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ON THE UNIFORM CONVERGENCE OF SINGULARLY PERTURBED REACTION–DIFFUSION PROBLEMS WITH NON–SMOOTH DATA

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Abstract. In this paper singularly perturbed parabolic problems of reaction–diffusion type with a discontinuity between the initial and boundary conditions are considered. We approximate the solution of this class of problems with a fitted operator method defined on special meshes of Shishkin and Bakhvalov type. Global approximations of the solution are constructed with a non-linear interpolation operator and some numerical results are given showing the parameter–uniform convergence of the numerical methods.

Keywords: Singular perturbation, discontinuous data, Bakhvalov and Shishkin meshes, nodal and global convergence.

AMS classification: 65L11, 65L12.

§1. Introduction

In this paper we consider the following class of singularly perturbed problems

$$u_t - \varepsilon u_{xx} + a(t)u = f(x,t),\ G := (x,t) \in (0,1) \times (0,1], \tag{1.1a}$$

$$u(x,0) = \phi_B(x),\ u(0,t) = \phi_L(t),\ u(1,t) = \phi_R(t), \tag{1.1b}$$

$$a \geq 0, \quad \phi_L(0) \neq \phi_B(0), \tag{1.1c}$$

where ε, the singular perturbation parameter, is a positive parameter which can take arbitrary small values and ϕ_B, ϕ_L and ϕ_R are given functions (the initial and boundary values of the function u). The solution of the class of problems (1.1) is characterized by the singular nature of the operator and the classical singularity caused by the discontinuity at the corner $(0,0)$ where the initial and boundary conditions intersect. So, large derivatives can appear in narrow subregions of the domain, called layer regions, and then classical numerical methods are not appropriate to numerically solve singularly perturbed equations [2, 6].

In [4] and [5] the asymptotic behavior associated with $\phi_L(0) \neq \phi_B(0)$ is shown to be related to the following function

$$(\phi_B(0) - \phi_L(0))w(x,t)e^{-\int_0^t a(s)ds},$$

where

$$w(x,t) := \frac{1}{2}\mathrm{erf}(\frac{x}{2\sqrt{\varepsilon t}}), \qquad \mathrm{erf}(\zeta) := \frac{2}{\sqrt{\pi}}\int_0^{\zeta} \exp(-\alpha^2)\, d\alpha. \tag{1.2}$$

For $t = 0$, in $x = 0$, the error function is defined by continuous extension.

Based on this information, in [5] the authors prove the uniform nodal convergence of a special finite difference method by using a fitted operator method defined on a uniform mesh where the fitting coefficient is chosen in such a way that the discrete scheme is exact for the parameter–dependent error function $w(x,t)$.

Nevertheless, global approximations of the solution in the whole domain are required due to the multiscale character of the solution. The global convergence of the numerical method given by Hemker and Shishkin [5] was analyzed in [3]; where it was reported that a uniformly global convergent approximation can be obtained from a nodal approximation on a mesh of Shishkin type [2, 6] and using a suitable nonlinear interpolation, which is exact for the error function. On the other hand, this nonlinear interpolation and the approximation of the solution on a uniform mesh do not provide, in general, a global approximation. Therefore, the use of a special mesh condensing in the layer region and a special nonlinear interpolation are crucial to generate global approximations of the solution.

In this paper we focus our analysis in a different approach that in [3] where the numerical scheme proposed by Hemker and Shishkin [5] is defined on a mesh of Bakhvalov type [1] instead of a Shishkin mesh. With this purpose, in Section §2 we recall the method proposed by Hemker and Shishkin in [5] on a uniform mesh and it is extended to a mesh of Bakhvalov type. In Section §3 we display the numerical results for a test problem and they suggest that this graded mesh provides nodal approximations of the solution. In addition, a global approximation is constructed from the nodal values of the solution defined on the Bakhvalov mesh and using an appropriate nonlinear interpolation. In summary, similar conclusions are deduced for both kind of meshes which are widely used in the context of singularly perturbed problems.

