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STABILITY OF THE SOLUTIONS FOR A SINGULAR AND SUBLINEAR ELLIPTIC PROBLEM

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Abstract. We give here some additional stability and qualitative properties for the solutions of a singular and sublinear elliptic absorption problem which has already been studied in Giacomoni-Mâagli-Sauvy [5].

Keywords: Stability, elliptic problem, sub and supersolutions method.

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§1. Introduction and recalls

The objective of the present paper is to give some additional results concerning the stability and other qualitative properties of the solutions of a quasilinear and singular problem (P_λ), which has been studied in Giacomoni-Mâagli-Sauvy [5]. Before giving more details about our main results, let us start by recalling the framework of [5].

In [5], the authors have considered the following quasilinear and singular problem:

$$-\Delta_p u = \mathbf{1}_{\{u>0\}} K(x)(\lambda u^q - u^r) \quad \text{in } \Omega; \quad u|_{\partial\Omega} = 0, \quad u \geq 0 \quad \text{in } \Omega, \tag{P_λ}$$

where Ω is a C^2 bounded domain of $\mathbb{R}^N$, $1 < p < \infty$, $\lambda > 0$ is a positive parameter. In the right-hand side of the equation, the exponents q and r satisfy $-1 < r < q < p-1$ and $K \in C(\Omega)$ is a positive function having a singular behaviour near the boundary $\partial\Omega$. Precisely, $K(x) = d(x)^{-k} L(d(x))$ in Ω, with $d(x)$ the distance from $x \in \Omega$ to the boundary, $0 < k < p$ and L a Karamata function, which is a lower positive perturbation satisfying $L \in C^2((0, D])$ a positive function, with $D := \mathrm{diam}(\Omega)$, defined as follows:

$$L(t) = \exp\left(\int_t^D \frac{z(s)}{s} ds\right), \tag{1.1}$$

with $z \in C([0, D]) \cap C^1((0, D])$ and $z(0) = 0$. Let us just recall that (1.1) implies that

$$\forall \varepsilon > 0, \quad \lim_{t \to 0^+} t^{\varepsilon} L(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} t^{-\varepsilon} L(t) = +\infty. \tag{1.2}$$

The authors have discussed the existence of positive or compact-support solutions of (P_λ) with respect to the blow-up rate k of the singularity $K(x)$. Precisely, they have proved the existence of a critical value for the blow-up rate k separating existence and non-existence of positive solutions for problem (P_λ). In particular, the first case (existence of positive solutions) is investigated in the following theorem:

Theorem 1 (See [5, Theorem 2.1]). *When $k < 1+r$, there exists a constant $\Lambda_1 > 0$ such that:*

1. *For $\lambda > \Lambda_1$, (P_λ) admits a positive weak solution.*
2. *Any weak solution of (P_λ) is $C^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$.*
3. *For $\lambda < \Lambda_1$, (P_λ) has no positive solution.*

The critical parameter $\Lambda_1 > 0$ is defined as follows: $\Lambda_1 := \inf\{\lambda > 0 \mid u_\lambda > 0 \text{ a.e. in } \Omega\}$, where $u_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a maximal solution to (P_λ) obtained by a sub and supersolution method. In particular, we have $u_\lambda \leq \overline{u}_\lambda$ a.e. in Ω, where $\overline{u}_\lambda \in W_0^{1,p}(\Omega)$ is a supersolution to (P_λ). Precisely, $\overline{u}_\lambda := \overline{M}v$ in Ω, where $\overline{M}$ is a positive constant sufficiently large and v is the unique solution of problem

$$-\Delta_p v = K(x)v^q \quad \text{in } \Omega; \quad v|_{\partial\Omega} = 0, \quad v \geq 0 \quad \text{in } \Omega. \tag{Q}$$

Moreover, from Moser iterations technique, we can prove that $v \in L^\infty(\Omega)$ and from Lieberman [9], $v \in C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Then, v behaves like the distance to the boundary function in Ω (see [5, Lemma 3.3]). It is also proved in this paper the existence of a parameter $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$, $u_\lambda \equiv 0$ in Ω.

Accordingly, natural issues deriving from Theorem 1 for problem (P_λ) concern the precise behaviour (with respect to the distance to the boundary) of the positive solution u_λ with $\lambda > \Lambda_1$, the existence or non-existence of a non-trivial solution for the critical problem (P_{Λ_1}) and the stability of the solutions u_λ with $\lambda \geq \Lambda_1$ to (P_λ). In the general case studied in [5], those above questions have not been reached and remain open. In this paper, our goal is to answer these questions in the particular case of the Laplacian operator (*i.e.* when $p = 2$) with a concave right hand side $K(x)(\lambda u^q - u^r)$ with respect to u; that is to say with

$$-1 < r < 0 \quad \text{and} \quad 0 < q < 1. \tag{1.3}$$

Precisely, in the next section (Section 2), we prove that the positive solutions constructed in Theorem 1 (point 1.) behave like the distance to the boundary function. Next, in Section 3, we investigate the critical case $\lambda = \Lambda_1$. We prove the existence of a unique almost everywhere positive solution of (P_{Λ_1}). Finally, in Section 4, we prove the stability of the positive solutions of problem (P_λ) with $\lambda > \Lambda_1$ and the semi-stability of the almost everywhere positive solution of (P_{Λ_1}).

So, from now, in problem (P_λ)we suppose that $p = 2$ and that the exponents q and r satisfy the assumption (1.3).

§2. Behaviour of the solution u_λ

In this context we first get a precise behaviour in Ω of our maximal solution u_λ for $\lambda > \Lambda_1$.

Proposition 2. *Assume that $\lambda > \Lambda_1$. Then, there exist two constants $C_1, C_2 > 0$ (depending on λ) such that, for all $x \in \Omega$, $C_1 d(x) \leq u_\lambda(x) \leq C_2 d(x)$.*

Proof. Let us choose $\lambda' \in (\Lambda_1, \lambda)$ and consider $\varphi \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, solution to

$$-\Delta\varphi = K(x)u_{\lambda'}^q \quad \text{in } \Omega; \quad \varphi|_{\partial\Omega} = 0.$$

By the Hopf's Lemma (see for instance Evans [4, Lemma p. 330]), φ behaves like the distance function in Ω. Moreover, for $\varepsilon > 0$ sufficiently small, $w := u_{\lambda'} + \varepsilon\varphi$ is a subsolution of (P_λ) in Ω. Indeed, if (1.3) is satisfied and $\lambda' + \varepsilon \leq \lambda$, we have

$$-\Delta w = K(x)\{(\lambda' + \varepsilon)u_{\lambda'}^q - u_{\lambda'}^r\} \leq K(x)(\lambda w^q - w^r) \quad \text{in } \Omega.$$

Then, choosing M sufficiently large in the definition of $\overline{u}_\lambda$ and using the same lower- and upper-solution method as in [5, Proposition 4.1], we get $w \leq u_\lambda \leq \overline{u}_\lambda$ in Ω. Since both w and $\overline{u}_\lambda$ behave like the distance function, the proof of Proposition 2 is now complete. □

§3. About the critical problem (P_{Λ_1})

In Theorem 1, the existence of a critical value $\Lambda_1 > 0$ separating existence and non-existence of a positive solution to (P_λ) is proved. However, it is not clear if there exists a positive solution u_{Λ_1} to (P_{Λ_1}). The present section deals with the positiveness of u_{Λ_1}.

