Monografías Matemáticas *García de Galdeano* **39**, 75–85 (2014)

AN INVERSE PROBLEM FOR A TIME-DEPENDENT SCHRÖDINGER OPERATOR IN AN UNBOUNDED STRIP

Laure Cardoulis

Abstract. In this article we prove a stability result for two independent coefficients (each one depending on only one space variable and the potential also depending on the time variable) for a time-dependent Schrödinger operator in an unbounded strip with one observation on an unbounded subset of the boundary. Using an adapted Carleman estimate and an energy estimate, we obtain the simultaneous identification of the diffusion coefficient and the time-dependent potential with one observation and the data of the solution at a fixed time.

Keywords: Schrödinger operators, Carleman estimate, inverse problem.

AMS classification: 35J10.

§1. Introduction

This paper is an improvement of the work [5] in the sense that we determine two independent coefficients, the diffusion coefficient $a := a(x_2)$ and the potential $b := b(x_1, t) = f(t)g(x_1)$, one of them depending on the time variable, with one observation .

Let $\Omega = \mathbb{R} \times (0, d)$ be an unbounded strip of $\mathbb{R}^2$ with a fixed width d. Let ν be the outward unit normal to Ω on $\Gamma = \partial\Omega$. We denote $x = (x_1, x_2)$ and $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \{x \in \Gamma;\ x_2 = d\}$ and $\Gamma^- = \{x \in \Gamma;\ x_2 = 0\}$. We consider the following Schrödinger equation

$$\begin{cases} Hq(x,t) := i\partial_t q(x,t) + \nabla\cdot(a(x_2)\nabla q(x,t)) + b(x_1,t)q(x,t) = 0 \text{ in } Q = \Omega \times (0,T), \\ q(x,t) = F(x,t) \text{ on } \Sigma = \partial\Omega \times (0,T), \\ q(x,0) = q_0(x) \text{ in } \Omega. \end{cases} \tag{1.1}$$

where $a \in C^3(\overline{\Omega})$, $b \in C^2(\overline{\Omega \times (0,T)})$ and $a(x) \geq a_{min} > 0$. Moreover, we assume that a (resp. b) and all its derivatives up to order three (resp. two) are bounded.

Our problem can be stated as follows: Is it possible to determine the coefficients a and b from the measurement of $\partial_\nu(q)$, $\partial_\nu(\partial_t q)$ and $\partial_\nu(\partial_t^2 q)$ on Γ^+?

We will consider two cases for the potential $b := b(x_1, t) = f(t)g(x_1)$.
In a first case we consider q (resp. $\widetilde{q}$) a solution of (1.1) associated with (a, f, g, F, q_0) (resp. $(\widetilde{a}, \widetilde{f}, g, F, q_0)$) satisfying some regularity properties:

Assumption 1.

- *$\widetilde{q}$ and all its derivatives up to order four are bounded.*
- *q_0 is a real valued function in $C^3(\overline{\Omega})$.*
- *q_0 and all its derivatives up to order three are bounded.*

In a second case we consider q (resp. $\widetilde{q}$) a solution of (1.1) associated with (a, g, f, F, q_0) (resp. $(\widetilde{a}, \widetilde{g}, f, F, q_0)$) satisfying the regularity properties of Assumption 1.

We now specify our hypotheses on the diffusion coefficient a and the potential b. Let R_1 be a strictly positive and fixed real. Consider the set

$$\Lambda(R_1) := \{f \in L^\infty(\overline{\Omega \times (0,T)}),\ \|f\|_{L^\infty(\overline{\Omega\times(0,T)})} < R_1\}.$$

Let a be a real-valued function in $C^3(\overline{\Omega})$ and b be a real-valued function in $C^2(\overline{\Omega \times (0,T)})$ such that

Assumption 2.