§2. The fitted mesh method

In this section we firstly recall the fitted operator method given in [5] on the uniform mesh

$$\overline{G}_u^{N,M} := \{x_i\} \times \{t_j\} = \{\frac{i}{N}\}_{i=0}^{N} \times \{\frac{j}{M}\}_{j=0}^{M}, \tag{2.1a}$$

where N and M are the space and time discretization parameters. The fitting difference scheme is given by

$$D_t^- U - \varepsilon\kappa(x,t)\delta_x^2 U + a(t)U = f(x,t), \quad (x,t) \in G_u^{N,M}, \tag{2.1b}$$

$$U = u, \quad (x,t) \in \Gamma_u^{N,M}, \tag{2.1c}$$

where

$$G_u^{N,M} := \overline{G}_u^{N,M} \cap G, \quad \Gamma_u^{N,M} := \overline{G}_u^{N,M} \cap (\overline{G} \setminus G),$$

and

$$D_t^-\Upsilon(x_i,t_j) := \frac{\Upsilon(x_i,t_j)-\Upsilon(x_i,t_{j-1})}{t_j-t_{j-1}},\ D_x^-\Upsilon(x_i,t_j) := \frac{\Upsilon(x_i,t_j)-\Upsilon(x_{i-1},t_j)}{x_i-x_{i-1}},$$

$$D_x^+\Upsilon(x_i,t_j) := \frac{\Upsilon(x_{i+1},t_j)-\Upsilon(x_i,t_j)}{x_{i+1}-x_i},$$

$$\delta_x^2\Upsilon(x_i,t_j) := \frac{2}{h_i+h_{i+1}}(D_x^+\Upsilon(x_i,t_j)-D_x^-\Upsilon(x_i,t_j)).$$

The fitting coefficient κ is specified by (see [5] for more details)

$$\kappa(x,t) = \frac{D_t^- w_0 + D_t^- u_0}{\varepsilon\delta_x^2 w_0 + \varepsilon\delta_x^2 u_0}, \quad (x,t)\in G_u^{N,M}, \tag{2.1d}$$

with $u_0 = -x^3 - 6\varepsilon xt$. In [5] it was proved the following result of convergence

$$\|u(x,t)-U(x,t)\|_{\overline{G}_u^{N,M}} \le C((h+\tau)^\nu + \frac{\tau^{3/2}}{h}),$$

for any $\nu \in (0,1/3)$ with $h = 1/N$ and $\tau = 1/M$, and C is a constant independent of the singular perturbation and discretizations parameters.

In general, the solution can exhibit a layer region at the edges $x = 0,1$ and, for this reason, in [3] the uniform mesh $\overline{G}_u^{N,M}$ defined in (2.1a) was replaced by the mesh

$$\overline{G}_S^{N,M} := \{x_i\}\times\{t_j\},$$

where the mesh in time is also uniform but the mesh in space is a piecewise uniform mesh of Shishkin type. The interval $[0,1]$ is split in three subintervals

$$[0,\sigma]\cup[\sigma,1-\sigma]\cup[1-\sigma,1],$$

with

$$\sigma := \min\{\frac{1}{4}, 2\sqrt{\varepsilon}\ln N\}, \tag{2.2}$$

and $x_0 = 0$, $x_{N/4} = \sigma$, $x_{3N/4} = 1-\sigma$, and $x_N = 1$.

The fitting coefficient κ was defined in the subdomains $(0,\sigma)$ $(\sigma,1-\sigma)$ and $(1-\sigma,1)$, where the mesh is uniform, as in (2.1d). At the transition points σ and $1-\sigma$, the fitting coefficient κ was specially defined since the Shishkin mesh is, in general, very anisotropic. The coefficient $\kappa := \kappa(x_{N/4},t_j)$ at the transition point $(x_{N/4},t_j)$ is computed by using linear interpolation based on the values $\kappa(x_{N/4-1},t_j)$ and $\kappa(x_{N/4+1},t_j)$, i.e,

$$\kappa(x_{N/4},t_j) = \frac{(x_{N/4+1}-x_{N/4})\kappa(x_{N/4-1},t_j) + (x_{N/4}-x_{N/4-1})\kappa(x_{N/4+1},t_j)}{x_{N/4+1}-x_{N/4-1}}.$$

At the other transition point $x = 1-\sigma$ is similarly defined.

The Bakhvalov mesh [1]

$$\overline{G}_B^{N,M} := \{x_i\}\times\{t_j\},$$

is uniform in the time variable variable and it condenses in the layer regions $x = 0, 1$. In particular, it is uniform outside of the layers regions and it is a graded mesh in the layer regions. The grid points x_i are defined by means of a mesh generating function $x_i = \varphi(t_i)$, where $t_i = i/N$, $i = 0, \dots, N$, and

$$\varphi(t) = \begin{cases} \chi(t) := -2\sqrt{\varepsilon}\ln\left(\dfrac{q-t}{q}\right), & \text{for } t \in [0, \tau], \\ \pi(t) := \chi(\tau) + \chi'(\tau)(t-\tau), & \text{for } t \in (\tau, 1/2], \end{cases}$$

with $q = 1/4$ and the point τ satisfies

$$\chi'(\tau) = \frac{1 - \chi(\tau)}{1 - \tau}.$$

In the subinterval $[1/2, 1]$ it is symmetrically defined. The fitting coefficient at the transition points of the Bakhvalov mesh is defined in a similar way to the Shishkin mesh.