First, let us prove the existence of a non-trivial solution of (P_{Λ_1}). For that, we use the precise behaviour of the solutions of (P_λ), for $\lambda > \Lambda_1$, given in Proposition 2. Let $v \in C_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, be the unique solution to (Q). Then for $\lambda > \Lambda_1$, we define

$$U_\lambda := \lambda^{\frac{1}{1-q}} v \quad \text{in } \Omega. \tag{3.1}$$

This function U_λ is the unique solution of the problem

$$-\Delta w = \lambda K(x) w^q \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega, \tag{$\overline{\mathrm{P}}_\lambda$}$$

and behaves like the distance function in Ω (see [5, Lemma 3.3]). Furthermore, U_λ is also a supersolution to problem (P_λ) and, from the lower- and upper-solutions method, we have that $\lambda \mapsto U_\lambda$ is increasing on $(\Lambda_1, +\infty)$. Then, we first prove the following lemma:

Lemma 3. *Let $\lambda > \Lambda_1$ and let $u_\lambda \in C^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$, be the positive maximal solution of (P_λ) we proved in Theorem 1. Then, $u_\lambda \leq U_\lambda$ in Ω.*

Proof. In the proof, we use the uniqueness of the solution to problem (Q). Precisely, let us notice that $\underline{v} := u_\lambda$ is a subsolution to (Q). Then, let us define $\overline{v} := MV$ in Ω, where $M > 0$ is taken large enough and V is the unique solution of problem

$$-\Delta V = K(x) \quad \text{in } \Omega; \quad V|_{\partial\Omega} = 0. \tag{3.2}$$

Using a regularity result due to Gui-Lin [8], $V \in C_0(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$ and thanks to the Hopf's Lemma, V behaves like the distance function in Ω. Then, for $M > 0$ large enough, using the sub-homogeneity of problem (Q), $\overline{v}$ is a supersolution to (Q). Moreover using the behaviour of u_λ given by Proposition 2, for M large enough, $\underline{v} \leq \overline{v}$ in Ω. Then, we consider the following monotone iterative scheme: for $n \in \mathbb{N}^*$,

$$-\Delta v_n = \lambda K(x) v_{n-1}^q \quad \text{in } \Omega; \quad v_n|_{\partial\Omega} = 0, \tag{Q_n}$$

with $v_0 := \underline{v}$ in Ω. By induction on n, (Q_n) admits a unique solution $v_n \in C_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Moreover, using the weak maximum principle, for all $n \in \mathbb{N}^*$,

$$u_\lambda = \underline{v} \leq v_n \leq v_{n+1} \leq \overline{v} \quad \text{in } \Omega. \tag{3.3}$$

So, for all $x \in \overline{\Omega}$, let us define $\tilde{v}(x) := \lim_{n\to+\infty} v_n(x)$. Moreover $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{H}_0^1(\Omega)$, then passing to the limit in (Q_n), $\tilde{v}$ is a weak solution to $(\overline{\mathrm{P}}_\lambda)$. Passing to the limit in (3.3), $\tilde{v}(x)$ behaves as the distance function in Ω. Therefore, from the uniqueness of the solution to $(\overline{\mathrm{P}}_\lambda)$, $\tilde{v} = U_\lambda$ in Ω. The proof is now complete. □

The next result shows the existence and the positivity of an extremal solution u_{Λ_1} for the problem (P_λ) (u_{Λ_1} may vanish on a Lebesgue's measure-zero set).

Proposition 4. *Problem* (P_{Λ_1}) *admits a non-trivial weak solution* $u_{\Lambda_1} \in C^{1,\beta}(\overline{\Omega})$*, for some* $0 < \beta < 1$. *Moreover,*

$$\int_\Omega K(x) u_{\Lambda_1}^r \varphi_1 dx < +\infty. \tag{3.4}$$

As a consequence, $u_{\Lambda_1} > 0$ a.e. *in* Ω.

Proof. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to Λ_1. For all $n \in \mathbb{N}$, let us consider u_{λ_n} the maximal solution to (P_{λ_n}) given in Theorem 1. So for all $n \in \mathbb{N}$, $u_{\lambda_{n+1}}$ is a subsolution of $(\overline{\mathrm{P}}_{\lambda_n})$ and using Lemma 3, $u_{\lambda_{n+1}} \le U_{\lambda_{n+1}} \le U_{\lambda_n}$ in Ω. Then, by the lower- and upper-solution method as it is used in the proof of Theorem 1, we construct $\tilde{u}_{\lambda_n}$ solution to (P_{λ_n}) between $u_{\lambda_{n+1}}$ and U_{λ_n}. Hence, by maximality of u_{λ_n}, it follows that

$$0 < u_{\lambda_{n+1}} \le u_{\lambda_n} \le U_{\lambda_0} \quad \text{in } \Omega. \tag{3.5}$$

So let us define for all $x \in \overline{\Omega}$, $u_{\Lambda_1}(x) := \lim_{n\to+\infty} u_{\lambda_n}(x) \in [0, U_{\lambda_0}(x)]$. To prove (3.4), let us choose $\gamma \in (0,1)$, $\varepsilon > 0$ (small enough) and consider the function $\psi := (\varphi_1+\varepsilon)^\gamma - \varepsilon^\gamma \in \mathrm{H}_0^1(\Omega)$ as a test function. Then, a direct computation gives

$$-\Delta\psi = -\gamma(\gamma-1)\left|\nabla\varphi_1\right|^2(\varphi_1+\varepsilon)^{\gamma-2} + \lambda_1\varphi_1\gamma(\varphi_1+\varepsilon)^{\gamma-1} \ge 0 \quad \text{in } \Omega.$$