- *$a \geq a_{min} > 0$, a and all its derivatives up to order three are in $\Lambda(R_1)$,*
- *b and all its derivatives up to order two are in $\Lambda(R_1)$.*

Moreover as the method employed in this paper uses a Carleman estimate, we will need a weight function $\widetilde{\beta}$ in connection with the diffusion coefficient a. Let $\widetilde{\beta}$ be a $C^4(\overline{\Omega})$ positive function such that there exist positive constants C_0, C_{pc} which satisfy

Assumption 3.

- *$|\nabla\widetilde{\beta}| \geq C_0 > 0$ in $\overline{\Omega}$, $\partial_\nu\widetilde{\beta} \leq 0$ on Γ^-.*
- *$\widetilde{\beta}$ and all its derivatives up to order four are in $\Lambda(R_1)$.*
- *$2\Re(D^2\widetilde{\beta}(\zeta,\bar{\zeta})) - a\nabla a \cdot \nabla\widetilde{\beta}|\zeta|^2 + 2a^2|\nabla\widetilde{\beta}\cdot\zeta|^2 \geq C_{pc}|\zeta|^2$, for all $\zeta \in \mathbb{C}$*

where

$$D^2\widetilde{\beta} = \begin{pmatrix} a\partial_{x_1}(a\partial_{x_1}\widetilde{\beta}) & a\partial_{x_1}(a\partial_{x_2}\widetilde{\beta}) \\ a\partial_{x_2}(a\partial_{x_1}\widetilde{\beta}) & a\partial_{x_2}(a\partial_{x_2}\widetilde{\beta}) \end{pmatrix}.$$

Note that the last assertion of Assumption 3 expresses the pseudo-convexity condition for the function $\widetilde{\beta}$. This Assumption imposes restrictive conditions for the choice of the functions $\widetilde{\beta}$ in connection with the function a (see [4, 5, 6]). Note that here a only depends on x_2 and if we consider $\widetilde{\beta}(x_1, x_2) := \widetilde{\beta}(x_2)$ then the last assertion of Assumption 3 can be rewritten on the following form: there exists a positive constant r_0 such that

$$\begin{cases} -a\,\partial_{x_2} a\partial_{x_2}\widetilde{\beta} \geq r_0 > 0, \\ a\,\partial_{x_2} a\partial_{x_2}\widetilde{\beta} + 2a^2(\partial^2_{x_2}\widetilde{\beta} + (\partial_{x_2}\widetilde{\beta})^2) \geq r_0 > 0. \end{cases}$$

Similar restrictive conditions have also been highlighted for the hyperbolic case (see [11]). And this above function $\widetilde{\beta}$ will also have to verify the following hypothesis in connection with the initial condition $Q_0 = (q_0, \partial_{x_2} q_0)$.

Assumption 4. *There exists a positive constant $k > 0$ such that $|Q_0 \cdot \nabla\widetilde{\beta}| \geq k > 0$ on $\overline{\Omega}$.*

Note that if $\widetilde{\beta}(x) := \widetilde{\beta}(x_2)$ this condition becomes $|\partial_{x_2} q_0 \partial_{x_2}\widetilde{\beta}| \geq cst > 0$.

For example, if we choose as classes of available functions a and q_0 the following classes

$$a \in \left\{ f \in C^3(\Omega); \exists r_0 \text{ positive constant}, \left\{ \begin{array}{l} -f\,\partial_{x_2} f \geq r_0 > 0, \\ f\,\partial_{x_2} f + 2f^2(1+e^{x_2}) \geq r_0 > 0. \end{array} \right. \right\}$$

and

$$q_0 \in \left\{ f \in C^3(\Omega); \exists r_0 \text{ positive constant}, |\partial_{x_2} f| \geq r_0 > 0 \right\}$$

then we can take $\widetilde{\beta}(x) = e^{x_2}$.

We also define $\beta := \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (-T, T)$, we define the following weight functions

$$\varphi(x,t) := \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)}, \quad \eta(x,t) := \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)},$$

and $\eta_0(x) := \eta(x,0)$, $\varphi_0(x) := \varphi(x,0)$. Denote also by $\varphi^{-1} = \frac{1}{\varphi}$.