§3. Numerical experiments

We consider the following variable coefficient problem

$$0.5(1 + e^{-2xt})u_t - \varepsilon u_{xx} + (1-x)^2 u = -(1-x)(2-x), \quad (x,t) \in (0,1) \times (0,1], \tag{3.1}$$

with the following initial and boundary conditions

$$u(x,0) = 1,\ x \in (0,1), \quad u(0,t) = 0,\ t \in [0,1],\ u(1,t) = 1,\ t \in [0,1].$$

Note that in this test problem there is a discontinuity between the initial and the boundary conditions at the corner $(0,0)$. In Figure 1 the numerical solution of problem (3.1) for $\varepsilon = 10^{-8}$ in the whole domain (left figure) and a zoom of the solution in the layer region (right figure) are given. Observe the boundary layers near both the edges $x = 0$ and $x = 1$ as well as the layer region caused by the discontinuity at the corner $(0,0)$ where the solution has steep gradients.

On the Shishkin $\overline{G}_S^{N,M}$ and Bakhvalov $\overline{G}_B^{N,M}$ meshes, we define the fitted operator method

$$0.5(1 + e^{-2xt})D_t^- U - \varepsilon\kappa\delta_x^2 U + (1-x)^2 U = -(1-x)(2-x), \tag{3.2}$$

where the fitting coefficient has been computed by using (2.1d) and, henceforth, it is independent of the coefficients of the differential equation (3.1).

The solution of problem (3.1) is unknown and the nodal errors are estimated by using the double mesh principle [2]: We compute the two-mesh nodal differences $d_\varepsilon^{N,M}$:

$$D_\varepsilon^{N,M} := \left\| U^{N,M} - \bar{U}^{2N,2M} \right\|_{\bar{G}^{N,M}},$$

where $U^{N,M}$, $U^{2N,2M}$ denote the discrete functions defined on the meshes $\bar{G}^{N,M}$ and $\bar{G}^{2N,2M}$, respectively; $\bar{U}^{2N,2M}$ is the bilinear interpolant of the numerical solution $U^{2N,2M}$. The computed orders of convergence $Q_\varepsilon^{N,M}$ are defined by

$$Q_\varepsilon^{N,M} := \log_2\left(\frac{D_\varepsilon^{N,M}}{D_\varepsilon^{2N,2M}}\right).$$

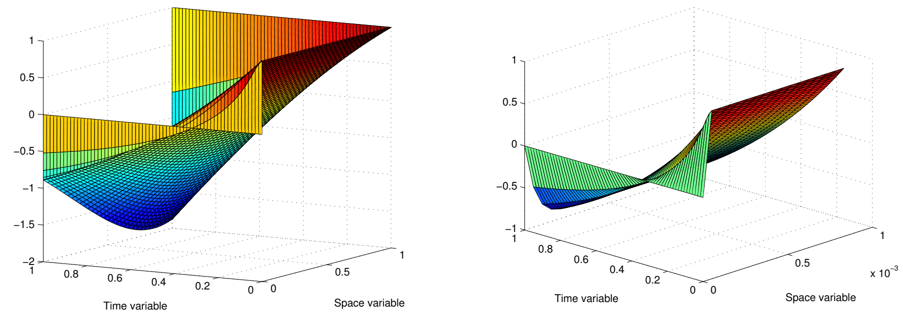

Figure 1: Numerical solution of problem (3.1) for $\varepsilon = 10^{-8}$ in the whole domain (left) and a detail of the solution in the layer region (right)

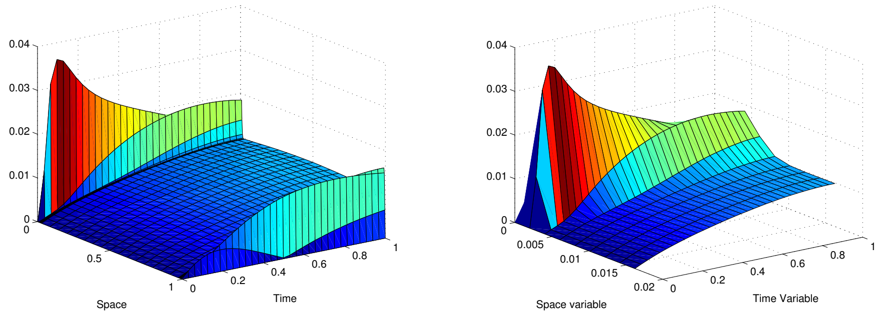

Figure 2: Two–mesh nodal difference surface for problem (3.1) using finite difference scheme (3.2) defined on the Shishkin mesh with $\varepsilon = 2^{-20}$, and $N = M = 32$ in the whole domain (left figure) and in the layer region (right figure).