For all $n \in \mathbb{N}$,

$$\langle -\Delta u_{\lambda_n}, \psi\rangle_{\mathrm{H}^{-1}(\Omega)\times\mathrm{H}_0^1(\Omega)} = \int_\Omega K(x)(\lambda_n u_{\lambda_n}^q - u_{\lambda_n}^r)\psi\, dx \ge 0.$$

Thus, we get

$$\int_\Omega K(x) u_{\lambda_n}^r \psi\, dx \le \lambda_n \int_\Omega K(x) u_{\lambda_n}^q \psi\, dx$$

and passing to the limit as $\varepsilon \to 0$ and as $\gamma \to 1$, the Lebegue's dominated convergence theorem yields

$$\int_\Omega K(x) u_{\lambda_n}^r \varphi_1\, dx \le \lambda_n \int_\Omega K(x) u_{\lambda_n}^q \varphi_1\, dx.$$

Finally, since for all $n \in \mathbb{N}$, $u_{\lambda_n} \le U_{\lambda_0}$ in Ω, we have

$$\int_\Omega K(x) u_{\lambda_n}^r \varphi_1\, dx \le \Lambda_1 \int_\Omega K(x) U_{\lambda_0}^q \varphi_1\, dx < +\infty. \tag{3.6}$$

Passing to the limit in (3.6), the monotone convergence theorem provides estimate (3.4).

To complete the proof, we still have to show that u_{Λ_1} is a non-trivial weak solution of the extremal problem (P_{Λ_1}). First, notice that $(u_{\lambda_n})_{n\in\mathbb{N}}$ is bounded in $\mathrm{H}_0^1(\Omega)$. Indeed, we have for all $n \in \mathbb{N}$,

$$\int_\Omega |\nabla u_{\lambda_n}|^2\,dx \le \int_\Omega \lambda_n K(x) u_{\lambda_n}^{q+1}\,dx \le \int_\Omega \lambda_0 K(x) U_{\lambda_0}^{q+1}\,dx < +\infty.$$

So, identifying the limits in $\mathcal{D}'(\Omega)$, up to a subsequence denoted in the same way, $u_{\lambda_n} \xrightarrow[n\to+\infty]{} u_{\Lambda_1}$ in $\mathrm{H}_0^1(\Omega)$ and a.e. in Ω. Let $\varphi \in \mathcal{D}(\Omega)$, then we get

$$\forall n \in \mathbb{N}, \qquad \int_\Omega \nabla u_{\lambda_n}.\nabla\varphi\,dx = \int_\Omega K(x)(\lambda_n u_{\lambda_n}^q - u_{\lambda_n}^r)\varphi\,dx. \tag{3.7}$$

In (3.7), it is easy to get the convergence of both the left hand side and the positive part of the right hand side. Concerning the negative part, since $u_{\Lambda_1} > 0$ a.e. in Ω, we have that $K(x)u_{\lambda_n}^r\varphi \xrightarrow[n\to+\infty]{} K(x)u_{\Lambda_1}^r\varphi$ a.e in Ω. Moreover, from estimate (3.6), for almost every $x \in \Omega$, $\left|K(x)u_{\lambda_n}^r\varphi\right| \le K(x)u_{\Lambda_1}^r|\varphi| \in \mathrm{L}^1(\Omega)$. So, the Hölder inequality ensures that (3.4) holds. Hence, by the Lebesgue's dominated convergence theorem we pass to the limit as $n \to +\infty$ in (3.7) and it follows that u_{Λ_1} is a non-trivial weak solution to (P_{Λ_1}). Finally, the $C^{1,\beta}(\overline{\Omega})$ regularity of u_{Λ_1} follows from Theorem 1. □

Now, we show the uniqueness of the extremal positive solution u_{Λ_1} to (P_{Λ_1}).

Proposition 5. *Let $v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a positive solution to (P_{Λ_1}). Then, $v = u_{\Lambda_1}$ a.e. in Ω.*

Proof. Let $v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a positive solution to (P_{Λ_1}) such that $v \not\equiv u_{\Lambda_1}$ in Ω. Since the mapping $t \mapsto \Lambda_1 t^q - t^r$ is (strictly) concave on $(0, +\infty)$, the convex combination $w := tu_{\Lambda_1} + (1-t)v$, with $0 < t < 1$, is a (strict) subsolution of (P_{Λ_1}) in Ω. We now prove that it implies the existence of a positive solution to a problem $(\mathrm{P}_{\lambda'})$ with $\lambda' < \Lambda_1$ close enough to Λ_1, from which we get a contradiction. Let $\varphi \in C^{1,\alpha}(\overline{\Omega})$, for a fixed $0 < \alpha < 1$, the unique solution to

$$-\Delta\varphi = K(x)(\Lambda_1 w^q - w^r) \quad \text{in } \Omega; \quad \varphi|_{\partial\Omega} = 0.$$

By the weak maximum principle, $\varphi \ge w$ in Ω and by the strong maximum principle of Brézis-Nirenberg [2], there exists $\varepsilon > 0$ small enough such that $\varphi(x) \ge (w + \varepsilon V)(x)$ and $(\varphi - \varepsilon V)(x) \ge \varepsilon d(x)$, for $x \in \Omega$. Furthermore, $-\Delta(\varphi - \varepsilon V) \le K(x)[\Lambda_1(\varphi - \varepsilon V)^q - (\varphi - \varepsilon V)^r - \varepsilon]$ in Ω, where V is defined in (3.2). Thus, using lower- and upper-solutions method as in [5, Proposition 4.1] of this chapter, we prove the existence of $w_1 \in C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, solution of

$$-\Delta w_1 = K(x)(\Lambda_1 w_1^q - w_1^r - \varepsilon) \quad \text{in } \Omega; \quad w_1|_{\partial\Omega} = 0.$$

It follows from the weak maximum principle that $w_1(x) \ge (\varphi - \varepsilon V)(x) \ge \varepsilon d(x)$ in Ω. Then, let $\lambda' \in (0, \Lambda_1)$ and $\varepsilon' \in (0, (\lambda'/\Lambda_1)\varepsilon)$ be such that

$$(\lambda'/\Lambda_1)^{\frac{1}{r}} w_1 \le \varepsilon' V + (\lambda'/\Lambda_1) w_1 \quad \text{in } \Omega.$$

Setting $w_2 := \varepsilon' V + (\lambda'/\Lambda_1) w_1$ in Ω, we get

$$-\Delta w_2 \le K(x)\left(\lambda' w_1^q - (\lambda'/\Lambda_1) w_1^r - (\lambda'/\Lambda_1)\varepsilon + \varepsilon'\right) \le K(x)\left(\lambda' w_2^q - w_2^r\right) \quad \text{in } \Omega.$$

By choosing λ' close enough to Λ_1, $w_2 \geq w_1$ in Ω. Finally, by a sub and supersolution method, we conclude on the existence of a positive solution of the problem $(\mathrm{P}_{\lambda'})$, which proves the uniqueness of u_{Λ_1} among the almost everywhere positive solution to (P_{Λ_1}). □

Remark 1. This kind of argument has been introduced by Brezis *et al.* [1] for convex nonlinearities.

