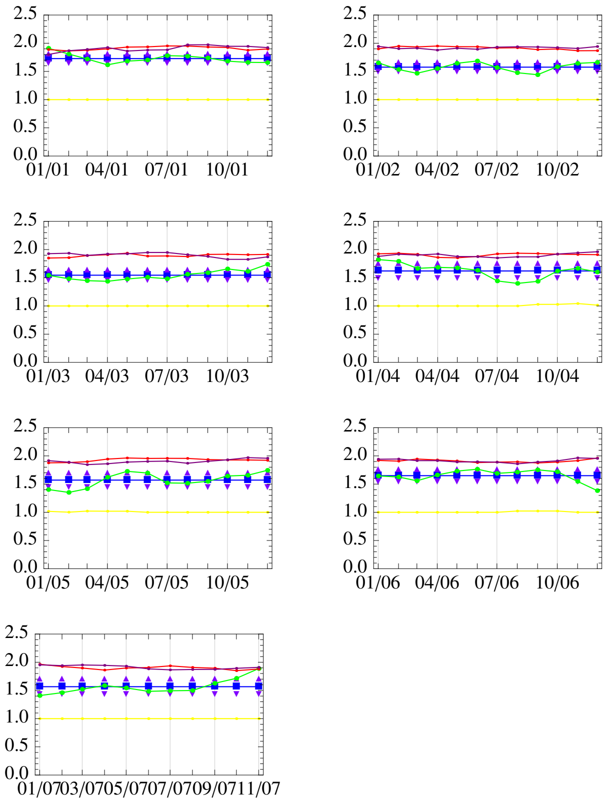

Figure 1: Annual graphics of the values of the weighted fractal dimension in the period 2001-2007

The least difference between IBEX and random variables was obtained in 2001.

We performed a statistical study in order to elucidate if the differences between the dimensions of the IBEX and the rest of the variables were significant. The normality of the dimension data was established by means of a Shapiro & Wilks test. The statistics was done yearly, comparing the data of weighted fractal dimensions of IBEX with the values of the random and deterministic signals. This was done by means of a t-Student test of mean comparisons. The study determined that the differences were pairwise significant at the level 0.01. These results aim at a fractal structure of the IBEX series, which we check further in the following procedure.

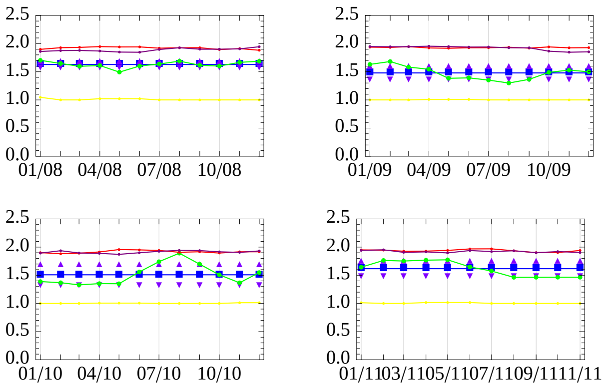

Figure 2: Annual graphics of the values of the weighted fractal dimension in the period 2008-2011

§2. Hurst exponents: IBEX 35 as fractional Brownian function

The second parameter computed by our group is the Hurst exponent (H) of the IBEX series. H is an indicator of self-similarity of the recording. It was proposed by H. E. Hurst [4], who searched a formula to quantify the levels of the Nile River. The value $H = 0$ is associated to a white noise, $H = 1$ corresponds to a deterministic signal, and $H = 0.5$ is related to a Brownian motion. For stock values, H is a measure of the trend of the asset. $H < 0.5$ is interpreted as index of a high volatility, $H = 0.5$ as a red noise and if $H > 0.5$ the series displays a definite trend. From the point of view of the theory, the exponent is related to a fractional Brownian motion.

It is a proved fact that the Brownian motion is a good model for experimental series and, in particular, for economic historical data. The fractional (or fractal) Brownian motions (fBm) were studied by Mandelbrot (see for instance [6], [5]). They are random functions containing both independence and asymptotic dependence and admit the possibility of a long-term autocorrelation. Another characteristic feature of fBm's is the self-similarity [6]. In words of B. Mandelbrot & Van Ness [6]:"fBm falls outside the usual dichotomy between causal trends and random perturbation". The fractional Brownian motion is generally associated with a spectral density proportional to $1/f^{2H+1}$, where f is the frequency. It is then a "coloured noise". For $H = 1/2$ one has an $1/f^2$−noise (Brownian or red).