The uniform nodal two-mesh differences and their corresponding orders of convergence are defined by

$$D^{N,M} := \max_{\varepsilon \in S_\varepsilon} D_\varepsilon^{N,M} \qquad Q^{N,M} := \log_2 \left(\frac{D^{N,M}}{D^{2N,2M}} \right),$$

where $S_\varepsilon = \{2^0, 2^{-2}, 2^{-4}, \ldots, 2^{-20}\}$.

In Figures 2 and 3 we display the two–mesh nodal differences of the finite difference scheme (3.2) defined on the Shishkin and Bakhvalov meshes, respectively, for the parameters settings $\varepsilon = 2^{-20}$ and $N = M = 32$. A zoom of the layer region is showed in the right figure; and we note the small scale in the vertical axis. From these figures, we observe that the maximum two–mesh differences for the test problem (3.1) occur along the vicinity of the edge $x = 0$.

In Tables 1 and 2 we show the two–mesh and uniform nodal differences for the test problem (3.1) associated to the finite difference scheme (3.2) defined on the Shishkin and Bakhvalov meshes, respectively. We observe that both the methods are uniformly nodal con-

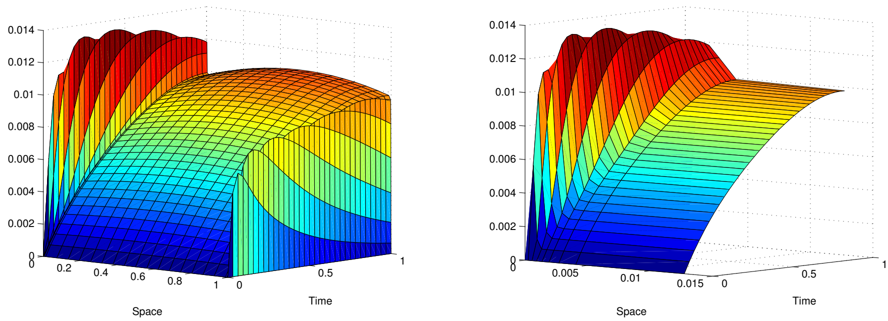

Figure 3: Two–mesh nodal difference surface for problem (3.1) using finite difference scheme (3.2) defined on the Bakhvalov mesh with $\varepsilon = 2^{-20}$, and $N = M = 32$ in the whole domain (left figure) and in the layer region (right figure).

	N=8 M=8	N=16 M=16	N=32 M=32	N=64 M=64	N=128 M=128	N=256 M=256	N=512 M=512
$\varepsilon = 2^{0}$	6.241E-002	3.318E-002	1.704E-002	8.625E-003	4.317E-003	2.186E-003	1.303E-003
	0.911	0.961	0.983	0.999	0.982	0.747	
$\varepsilon = 2^{-2}$	5.482E-002	2.770E-002	1.405E-002	7.094E-003	3.565E-003	1.786E-003	8.930E-004
	0.985	0.979	0.986	0.993	0.997	1.000	
$\varepsilon = 2^{-4}$	9.829E-002	2.862E-002	1.357E-002	6.662E-003	3.304E-003	1.646E-003	8.216E-004
	1.780	1.076	1.027	1.012	1.005	1.002	
$\varepsilon = 2^{-6}$	2.714E-001	9.473E-002	4.009E-002	2.563E-002	1.270E-002	6.549E-003	3.366E-003
	1.519	1.241	0.646	1.012	0.956	0.960	
$\varepsilon = 2^{-8}$	5.472E-001	2.827E-001	1.852E-001	9.267E-002	4.592E-002	2.858E-002	1.380E-002
	0.953	0.610	0.999	1.013	0.684	1.050	
$\varepsilon = 2^{-10}$	4.295E-001	3.372E-001	3.432E-001	3.293E-001	2.004E-001	9.753E-002	4.733E-002
	0.349	-0.025	0.060	0.716	1.039	1.043	
$\varepsilon = 2^{-12}$	4.288E-001	3.370E-001	3.295E-001	3.171E-001	2.654E-001	1.931E-001	1.233E-001
	0.348	0.033	0.055	0.257	0.459	0.647	
$\varepsilon = 2^{-14}$	4.286E-001	3.369E-001	3.295E-001	3.171E-001	2.654E-001	1.931E-001	1.233E-001
	0.347	0.032	0.055	0.257	0.459	0.647	
$\varepsilon = 2^{-16}$	4.284E-001	3.369E-001	3.295E-001	3.171E-001	2.654E-001	1.931E-001	1.233E-001
	0.347	0.032	0.055	0.257	0.459	0.647	
$\varepsilon = 2^{-18}$	4.284E-001	3.369E-001	3.295E-001	3.171E-001	2.654E-001	1.931E-001	1.233E-001
	0.347	0.032	0.055	0.257	0.459	0.647	
$\varepsilon = 2^{-20}$	4.284E-001	3.369E-001	3.295E-001	3.171E-001	2.654E-001	1.931E-001	1.233E-001
	0.347	0.032	0.055	0.257	0.459	0.647	
$D^{N,M}$	5.472E-001	3.372E-001	3.432E-001	3.293E-001	2.654E-001	1.931E-001	1.233E-001
$Q^{N,M}$	0.699	-0.025	0.060	0.311	0.459	0.647	