§4. About the stability of the solution u_λ

Now for $\lambda > \Lambda_1$, let us focus on the stability of the maximal solutions u_λ of Theorem 1. For that, we use some variational methods extracted from [6] and [7]. Let us define the energy functional $\mathcal{E}_\lambda$ by

$$\mathcal{E}_\lambda(v) := \int_\Omega |\nabla v|^2\, dx + r\int_\Omega K(x)u_\lambda^{r-1}v^2\,dx - \lambda q\int_\Omega K(x)u_\lambda^{q-1}v^2\,dx,$$

for all $\lambda > \Lambda_1$ and all $v \in \mathrm{H}_0^1(\Omega)$; and set $\Lambda(\lambda) := \inf\{\mathcal{E}_\lambda(v) \mid v \in \mathrm{H}_0^1(\Omega),\ \|v\|_{\mathrm{L}^2(\Omega)} = 1\}$, the first eigenvalue of the linearised operator associated to (P_λ).

Definition 1. The maximal solution u_λ of problem (P_λ) is said to be *stable* (resp. *semi-stable*) if and only if $\Lambda(\lambda) > 0$ (resp. $\Lambda(\lambda) \geq 0$).

For more details concerning stability of solutions, we refer to the book of L. Dupaigne [3]. First, we observe that $\Lambda(\lambda)$ is well defined thanks to Proposition 2 and Hardy's inequality. Indeed, for all $v \in \mathrm{H}_0^1(\Omega)$ and $\varepsilon > 0$ small enough,

$$\begin{aligned}\mathcal{E}_\lambda(v) &\geq \|v\|_{\mathrm{H}_0^1(\Omega)} - \lambda q\int_{\Omega\setminus\Omega_\varepsilon} K(x)u_\lambda^{q-1}v^2\,dx - \lambda q\int_{\Omega_\varepsilon} K(x)u_\lambda^{q-1}v^2\,dx\\ &\geq \|v\|_{\mathrm{H}_0^1(\Omega)} - \lambda q C_\varepsilon\|v\|_{\mathrm{L}^2(\Omega)} - \lambda q\varepsilon^{q+1-k}C\|v\|_{\mathrm{H}_0^1(\Omega)}\\ &\geq \tfrac{1}{2}\|v\|^2_{\mathrm{H}_0^1(\Omega)} - \lambda q C_\varepsilon\|v\|_{\mathrm{L}^2(\Omega)} \geq C_0 > -\infty,\end{aligned} \tag{4.1}$$

with $\Omega_\varepsilon := \{x \in \Omega \mid d(x) < \varepsilon\}$. Using the maximality of the solution u_λ, we now prove that $\Lambda(\lambda) > 0$, for every $\lambda > \Lambda_1$.

4.1. Study of a regularised problem

Let $\varepsilon_0 > 0$. So, for $\varepsilon \in (0, \varepsilon_0)$, we consider the following perturbed problem:

$$-\Delta u_\varepsilon = \lambda K(x)(u_\varepsilon+\varepsilon)^q - \frac{K(x)u_\varepsilon}{(u_\varepsilon+\varepsilon)^{1-r}} \quad \text{in } \Omega, \quad u_\varepsilon|_{\partial\Omega} = 0, \quad u_\varepsilon \geq 0 \quad \text{in } \Omega. \tag{$\mathrm{P}_{\lambda,\varepsilon}$}$$

Let us prove that $(\mathrm{P}_{\lambda,\varepsilon})$ admits a maximal solution. Observe that u_λ the maximal solution to (P_λ) constructed in Theorem 1 is a subsolution of $(\mathrm{P}_{\lambda,\varepsilon})$. To get a suitable supersolution of problem $(\mathrm{P}_{\lambda,\varepsilon})$, we consider the following problem:

$$-\Delta v = \lambda K(x)(v+\varepsilon)^q \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0, \quad v \geq 0 \quad \text{in } \Omega. \tag{$\overline{\mathrm{P}}_{\lambda,\varepsilon}$}$$

Proposition 6. *Problem* $(\overline{\mathrm{P}}_{\lambda,\varepsilon})$ *has a maximal solution* $\overline{u}_{\lambda,\varepsilon} \in C^{1,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$ *satisfying* $U_\lambda \le \overline{u}_{\lambda,\varepsilon} \le MV$ *in* Ω, *where* U_λ *and* V *are respectively defined in* (3.1) *and* (3.2) *and* $M > 0$ *is chosen large enough. Moreover, for* $0 < \varepsilon' \le \varepsilon < \varepsilon_0$, $\overline{u}_{\lambda,\varepsilon'} \le \overline{u}_{\lambda,\varepsilon}$ *in* Ω.

Proof. Proposition 6 follows from the lower- and upper-solution method. Indeed, U_λ is a subsolution of $(\overline{\mathrm{P}}_{\lambda,\varepsilon})$ independent of ε. Moreover, since V is bounded in Ω, there exists $C > 0$ independent of M and ε such that $\lambda(V + \varepsilon/M)^q \le C$ in Ω. Then,

$$-\Delta(MV) - K(x)(MV + \varepsilon)^q \ge M^q K(x)\left[M^{1-q} - C\right] \ge 0 \quad \text{in } \Omega,$$

for $M > 0$ large enough. Thus, MV is a supersolution to $\left(\overline{P}_{\lambda,\varepsilon}\right)$ and the existence of the maximal solution $\overline{u}_{\lambda,\varepsilon}$ follows. For $\varepsilon' \in (0, \varepsilon)$, $\overline{u}_{\lambda,\varepsilon'}$ is a subsolution of $(\overline{\mathrm{P}}_{\lambda,\varepsilon})$ such that $\overline{u}_{\lambda,\varepsilon'} \le MV$ in Ω. Therefore, from the maximality of the solution $\overline{u}_{\lambda,\varepsilon}$, the second inequality follows. □

Proposition 7. *Problem* $(P_{\lambda,\varepsilon})$ *has a maximal solution* $u_{\lambda,\varepsilon} \in C^{1,\alpha}(\overline{\Omega})$, *for some* $0 < \alpha < 1$, *such that* $u_\lambda \le u_{\lambda,\varepsilon} \le \overline{u}_{\lambda,\varepsilon}$ *in* Ω. *Moreover for* $0 < \varepsilon' \le \varepsilon < \varepsilon_0$, *we have* $u_{\lambda,\varepsilon'} \le u_{\lambda,\varepsilon}$ *in* Ω.