Since we suppose that Assumption 1 is checked throughout all the paper, we can extend the functions q (resp. b) on $\widetilde{Q} = \Omega \times (-T, T)$ by the formula $q(x,t) = \overline{q}(x,-t)$ (resp. $b(x,t) = b(x,-t)$) for every $(x,t) \in \Omega \times (-T, 0)$. With all these hypotheses, we obtain our main result for the first case.

Theorem 5. *Let q and $\widetilde{q}$ be solutions of (3.1), respectively associated with $(a,\ f,\ g,\ F,\ q_0)$ and $(\widetilde{a},\ \widetilde{f},\ g,\ F,\ q_0)$, such that $a - \widetilde{a} \in H_0^1(\Omega)$, $\partial_{x_2}(a-\widetilde{a}) \in H_0^1(\Omega)$, $b_0 - \widetilde{b}_0 \in H_0^1(\Omega)$ and $b_1 - \widetilde{b}_1 \in H_0^1(\Omega)$ with $b_0(x) = b(x,0)$, $\widetilde{b}_0(x) = \widetilde{b}(x,0)$, $b_1(x) = \partial_t b(x,0)$, $\widetilde{b}_1(x) = \partial_t \widetilde{b}(x,0)$. We assume that Assumptions 1-4 are satisfied.*
If $(f - \widetilde{f})(0) \neq 0$, then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1)$ such that for s and λ large enough,

$$\int_\Omega \varphi_0\, e^{-2s\eta_0}(|a-\widetilde{a}|^2 + |\nabla(a-\widetilde{a})|^2) + \int_{-T}^{T}\int_\Omega e^{-2s\eta}(|b-\widetilde{b}|^2 + |\partial_t(b-\widetilde{b})|^2)$$
$$\leq C \int_{-T}^{T}\int_{\Gamma^+} \varphi\, e^{-2s\eta}\partial_\nu\beta\, [|\partial_\nu(q-\widetilde{q})|^2 + |\partial_\nu(\partial_t q - \partial_t\widetilde{q})|^2]\, d\sigma\, dt. \tag{1.2}$$

If $(f-\widetilde{f})(0) = 0$ and $(f-\widetilde{f})'(0) \neq 0$, then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1)$ such that for s and λ large enough,

$$\int_\Omega \varphi_0\, e^{-2s\eta_0}(|a-\widetilde{a}|^2 + |\nabla(a-\widetilde{a})|^2) + \int_{-T}^{T}\int_\Omega e^{-2s\eta}(|b-\widetilde{b}|^2 + |\partial_t(b-\widetilde{b})|^2 + |\partial_t^2(b-\widetilde{b})|^2)$$

$$\leq C \int_{-T}^{T} \int_{\Gamma^+} \varphi\, e^{-2s\eta} \partial_\nu \beta\, [|\partial_\nu (q-\widetilde{q})|^2 + |\partial_\nu (\partial_t q - \partial_t \widetilde{q})|^2 + |\partial_\nu (\partial_t^2 q - \partial_t^2 \widetilde{q})|^2]\, d\sigma\, dt \qquad (1.3)$$

$$+C \int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0} [|\nabla \partial_{x_1} \partial_t (q-\widetilde{q})(.,0)|^2 + |\Delta \partial_{x_1} \partial_t (q-\widetilde{q})(.,0)|^2].$$

Note that we impose restrictive conditions on $(f-\widetilde{f})(0)$ where $\widetilde{f}$ is a given function in $[0,T]$ and these conditions imply that f is a non null perturbation of $\widetilde{f}$ in a neighbourhood of $t=0$. For the second case, we obtain the same results as above by considering the case $f(0) \neq 0$ on one hand and the case $f(0)=0$, $f'(0) \neq 0$ on the other hand, with this time $b = fg, \widetilde{b} = f\widetilde{g}$. This is the following result.