Table 1: Test problem (3.1): Two–mesh nodal differences $D_\varepsilon^{N,M}$ and $D^{N,M}$ and their computed orders of convergence $Q_\varepsilon^{N,M}$ and $Q^{N,M}$ for finite difference scheme (3.2) defined on the Shishkin mesh.

	N=8 M=8	N=16 M=16	N=32 M=32	N=64 M=64	N=128 M=128	N=256 M=256	N=512 M=512
$\varepsilon = 2^{0}$	6.241E-002	3.318E-002	1.704E-002	8.625E-003	4.317E-003	2.186E-003	1.303E-003
	0.911	0.961	0.983	0.999	0.982	0.747	
$\varepsilon = 2^{-2}$	5.482E-002	2.770E-002	1.405E-002	7.094E-003	3.565E-003	1.786E-003	8.930E-004
	0.985	0.979	0.986	0.993	0.997	1.000	
$\varepsilon = 2^{-4}$	5.096E-002	2.602E-002	1.306E-002	6.542E-003	3.275E-003	1.639E-003	8.198E-004
	0.970	0.994	0.998	0.998	0.999	0.999	
$\varepsilon = 2^{-6}$	6.593E-002	3.151E-002	1.464E-002	6.921E-003	3.346E-003	1.644E-003	8.150E-004
	1.065	1.106	1.081	1.049	1.025	1.012	
$\varepsilon = 2^{-8}$	7.355E-002	3.011E-002	1.352E-002	6.548E-003	3.248E-003	1.622E-003	8.118E-004
	1.289	1.155	1.046	1.011	1.002	0.999	
$\varepsilon = 2^{-10}$	7.049E-002	2.884E-002	1.346E-002	6.537E-003	3.244E-003	1.621E-003	8.114E-004
	1.289	1.099	1.042	1.011	1.001	0.998	
$\varepsilon = 2^{-12}$	6.820E-002	2.880E-002	1.343E-002	6.528E-003	3.240E-003	1.619E-003	8.106E-004
	1.244	1.100	1.041	1.011	1.001	0.998	
$\varepsilon = 2^{-14}$	6.706E-002	2.878E-002	1.341E-002	6.523E-003	3.237E-003	1.618E-003	8.100E-004
	1.220	1.101	1.040	1.011	1.001	0.998	
$\varepsilon = 2^{-16}$	6.640E-002	2.877E-002	1.340E-002	6.520E-003	3.236E-003	1.617E-003	8.097E-004
	1.206	1.102	1.040	1.011	1.001	0.998	
$\varepsilon = 2^{-18}$	6.597E-002	2.877E-002	1.340E-002	6.518E-003	3.235E-003	1.617E-003	8.096E-004
	1.197	1.102	1.040	1.011	1.001	0.998	
$\varepsilon = 2^{-20}$	6.756E-002	2.876E-002	1.340E-002	6.518E-003	3.235E-003	1.616E-003	8.095E-004
	1.232	1.102	1.039	1.011	1.001	0.998	
$D^{N,M}$	7.355E-002	3.318E-002	1.704E-002	8.625E-003	4.317E-003	2.186E-003	1.303E-003
$Q^{N,M}$	1.148	0.961	0.983	0.999	0.982	0.747	

Table 2: Test problem (3.1): Two–mesh nodal differences $D_\varepsilon^{N,M}$ and $D^{N,M}$ and their computed orders of convergence $Q_\varepsilon^{N,M}$ and $Q^{N,M}$ for finite difference scheme (3.2) defined on the Bakhvalov mesh.

vergent and we point out that the two–mesh differences obtained with the Bakhvalov mesh are smaller that those ones on the Shishkin mesh.