Proof. We consider the following iterative scheme:

$$-\Delta u_\varepsilon^n + \frac{K(x)u_\varepsilon^n}{(u_\varepsilon^{n-1} + \varepsilon)^{1-r}} = \lambda K(x)(u_\varepsilon^{n-1} + \varepsilon)^q \quad \text{in } \Omega; \quad u_\varepsilon^n|_{\partial\Omega} = 0, \quad u_\varepsilon^n \ge 0 \quad \text{in } \Omega, \qquad (\mathrm{P}^n_{\lambda,\varepsilon})$$

with $u_\varepsilon^0 = \overline{u}_{\lambda,\varepsilon}$. By induction on n, $(\mathrm{P}^n_{\lambda,\varepsilon})$ admits a unique solution $u_\varepsilon^n \in C^2(\Omega) \cap C(\overline{\Omega})$. Indeed, for $n = 1$ we get a solution u_ε^1 of $(\mathrm{P}^1_{\lambda,\varepsilon})$ as a minimizer of the functional E_1 defined for all $v \in \mathrm{H}_0^1(\Omega)$ by

$$E_1(v) := \int_\Omega |\nabla v|^2 \, dx + \frac{1}{2}\int_\Omega \frac{K(x)v^2}{(\overline{u}_{\lambda,\varepsilon} + \varepsilon)^{1-r}}\, dx - \lambda \int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon} + \varepsilon)^q v \, dx.$$

Moreover,

$$-\Delta\left(u_\varepsilon^1 - \overline{u}_{\lambda,\varepsilon}\right) + K(x)\left[\frac{u_\varepsilon^1 - \overline{u}_{\lambda,\varepsilon}}{(\overline{u}_{\lambda,\varepsilon} + \varepsilon)^{1-r}}\right] \le 0$$

in $\mathrm{H}^{-1}(\Omega)$. Then by the weak maximum principle, $u_\varepsilon^1 \le \overline{u}_{\lambda,\varepsilon}$ in Ω. And similarly, $u_\lambda \le u_\varepsilon^1$ in Ω. Now, let $n \in \mathbb{N}^*$. By the same method we prove the existence of u_ε^n solution of $(\mathrm{P}^n_{\lambda,\varepsilon})$ such that $u_\lambda \le u_\varepsilon^n \le \overline{u}_{\lambda,\varepsilon}$ in Ω. Moreover, we have

$$-\Delta\left(u_\varepsilon^{n+1} - u_\varepsilon^n\right) + K(x)\left[\frac{u_\varepsilon^{n+1}}{(u_\varepsilon^n + \varepsilon)^{1-r}} - \frac{u_\varepsilon^n}{(u_\varepsilon^{n-1} + \varepsilon)^{1-r}}\right] = K(x)\left[(u_\varepsilon^n)^q - (u_\varepsilon^{n-1})^q\right] \quad \text{in } \mathrm{H}^{-1}(\Omega).$$

So choosing $(u_\varepsilon^{n+1} - u_\varepsilon^n)^+ \in \mathrm{H}_0^1(\Omega)$, we get

$$\int_\Omega K(x)\left[(u_\varepsilon^n)^q - (u_\varepsilon^{n-1})^q\right](u_\varepsilon^{n+1} - u_\varepsilon^n)^+ \, dx \le 0$$

and

$$\int_\Omega K(x)\left[\frac{u_\varepsilon^{n+1}}{(u_\varepsilon^n+\varepsilon)^{1-r}}-\frac{u_\varepsilon^n}{(u_\varepsilon^{n-1}+\varepsilon)^{1-r}}\right](u_\varepsilon^{n+1}-u_\varepsilon^n)^+\,dx \geq \int_\Omega K(x)\left[\frac{u_\varepsilon^{n+1}-u_\varepsilon^n}{(u_\varepsilon^n+\varepsilon)^{1-r}}\right](u_\varepsilon^{n+1}-u_\varepsilon^n)^+\,dx \geq 0.$$

Hence finally, for any $n \in \mathbb{N}^*$, we get $u_\lambda \leq u_\varepsilon^{n+1} \leq u_\varepsilon^n \leq \overline{u}_{\lambda,\varepsilon}$ in Ω. For all $x \in \Omega$, we define $u_{\lambda,\varepsilon}(x) = \lim_{n\to\infty} u_\varepsilon^n(x)$. We also have for all $n \in \mathbb{N}^*$,

$$\int_\Omega \left|\nabla u_{\lambda,\varepsilon}^n\right|^2\,dx \leq \int_\Omega K(x)(u_\varepsilon^{n-1}+\varepsilon)^q u_\varepsilon^n\,dx \leq \int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^q\overline{u}_{\lambda,\varepsilon}\,dx < +\infty.$$

Hence, $(u_\varepsilon^n)_{n\in\mathbb{N}^*}$ is bounded in $\mathrm{H}_0^1(\Omega)$. Therefore, $u_{\lambda,\varepsilon} \in \mathrm{H}_0^1(\Omega)$ and up to a subsequence denoted in the same way, $u_\varepsilon^n \xrightarrow[n\to+\infty]{} u_{\lambda,\varepsilon}$ in $\mathrm{H}_0^1(\Omega)$ and a.e. in Ω. So passing to the limit in $(\mathrm{P}_{\lambda,\varepsilon}^n)$, $u_{\lambda,\varepsilon}$ is a weak solution of $(\mathrm{P}_{\lambda,\varepsilon})$ satisfying the first inequality of the statement. Finally, the $C^{1,\alpha}(\overline{\Omega})$ regularity of $u_{\lambda,\varepsilon}$ follows from Gui-Lin [8, Theorem 1.1]. Now, let $\varepsilon' \in (0,\varepsilon)$. Then, for $n = 1$ we have

$$-\Delta\left(u_{\varepsilon'}^1-u_\varepsilon^1\right)+K(x)\left[\frac{u_{\varepsilon'}^1-u_\varepsilon^1}{(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^{1-r}}\right] \leq \lambda K(x)\left[(\overline{u}_{\lambda,\varepsilon}+\varepsilon')^q-(\overline{u}_{\lambda,\varepsilon}+\varepsilon)^q\right] \leq 0.$$