Theorem 6. *Let q and $\widetilde{q}$ be solutions of (3.1), respectively associated with (a, g, f, F, q_0) and $(\widetilde{a}, \widetilde{g}, f, F, q_0)$, such that $a-\widetilde{a} \in H_0^1(\Omega)$, $\partial_{x_2}(a-\widetilde{a}) \in H_0^1(\Omega)$, $b_0 - \widetilde{b}_0 \in H_0^1(\Omega)$ and $b_1 - \widetilde{b}_1 \in H_0^1(\Omega)$. We assume that Assumptions 1-4 are satisfied.*
If $f(0) \neq 0$, then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1)$ such that for s and λ large enough (1.2) *is satisfied.*
If $f(0) = 0$ and $f'(0) \neq 0$, then there exists a positive constant $C = C(\Omega, \Gamma, T, R_1)$ such that for s and λ large enough (1.3) *is satisfied.*

The major novelty of this paper is to consider the coefficient b depending on the time variable. In several works, the problem of the identification of coefficients for the Schrödinger operator have been studied (see [1] in bounded domains and [4, 5, 6] in unbounded domains) but in all of these works, the coefficients only depended on the space variable. Note also that in [7], for a magnetic Schrödinger operator with a time-dependent magnetic potential $\chi(t)a$, there is a result for the identification of the coefficient a which does not depend on the time variable. For the problem of the identification of a potential $b(x,t)$, to our knowledge, there is no result. The main difficulty comes from the time dependent potential $b(x,t)$. Indeed we consider $u = q - \widetilde{q}$, $v = \partial_t u$, $w = \partial_t v$. Thus time dependent terms appear in the right-hand sides of the equations (3.3)-(3.4) satisfied by v, w and the difficulty is to control them with the estimates given by the Carleman inequality (2.4). Because these terms did not appear in earlier papers, we cannot adapt the previous works to our case. So we present a first result for a time dependent potential but open questions are numerous: the removal of any conditions on $(f-\widetilde{f})(0)$ contrary to the above theorem ; the general case of a potential $b(x,t)$ with no particular form and no separated variables ; the case of $a = a(x_1, x_2)$ and $b = (x_1, x_2, t)$; the case of a time dependent diffusion coefficient a.

We will use the global Carleman estimate given in [4, 5, 6]. Indeed, Carleman inequalities constitute a very efficient tool to derive observability estimates. We recall that the method of Carleman estimates has been introduced in the field of inverse problems by Bukhgeim and Klibanov ([2, 3, 9, 10]). These methods give a local Lipschitz stability around a single known solution (see also [11]).

We then use an energy estimate for the operator H given in [4] and a Carleman type estimate for a first order differential equation proved in [8] for bounded domains and in [4] for unbounded domains.

This paper is organized as follows. In section 2, we recall an adapted global Carleman estimate for the operator H (see [4, 5, 6]) and also an energy estimate (see [4, 5]). In section 3 we prove our two main results, a stability result for the coefficients a and f on the one hand and a stability result for the coefficients a and g on the other hand.

§2. Some Useful Estimates

2.1. Global Carleman Inequality

Let a be a real-valued function in $C^3(\overline{\Omega})$ and b be a real-valued function in $C^2(\overline{\Omega \times (0,T)})$ which satisfy Assumption 2.

Let q be a function equals to zero on $\partial\Omega\times(-T,T)$ and solution of the Schrödinger equation

$$\begin{cases} Hq := i\partial_t q + \nabla\cdot(a\nabla q) + bq = f \ \text{ in } \ Q = \Omega\times(-T,T), \\ q = 0 \ \text{ on } \ \Sigma_T = \partial\Omega\times(-T,T). \end{cases} \tag{2.1}$$

We recall here a global Carleman-type estimate for q with a single observation acting on the upper part Γ^+ of the boundary Γ in the right-hand side of the estimate (see [4, 5, 6]). Even if the potential b did not depend on the time variable in [4, 5, 6], we can exactly proceed as in these previous papers. Indeed the Carleman inequality is obtained in [4, 5, 6] by a decomposition of the operator $Lq := i\partial_t q + \nabla\cdot(a\nabla q)$. So the fact that here the potential b is a time dependent one does not change the results. Otherwise, if the diffusion coefficient a depended on the time variable we could not apply the previous Carleman inequality. Recall that for the Carleman inequality we need weight functions. So let $\widetilde{\beta}$ be a $C^4(\overline{\Omega})$ positive function such that there exist positive constants C_0, C_{pc} which satisfy Assumption 3.