Next, we deal with the global approximation associated to the finite difference scheme (3.2). We define

$$d_{1,\varepsilon}^{N,M} := \left\|\bar{U}^{N,M} - \bar{U}^{2N,2M}\right\|_{\bar{G}^{N,M}\cup\bar{G}^{2N,2M}}, \quad d_1^{N,M} := \max_{\varepsilon\in S_\varepsilon} d_{1,\varepsilon}^{N,M},$$

where $U^{N,M}, U^{2N,2M}$ denote mesh functions defined, respectively, on the meshes $\bar{G}^{N,M}$ and $\bar{G}^{2N,2M}$; and $\bar{U}^{N,M}, \bar{U}^{2N,2M}$ the bilinear global functions. From these interpolated values we calculate computed orders of global convergence $q_{1,\varepsilon}^{N,M}$ and uniform computed orders of global convergence $q_1^{N,M}$ using

$$q_{1,\varepsilon}^{N,M} := \log_2\left(\frac{d_{1,\varepsilon}^{N,M}}{d_{1,\varepsilon}^{2N,2M}}\right), \quad q_1^{N,M} := \log_2\left(\frac{d_1^{N,M}}{d_1^{2N,2M}}\right).$$

In Table 3 we show the two–mesh global differences using bilinear interpolation for the test problem (3.1) associated to the finite difference scheme (3.2) defined on the Bakhvalov mesh.

	N=8 M=8	N=16 M=16	N=32 M=32	N=64 M=64	N=128 M=128	N=256 M=256	N=512 M=512
$\varepsilon = 2^{0}$	3.386E-001	3.788E-001	4.116E-001	4.365E-001	4.547E-001	4.679E-001	4.772E-001
	-0.162	-0.120	-0.085	-0.059	-0.041	-0.029	
$\varepsilon = 2^{-2}$	2.004E-001	2.655E-001	3.259E-001	3.740E-001	4.099E-001	4.359E-001	4.545E-001
	-0.406	-0.296	-0.198	-0.132	-0.089	-0.060	
$\varepsilon = 2^{-4}$	9.829E-002	9.344E-002	1.772E-001	2.576E-001	3.232E-001	3.730E-001	4.095E-001
	0.073	-0.923	-0.540	-0.328	-0.207	-0.135	
$\varepsilon = 2^{-6}$	2.893E-001	1.997E-001	9.225E-002	8.134E-002	1.725E-001	2.558E-001	3.226E-001
	0.535	1.114	0.182	-1.084	-0.569	-0.335	
$\varepsilon = 2^{-8}$	4.162E-001	2.493E-001	1.124E-001	7.297E-002	1.683E-001	2.541E-001	3.219E-001
	0.739	1.149	0.623	-1.206	-0.594	-0.341	
$\varepsilon = 2^{-10}$	3.972E-001	2.493E-001	1.124E-001	7.297E-002	1.683E-001	2.541E-001	3.219E-001
	0.672	1.149	0.623	-1.206	-0.594	-0.341	
$\varepsilon = 2^{-12}$	3.972E-001	2.493E-001	1.124E-001	7.297E-002	1.683E-001	2.541E-001	3.219E-001
	0.672	1.149	0.623	-1.206	-0.594	-0.341	
$\varepsilon = 2^{-14}$	3.972E-001	2.493E-001	1.124E-001	7.297E-002	1.683E-001	2.541E-001	3.219E-001
	0.672	1.149	0.623	-1.206	-0.594	-0.341	
$\varepsilon = 2^{-16}$	3.972E-001	2.493E-001	1.124E-001	7.297E-002	1.683E-001	2.541E-001	3.219E-001
	0.672	1.149	0.623	-1.206	-0.594	-0.341	
$\varepsilon = 2^{-18}$	3.972E-001	2.493E-001	1.124E-001	7.297E-002	1.683E-001	2.541E-001	3.219E-001
	0.672	1.149	0.623	-1.206	-0.594	-0.341	
$\varepsilon = 2^{-20}$	3.972E-001	2.493E-001	1.124E-001	7.297E-002	1.683E-001	2.541E-001	3.219E-001
	0.672	1.149	0.623	-1.206	-0.594	-0.341	
$d_1^{N,M}$	4.162E-001	3.788E-001	4.116E-001	4.365E-001	4.547E-001	4.679E-001	4.772E-001
$q_1^{N,M}$	0.136	-0.120	-0.085	-0.059	-0.041	-0.029	

Table 3: Test problem (3.1): Two–mesh global differences $d_{1,\varepsilon}^{N,M}$ and $d_1^{N,M}$ and their computed orders of convergence $q_{1,\varepsilon}^{N,M}$ and $q_1^{N,M}$ for finite difference scheme (3.2) defined on the Bakhvalov mesh and using bilinear interpolation.