Then by the weak maximum principle, $u_{\varepsilon'}^1 \leq u_\varepsilon^1$ in Ω. For $n \in \mathbb{N}^*$, by induction we have

$$\begin{aligned} &-\Delta(u_{\varepsilon'}^n-u_\varepsilon^n)+K(x)\left[\frac{u_{\varepsilon'}^n-u_\varepsilon^n}{(u_\varepsilon^{n-1}+\varepsilon)^{1-r}}\right] \\ &\quad\leq -\Delta(u_{\varepsilon'}^n-u_\varepsilon^n)+K(x)\left[\frac{u_{\varepsilon'}^n}{(u_{\varepsilon'}^{n-1}+\varepsilon')^{1-r}}-\frac{u_\varepsilon^n}{(u_\varepsilon^{n-1}+\varepsilon)^{1-r}}\right] \\ &\quad= \lambda K(x)\left[(u_{\varepsilon'}^{n-1}+\varepsilon')^q-(u_\varepsilon^{n-1}+\varepsilon)^q\right] \leq 0. \end{aligned}$$

Hence, $u_{\varepsilon'}^n \leq u_\varepsilon^n$ in Ω and passing to the limit as $n \to +\infty$, we finally get the second inequality. □

4.2. Semi-stability of the maximal solution $u_{\lambda,\varepsilon}$

Let $u_{\lambda,\varepsilon}$ be the maximal solution of $(\mathrm{P}_{\lambda,\varepsilon})$ obtained above and let us define the first eigenvalue of the linearised operator associated to $(\mathrm{P}_{\lambda,\varepsilon})$: $\Lambda_\varepsilon(\lambda) := \inf\{\mathcal{E}_{\lambda,\varepsilon}(v) \mid v \in \mathrm{H}_0^1(\Omega),\ \|v\|_{\mathrm{L}^2(\Omega)} = 1\}$, where $\mathcal{E}_{\lambda,\varepsilon}(v)$ is defined for all $v \in \mathrm{H}_0^1(\Omega)$ by

$$\begin{aligned} \mathcal{E}_{\lambda,\varepsilon}(v) := &\int_\Omega |\nabla v|^2\,dx - \lambda q\int_\Omega \frac{K(x)v^2}{(u_{\lambda,\varepsilon}+\varepsilon)^{1-q}}\,dx \\ &+\int_\Omega \frac{K(x)v^2}{(u_{\lambda,\varepsilon}+\varepsilon)^{1-r}}\,dx+(r-1)\int_\Omega \frac{K(x)u_{\lambda,\varepsilon}v^2}{(u_{\lambda,\varepsilon}+\varepsilon)^{2-r}}\,dx. \end{aligned} \tag{4.2}$$

Proposition 8. *There exists $\Phi_\varepsilon \in \mathcal{H} := \{v \in \mathrm{H}^1_0(\Omega) \mid \|v\|_{\mathrm{L}^2} = 1\}$, non-negative a.e. in Ω such that $\mathcal{E}_{\lambda,\varepsilon}(\Phi_\varepsilon) = \min_{v\in\mathcal{H}} \mathcal{E}_{\lambda,\varepsilon}(v)$. Hence, $\Phi_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, and satisfies*

$$-\Delta\Phi_\varepsilon = \Lambda_\varepsilon(\lambda)\Phi_\varepsilon + f'_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})\Phi_\varepsilon \quad \text{in } \Omega, \quad \Phi_\varepsilon|_{\partial\Omega} = 0, \quad \Phi_\varepsilon \geq 0 \quad \text{in } \Omega, \tag{4.3}$$

where for any $v \in \mathrm{H}^1_0(\Omega)$, $f_{\lambda,\varepsilon}(v) := \lambda K(x)(v + \varepsilon)^q - K(x)v/(v + \varepsilon)^{1-r}$.

Proof. For sake of clarity, we denote in (4.2), $\mathcal{E}_{\lambda,\varepsilon}(v) := \|v\|^2_{\mathrm{H}^1_0(\Omega)} - \lambda q I^\varepsilon_1(v) + I^\varepsilon_2(v) + (r-1)I^\varepsilon_3(v)$. Using Hardy's inequality, we get a similar estimate to (4.1) for $\mathcal{E}_{\lambda,\varepsilon}$ and $\Lambda_\varepsilon(\lambda) \in \mathbb{R}$. So, let $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ be an associated minimizing sequence. We have, $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$ (see (4.1)). Therefore, there exist $\Phi_\varepsilon \in \mathcal{H}$ and a subsequence still denoted $(v_n)_{n\in\mathbb{N}}$ such that $v_n \xrightarrow[n\to+\infty]{} \Phi_\varepsilon$ weakly in $\mathrm{H}^1_0(\Omega)$ and strongly in $\mathrm{L}^2(\Omega)$ and a.e. in Ω. Then we get $\|\Phi_\varepsilon\|_{\mathrm{H}^1_0(\Omega)} \leq \liminf_{n\to+\infty} \|v_n\|_{\mathrm{H}^1_0(\Omega)}$. Moreover, by Hardy's inequality $(v_n/d(x))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$; therefore up to a subsequence $v_n/d(x) \xrightarrow[n\to+\infty]{} \Phi_\varepsilon/d(x)$ in $\mathrm{L}^2(\Omega)$. Then, writing

$$I^\varepsilon_1(v_n) = \int_\Omega K(x)d(x)(u_{\lambda,\varepsilon} + \varepsilon)^{q-1}\,(v_n/d(x))\,v_n\,dx$$

we get that $I^\varepsilon_1(v_n) \xrightarrow[n\to+\infty]{} I^\varepsilon_1(\Phi_\varepsilon)$. And similarly, $I^\varepsilon_2(v_n) \xrightarrow[n\to+\infty]{} I^\varepsilon_2(\Phi_\varepsilon)$ and $I^\varepsilon_3(v) \xrightarrow[n\to+\infty]{} I^\varepsilon_3(\Phi_\varepsilon)$. Hence, $\mathcal{E}_{\lambda,\varepsilon}(\Phi_\varepsilon) \leq \liminf_{n\to+\infty} \mathcal{E}_{\lambda,\varepsilon}(v_n)$ and $\mathcal{E}_{\lambda,\varepsilon}(\Phi_\varepsilon) = \min_{v\in\mathcal{H}} \mathcal{E}_{\lambda,\varepsilon}(v) = \Lambda_\varepsilon(\lambda)$. Since for all $v \in \mathcal{H}$, $\mathcal{E}_{\lambda,\varepsilon}(v) = \mathcal{E}_{\lambda,\varepsilon}(|v|)$, we can assume $\Phi_\varepsilon \geq 0$ a.e in Ω. From variational arguments, Φ_ε is a weak solution to (4.3). Finally the $C^{1,\alpha}(\overline{\Omega})$ Hölder regularity of Φ_ε follows from Gui-Lin [8, Theorem 1.1]. □