If $B(t,\omega)$ is an ordinary Brownian motion, its increments $B(t_2,\omega) - B(t_1,\omega)$ are Gaussian with mean zero and variance $|t_2 - t_1|$ (see for instance [8]):

$$\{B(t_2,\omega) - B(t_1,\omega)\} \sim \mathcal{N}(0, |t_2 - t_1|).$$

A fractional Brownian motion with Hurst exponent H, $B_H(t,\omega)$, owns the following properties:

1. $B_H(t,\omega)$ has almost all sample paths continuous (when t lies in a compact interval I).
2. With probability one, the graph of $B_H(t,\omega)$ has both Hausdorff and box dimension equal to $2 - H$.
3. If $H = \frac{1}{2}$ then $B_H(t,\omega)$ is an ordinary Brownian function (or Wiener process). In this case the increments in disjoint intervals are independent.
4. The increments $\{B_H(t_0 + T,\omega) - B_H(t_0,\omega)\}$ are Gaussian with mean zero and variance proportional to T^{2H} [2], [6] (Corollary 3.4).

The goal of the our numerical experiment is to inquire about the structure of the IBEX daily data as fractional Brownian motion. Is IBEX really a variable of this type? In order to answer this question we considered the close daily data of the index of each year (2000-2011). Let us denote by $x(i)$ the value of the i-th day of one fixed year ($i = 1, 2, \ldots, N$, where N is the number of the data). Using different steps h_k, we defined the increment variables:

$$\psi_i^{h_k} = x(i + h_k) - x(i).$$

where $h_k = k\delta$. In this particular case we consider $\delta = 1$, corresponding to one day, because it is a natural period for this type of recordings.

For each k-th increment variable we computed the mean and the variance v^{h_k}. If the IBEX admits a model of fractional Brownian function the increments must follow the model:

$$\psi_i^{h_k} \simeq \mathcal{N}(0, v^{h_k}),$$

where $\mathcal{N}(0, v^{h_k})$ is Gaussian with variance v^{h_k} proportional to h_k^{2H} [2], [6], where H is the Hurst exponent [4]. H is also called Hurst parameter or index of the time series. If the IBEX is a fBm, a log-log plot of the variances v^{h_k} as function of h_k must display a collection of points on (or close to) a line (see Figure 3). This is due to the relation between the k-th variance and the corresponding step h_k described in the above property 4 [2].

The Hurst exponent is then the half of the slope of the line. We performed a linear regression and computed the corresponding parameters and correlation coefficients. The data considered are the k-th steps and the variances of the variables with increment h_k. The correlation between both coordinates is very high, since the coefficients are very close to one. The values of H of each year are displayed in Figure 4. The hypothesis of null correlation was rejected by means of a t-Student test with a significance level of 5% in all the periods.

Our numerical results show that the IBEX daily records admit a model of fractional Brownian function with Hurst parameter H in the range (0.4-0.6) approx. The time series is close to an ordinary Brownian motion ($H = 0.5$). However, depending on the year one can observe a slight variation in the self-similarity parameter H. In particular, we found a global

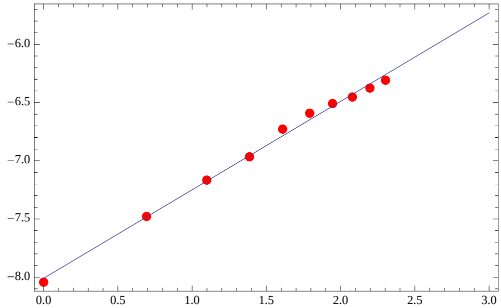

Figure 3: Log-log plot of the variances as function of the steps (h_k, v^{h_k}) at the year 2008. The line represents the regression of these data.

Year	Correlation Coef.
2000	0.999650
2001	0.999101
2002	0.999221
2003	0.998102
2004	0.999567
2005	0.999856
2006	0.999165
2007	0.999566
2008	0.997753
2009	0.999887
2010	0.999151
2011	0.991496

Table 1: Correlation coefficient of the regression for the computation of the Hurst parameter (years 2000-2011)

minimum at the year 2008 with value 0.380001. In general, lower values of the exponent are associated with variability, antipersistence and short term irregular cycles that hinder the financial forecasting attempts. These results may be useful for defining new risk measures, for coverage strategies and for financial asset pricing.

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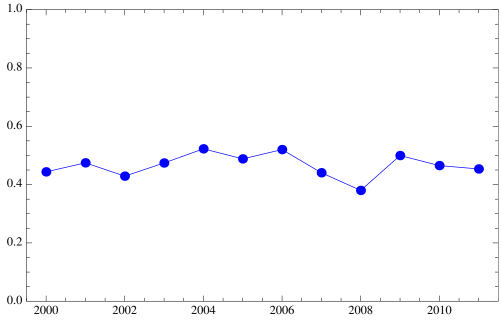

Figure 4: Yearly Hurst exponents in the range 2000-2011.

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