These results show that bilinear interpolation cannot be used to generate robust global approximations of the solution. The same behavior was observed when the Shishkin mesh is used instead of the Bakhvalov mesh. Therefore, although the Bakhvalov mesh is graded, it does not suffice to obtain an accurate global approximation.

Similarly to [3], we replace the bilinear interpolation by a non–linear interpolation which is exact for the parameter error function $w(x,t)$. Namely, if the point $(x,t) \in (x_i, x_{i+1}) \times (t_j, t_{j+1})$ with $x_i > 0$ and $t_j > 0$, we consider the following interpolation operator to approximate the value of $u(x,t)$

$$U_I^{N,M}(x,t) := \sum_{l,m=0}^{1} U_{i+l,j+m} T(t; x_{i+l}, t_{j+m}) S(x,t; x_{i+l}), \tag{3.3}$$

where

$$T(t; x_{i+l}, t_{j+m}) := \frac{w(x_{i+l}, t) - w(x_{i+l}, t_{j+1-m})}{w(x_{i+l}, t_{j+m}) - w(x_{i+l}, t_{j+1-m})},$$

$$S(x,t; x_{i+l}) := \frac{w(x,t) - w(x_{i+1-l}, t)}{w(x_{i+l}, t) - w(x_{i+1-l}, t)}.$$

	N=8 M=8	N=16 M=16	N=32 M=32	N=64 M=64	N=128 M=128	N=256 M=256	N=512 M=512
$\varepsilon = 2^{0}$	1.021E-001	7.899E-002	4.957E-002	2.691E-002	1.400E-002	7.140E-003	3.573E-003
	0.371	0.672	0.881	0.943	0.971	0.999	
$\varepsilon = 2^{-2}$	1.742E-001	9.761E-002	5.004E-002	2.517E-002	1.257E-002	6.314E-003	3.159E-003
	0.835	0.964	0.991	1.001	0.994	0.999	
$\varepsilon = 2^{-4}$	2.574E-001	1.021E-001	6.199E-002	3.390E-002	1.424E-002	8.947E-003	4.527E-003
	1.333	0.721	0.871	1.251	0.670	0.983	
$\varepsilon = 2^{-6}$	7.721E-001	3.964E-001	2.288E-001	1.242E-001	6.394E-002	3.297E-002	1.676E-002
	0.962	0.793	0.881	0.958	0.956	0.976	
$\varepsilon = 2^{-8}$	1.856E-001	1.249E-001	4.851E-002	2.381E-002	1.190E-002	5.967E-003	3.017E-003
	0.572	1.364	1.027	1.001	0.996	0.984	
$\varepsilon = 2^{-10}$	1.813E-001	1.359E-001	9.781E-002	4.698E-002	1.415E-002	5.885E-003	2.963E-003
	0.416	0.475	1.058	1.731	1.266	0.990	
$\varepsilon = 2^{-12}$	1.791E-001	1.363E-001	9.531E-002	4.782E-002	1.943E-002	6.970E-003	2.933E-003
	0.394	0.516	0.995	1.299	1.479	1.249	
$\varepsilon = 2^{-14}$	1.779E-001	1.365E-001	9.537E-002	4.783E-002	1.943E-002	6.971E-003	2.930E-003
	0.382	0.517	0.996	1.299	1.479	1.251	
$\varepsilon = 2^{-16}$	1.773E-001	1.366E-001	9.541E-002	4.784E-002	1.944E-002	6.971E-003	2.928E-003
	0.376	0.518	0.996	1.300	1.479	1.251	
$\varepsilon = 2^{-18}$	1.770E-001	1.367E-001	9.542E-002	4.784E-002	1.944E-002	6.972E-003	2.928E-003
	0.373	0.518	0.996	1.300	1.479	1.252	
$\varepsilon = 2^{-20}$	1.768E-001	1.367E-001	9.543E-002	4.785E-002	1.944E-002	6.972E-003	2.927E-003
	0.371	0.519	0.996	1.300	1.479	1.252	
$d_2^{N,M}$	7.721E-001	3.964E-001	2.288E-001	1.242E-001	6.394E-002	3.297E-002	1.676E-002
$q_2^{N,M}$	0.962	0.793	0.881	0.958	0.956	0.976	

Table 4: Test problem (3.1): Two–mesh global differences $d_{2,\varepsilon}^{N,M}$ and $d_2^{N,M}$ and their computed orders of convergence $q_{2,\varepsilon}^{N,M}$ and $q_2^{N,M}$ for finite difference scheme (3.2) defined on the Shishkin mesh and using nonlinear interpolation (3.3).