Proposition 9. *Let $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$. Then, the solution $u_{\lambda,\varepsilon}$ of $(P_{\lambda,\varepsilon})$ is semi-stable.*

Proof. Let us argue by contradiction. Suppose that $\Lambda_\varepsilon(\lambda) < 0$. Let $\varepsilon' > 0$ and consider $\underline{u}_{\lambda,\varepsilon} := u_{\lambda,\varepsilon} + \varepsilon'\Phi_\varepsilon$. Then, we have

$$-\Delta\underline{u}_{\lambda,\varepsilon} = f_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) + \varepsilon' f'_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})\Phi_\varepsilon + \varepsilon'\Lambda_\varepsilon(\lambda)\Phi_\varepsilon \quad \text{in } \Omega,$$

with $f_{\lambda,\varepsilon}$ defined in (4.3). And by a Taylor-Lagrange expansion

$$f_{\lambda,\varepsilon}(\underline{u}_{\lambda,\varepsilon}) = f_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) + \varepsilon' f'_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})\Phi_\varepsilon + \frac{1}{2}(\varepsilon'\Phi_\varepsilon)^2 f''_{\lambda,\varepsilon}(u_{\lambda,\varepsilon} + \theta\varepsilon'\Phi_\varepsilon) \quad \text{in } \Omega,$$

with $\theta \in (0, 1)$ and

$$\forall v \in H^1_0(\Omega),\ f''_{\lambda,\varepsilon}(v) = \lambda q(q-1)\frac{K(x)}{(v+\varepsilon)^{2-q}} + r(1-r)\frac{K(x)}{(v+\varepsilon)^{2-r}} - \varepsilon(1-r)(2-r)\frac{K(x)}{(v+\varepsilon)^{3-r}}.$$

By Theorem 2.1 in Gui-Lin [8], Φ_ε behaves like the distance function in Ω, therefore there exists a positive constant C independent of θ and ε' such that $\left|\Phi^2_\varepsilon f''_{\lambda,\varepsilon}(u_{\lambda,\varepsilon} + \theta\varepsilon'\Phi_\varepsilon)\right| \leq C$ in Ω. So, choosing ε' small enough,

$$\varepsilon'\Lambda_\varepsilon(\lambda)\Phi_\varepsilon < \frac{1}{2}(\varepsilon'\Phi_\varepsilon)^2 f''_{\lambda,\varepsilon}(u_{\lambda,\varepsilon} + \theta\varepsilon'\Phi_\varepsilon) \quad \text{in } \Omega$$

and $\underline{u}_{\lambda,\varepsilon}$ is a subsolution of $(P_{\lambda,\varepsilon})$. Moreover, using the Brézis-Nirenberg [2] strong maximum principle, $u_{\lambda,\varepsilon} < \overline{u}_{\lambda,\varepsilon_0}$ in Ω and since $\Phi_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$, for $\varepsilon' > 0$ sufficiently small, we have $\underline{u}_{\lambda,\varepsilon} \le \overline{u}_{\lambda,\varepsilon_0}$ in Ω. Hence, using the same sub and supersolution technique, we get the existence of $\tilde{u}_{\lambda,\varepsilon}$ weak solution of $(P_{\lambda,\varepsilon})$ such that $\underline{u}_{\lambda,\varepsilon} \le \tilde{u}_{\lambda,\varepsilon} \le \overline{u}_{\lambda,\varepsilon_0}$ in Ω, which contradicts the maximality of $u_{\lambda,\varepsilon}$ in Proposition 7. □

4.3. Semi-stability of the solution u_λ for $\lambda \geq \Lambda_1$

To prove the semi-stability of the maximal solution u_λ of (P_λ), we pass to the limit as $\varepsilon \to 0^+$. Indeed, from Proposition 6 and Proposition 7 let us define for all $x \in \Omega$, $\tilde{U}_\lambda(x) := \lim_{\varepsilon\to 0+} \overline{u}_{\lambda,\varepsilon}(x)$ and $\tilde{u}_\lambda(x) := \lim_{\varepsilon\to 0+} u_{\lambda,\varepsilon}(x)$. Then, passing to the limit in the inequality proved in Proposition 7, we get $u_\lambda \le \tilde{u}_\lambda \le \tilde{U}_\lambda$ in Ω. We also have, for all $\varepsilon \in (0,\varepsilon_0)$ and all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla\overline{u}_{\lambda,\varepsilon}.\nabla v\, dx = \lambda \int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon} + \varepsilon)^q v\, dx. \tag{4.4}$$

So choosing $\overline{u}_{\lambda,\varepsilon}$ as test function, we get

$$\int_\Omega \left|\nabla\overline{u}_{\lambda,\varepsilon}\right|^2 dx = \lambda \int_\Omega K(x)(\overline{u}_{\lambda,\varepsilon} + \varepsilon)^q \overline{u}_{\lambda,\varepsilon}\, dx \le \lambda M \int_\Omega K(x)(MV + \varepsilon_0)^q V\, dx < +\infty.$$

Then, $(\overline{u}_{\lambda,\varepsilon})_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$. So, up to a subsequence, passing to the limit as $\varepsilon \to 0^+$ in the first inequality of Proposition 6 and in (4.4), $\tilde{U}_\lambda$ is a weak solution of $(\overline{P}_\lambda)$ satisfying $U_\lambda \le \tilde{U}_\lambda \le MV$ in Ω. Hence, by uniqueness of a such solution of $(\overline{P}_\lambda)$, $U_\lambda = \tilde{U}_\lambda$ in Ω. Similarly, $(u_{\lambda,\varepsilon})_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$ and then $\tilde{u}_\lambda$ is a weak solution to (P_λ) such that $u_\lambda \le \tilde{u}_\lambda \le U_\lambda$ in Ω. Hence, since u_λ is a maximal solution, it follows that $\tilde{u}_\lambda \equiv u_\lambda$ in Ω. So finally, since $\Lambda_\varepsilon(\lambda) \ge 0$, $\mathcal{E}_{\lambda,\varepsilon}(v) \ge 0$ for all $v \in \mathcal{H}$. With the notations used in the previous proof,

$$I_1^\varepsilon(v) \xrightarrow[\varepsilon\to 0^+]{} \int_\Omega K(x)u_\lambda^{r-1}v^2\, dx \quad \text{and} \quad I_2^\varepsilon(v) + (r-1)I_3^\varepsilon(v) \xrightarrow[\varepsilon\to 0^+]{} r\int_\Omega K(x)u_\lambda^{q-1}v^2\, dx.$$

Hence, $\Lambda(\lambda) = \lim_{\varepsilon\to 0^+} \Lambda_\varepsilon(\lambda) \ge 0$, which proves the semi-stability of u_λ. Moreover, by inequality (3.5) and Dini's Theorem, $u_\lambda \xrightarrow[\lambda\to\Lambda_1^+]{} u_{\Lambda_1}$ in $L^\infty(\Omega)$. So, we also have $\Lambda(\Lambda_1) \ge 0$.

4.4. Stability of u_λ for $\lambda > \Lambda_1$

Finally, let us prove that $\Lambda(\lambda) > 0$ for $\lambda > \Lambda_1$. For that we introduce the following new perturbed problem:

$$-\Delta u = K(x)(\lambda u^q - u^r + \theta) \quad \text{in } \Omega; \quad u|_{\partial\Omega} = 0, \quad u \ge 0 \quad \text{in } \Omega, \tag{P_λ^θ}$$

with $\theta \in \mathbb{R}$. As above, we first show the existence of a branch of maximal solutions denoted $u_\lambda^\theta \in C^{1,\alpha}(\overline{\Omega})$ to problem (P_λ^θ) for $\lambda > \Lambda_{1,\theta}$, where $\Lambda_{1,\theta} := \inf\{\lambda > 0 \mid (P_\lambda^\theta)$ has a positive solution a.e. in $\Omega\}$. As above, we have $\Lambda^\theta(\lambda) := \inf_{v\in\mathcal{H}} \mathcal{E}^\theta(v) \ge 0$, with $v \in H_0^1(\Omega)$ and

$$\mathcal{E}^\theta(v) := \int_\Omega |\nabla u|^2\, dx + r\int_\Omega K(x)(u_\lambda^\theta)^{r-1}u^2\, dx - \lambda q\int_\Omega K(x)(u_\lambda^\theta)^{q-1}u^2\, dx.$$

Lemma 10. *Assume $\lambda > \Lambda_1$. Then,*

1. *there exists $\theta_0 < 0$ such that $u_\lambda^{\theta_0} > 0$ a.e. in Ω;*
2. *the mapping $\theta \longmapsto \Lambda^\theta(\lambda)$ is increasing on $(\theta_0, +\infty)$.*

Proof. By Proposition 2, for $\lambda > \Lambda_1$ there exist two constants $C_1, C_2 > 0$ depending on λ such that, for all $x \in \Omega$, $C_1 d(x) \leq u_\lambda(x) \leq C_2 d(x)$. Then, let us choose $\lambda' \in (\Lambda_1, \lambda)$ and ε small enough to satisfy $(\lambda/\lambda')u_{\lambda'} \geq u_{\lambda'} + \varepsilon V$ and $[\lambda/\lambda' - (\lambda/\lambda')^{1/r}]u_{\lambda'} \geq \varepsilon V$ a.e. in Ω, with V solution to (3.2). Defining $\underline{w} := (\lambda/\lambda')u_{\lambda'} - \varepsilon V$, we get $-\Delta \underline{w} \leq K(x)(\lambda \underline{w}^q - \underline{w}^r - \varepsilon)$ in Ω and as in the proof of Proposition 5, we prove the existence of $w \in C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, solution to

$$-\Delta w = K(x)(\lambda w^q - w^r - \varepsilon) \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0.$$

To complete the proof of point 1 of this lemma, it suffices to choose $\theta_0 \in (-\varepsilon, 0)$. The second assertion follows from the strong maximum principle from which we get that, for $\theta < \theta'$ the positive maximal solutions to (P_λ^θ) and $(P_\lambda^{\theta'})$ satisfy $u_\lambda^\theta < u_\lambda^{\theta'}$ in Ω. Then, noting that as previously, or every $\theta \in (\theta_0, +\infty)$, the infimum

$$\Lambda^\theta(\lambda) := \inf_{u \in \mathcal{H}} \left\{ \int_\Omega |\nabla u|^2\, dx + r \int_\Omega K(x) \left(u_\lambda^\theta\right)^{r-1} u^2\, dx - \lambda q \int_\Omega K(x) \left(u_\lambda^\theta\right)^{q-1} u^2\, dx \right\}$$

is achieved for an element $\Phi^\theta \in H_0^1(\Omega)$, we finally get $\Lambda^{\theta'}(\lambda) > \Lambda^\theta(\lambda)$. □

Thanks to this lemma, $\Lambda(\lambda) = \Lambda^0(\lambda) > \Lambda^{-\varepsilon}(\lambda) \geq 0$, which completes the proof.

Remark 2. When $\lambda > \Lambda_1$, u_λ is the unique positive and semi-stable solution of (P_λ). Indeed, let us suppose there exists another positive and semi stable solution $v_\lambda \in H_0^1(\Omega)$, therefore by the strong maximum principle $v_\lambda < u_\lambda$ in Ω. By hypothesis, for every $u \in H_0^1(\Omega)$,

$$\int_\Omega K(x) \left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right) u^2\, dx \leq \int_\Omega |\nabla u|^2\, dx.$$

Choosing $u = u_\lambda - v_\lambda \in H_0^1(\Omega)$ this estimate becomes

$$\int_\Omega K(x) \left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right) (u_\lambda - v_\lambda)^2\, dx \leq \int_\Omega |\nabla(u_\lambda - v_\lambda)|^2\, dx. \tag{4.5}$$

Since, u_λ and v_λ both are solution to (P_λ), then we also have

$$\int_\Omega K(x) \left[\lambda \left(u_\lambda^q - v_\lambda^q\right) - (u_\lambda^r - v_\lambda^r)\right] (u_\lambda - v_\lambda)\, dx = \int_\Omega |\nabla(u_\lambda - v_\lambda)|^2\, dx. \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\int_\Omega K(x)(u_\lambda - v_\lambda) \left\{ \left[\left(\lambda u_\lambda^q - u_\lambda^r\right) - \left(\lambda u_\lambda^q - u_\lambda^r\right)\right] - \left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right)(u_\lambda - v_\lambda) \right\} dx \geq 0,$$

which contradicts $(u_\lambda - v_\lambda) > 0$ in Ω because, by concavity of $t \mapsto \lambda t^q - t^r$,

$$\left[\left(\lambda u_\lambda^q - u_\lambda^r\right) - \left(\lambda u_\lambda^q - u_\lambda^r\right)\right] - \left(\lambda q v_\lambda^{q-1} - r v_\lambda^{r-1}\right)(u_\lambda - v_\lambda) \leq 0 \quad \text{in } \Omega.$$

Therefore u_λ is the unique solution among the positive and semi-stable solutions of (P_λ).

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