We denote the two–mesh global differences associated with this interpolation as follows

$$d_{2,\varepsilon}^{N,M} := \left\| U_I^{N,M} - U_I^{2N,2M} \right\|_{\bar{G}^{N,M} \cup \bar{G}^{2N,2M}}, \quad d_2^{N,M} := \max_{\varepsilon \in S_\varepsilon} d_{2,\varepsilon}^{N,M},$$

and their orders of convergence by

$$q_{2,\varepsilon}^{N,M} := \log_2 \left(\frac{d_{2,\varepsilon}^{N,M}}{d_{2,\varepsilon}^{2N,2M}} \right), \quad q_2^{N,M} := \log_2 \left(\frac{d_2^{N,M}}{d_2^{2N,2M}} \right).$$

The numerical results obtained by using the global approximation generated by this nonlinear interpolation are given in Tables 4 and 5, where the Shishkin and Bakhvalov meshes have been considered, respectively. Note that similar two–mesh global differences are obtained using the Shishkin and Bakhvalov meshes, although the orders of convergence are more regular when the Bakhvalov mesh is considered. These numerical results suggest that these finite difference schemes can provide parameter uniform approximations of the solution in the domain $\bar{G}$.

	N=8 M=8	N=16 M=16	N=32 M=32	N=64 M=64	N=128 M=128	N=256 M=256	N=512 M=512
$\varepsilon = 2^{0}$	1.021E-001	7.899E-002	4.957E-002	2.691E-002	1.400E-002	7.140E-003	3.573E-003
	0.371	0.672	0.881	0.943	0.971	0.999	
$\varepsilon = 2^{-2}$	1.742E-001	9.761E-002	5.004E-002	2.517E-002	1.257E-002	6.314E-003	3.159E-003
	0.835	0.964	0.991	1.001	0.994	0.999	
$\varepsilon = 2^{-4}$	2.574E-001	1.021E-001	6.199E-002	3.390E-002	1.424E-002	8.947E-003	4.527E-003
	1.333	0.721	0.871	1.251	0.670	0.983	
$\varepsilon = 2^{-6}$	7.721E-001	3.964E-001	2.288E-001	1.242E-001	6.394E-002	3.297E-002	1.676E-002
	0.962	0.793	0.881	0.958	0.956	0.976	
$\varepsilon = 2^{-8}$	1.943E-001	9.414E-002	4.758E-002	2.376E-002	1.190E-002	6.001E-003	3.020E-003
	1.045	0.984	1.002	0.997	0.988	0.991	
$\varepsilon = 2^{-10}$	1.920E-001	9.268E-002	4.691E-002	2.350E-002	1.177E-002	5.888E-003	2.961E-003
	1.051	0.982	0.997	0.998	0.999	0.992	
$\varepsilon = 2^{-12}$	1.912E-001	9.184E-002	4.654E-002	2.338E-002	1.172E-002	5.863E-003	2.933E-003
	1.058	0.980	0.993	0.997	0.999	0.999	
$\varepsilon = 2^{-14}$	1.908E-001	9.139E-002	4.635E-002	2.332E-002	1.169E-002	5.856E-003	2.930E-003
	1.062	0.979	0.991	0.996	0.998	0.999	
$\varepsilon = 2^{-16}$	1.906E-001	9.117E-002	4.626E-002	2.329E-002	1.168E-002	5.852E-003	2.928E-003
	1.064	0.979	0.990	0.995	0.998	0.999	
$\varepsilon = 2^{-18}$	1.905E-001	9.105E-002	4.621E-002	2.327E-002	1.168E-002	5.850E-003	2.928E-003
	1.065	0.979	0.990	0.995	0.997	0.999	
$\varepsilon = 2^{-20}$	1.903E-001	9.099E-002	4.618E-002	2.326E-002	1.168E-002	5.849E-003	2.927E-003
	1.065	0.978	0.989	0.995	0.997	0.999	
$d_2^{N,M}$	7.721E-001	3.964E-001	2.288E-001	1.242E-001	6.394E-002	3.297E-002	1.676E-002
$q_2^{N,M}$	0.962	0.793	0.881	0.958	0.956	0.976	

Table 5: Test problem (3.1): Two–mesh global differences $d_{2,\varepsilon}^{N,M}$ and $d_2^{N,M}$ and their computed orders of convergence $q_{2,\varepsilon}^{N,M}$ and $q_2^{N,M}$ for finite difference scheme (3.2) defined on the Bakhvalov mesh and using nonlinear interpolation (3.3).

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