Then, recall that we define $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (-T,T)$, we define the following weight functions $\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)}$, $\quad \eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)}$. Let H be the operator defined by

$$Hq := i\partial_t q + \nabla\cdot(a\nabla q) + bq \ \text{ in } \ \widetilde{Q} = \Omega\times(-T,T). \tag{2.2}$$

We set $\psi = e^{-s\eta}q$, $M\psi = e^{-s\eta}H(e^{s\eta}\psi)$ for $s > 0$ and we introduce the following operators

$$M_1\psi := i\partial_t\psi + \nabla\cdot(a\nabla\psi) + s^2 a|\nabla\eta|^2\psi + b\psi, \quad M_2\psi := is\partial_t\eta\psi + 2as\nabla\eta\cdot\nabla\psi + s\nabla\cdot(a\nabla\eta)\psi. \tag{2.3}$$

Note that the only difference with the Carleman estimate given in [4] is the presence of the term $b\psi$ in $M_1\psi$. And this term does not change anything in the calculus given in [4]. Then the following result holds.

Theorem 7. *Let H, M_1, M_2 be the operators defined respectively by (2.2), (2.3) . We assume that Assumptions 2 and 3 are satisfied. Then there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant $C = C(\Omega, \Gamma, T, C_0, C_{pc}, R_1)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the next inequality holds:*

$$\begin{aligned} s^3\lambda^4 \int_{-T}^{T}\int_\Omega e^{-2s\eta}\varphi^3|q|^2 + s\lambda \int_{-T}^{T}\int_\Omega e^{-2s\eta}\varphi|\nabla q|^2 + \|M_1(e^{-s\eta}q)\|^2_{L^2(\widetilde{Q})} + \|M_2(e^{-s\eta}q)\|^2_{L^2(\widetilde{Q})} \\ \leq C\left[s\lambda \int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi|\partial_\nu q|^2\, \partial_\nu\beta \, d\sigma\, dt + \int_{-T}^{T}\int_\Omega e^{-2s\eta}\, |Hq|^2\right], \end{aligned} \tag{2.4}$$

for all q satisfying $Hq \in L^2(\Omega \times (-T, T))$, $q \in L^2(-T, T; H_0^1(\Omega))$, $\partial_\nu q \in L^2(-T, T; L^2(\Gamma))$.

2.2. Energy Estimate

Let q be a function equals to zero on $\partial\Omega \times (-T, T)$ and solution of the Schrödinger equation (2.1). We recall here an energy estimate for q with a single observation acting on the upper part Γ^+ of the boundary Γ in the right-hand side of the estimate (see [4, 5]). As for the Carleman inequality, even if the potential b did not depend on the time variable in [4, 5], we can exactly proceed for the calculus as in these previous papers. Indeed this energy estimate is a consequence of the Carleman inequality and the result given in [4] remains valid.
We denote by

$$E_1(t) := \int_\Omega e^{-2s\eta(x,t)}|q(x,t)|^2 dx \text{ and } E_2(t) := \int_\Omega sa\, \varphi^{-1}(x,t)\, e^{-2s\eta(x,t)}|\nabla q(x,t)|^2 dx,$$

where $\varphi^{-1} = \frac{1}{\varphi}$. We give an estimate for $E_i(0)$ $i = 1, 2$ in Theorem 8.

Theorem 8. *Let q be solution of (2.1) in the following class $q \in C([0, T], H^1(\Omega))$, $\partial_\nu q \in L^2(0, T, L^2(\Gamma))$. We assume that Assumptions 2 and 3 are checked. Then there exists a positive constant $C = C(\Omega, \Gamma, T) > 0$ such that for s and λ sufficiently large*

$$E_1(0) + E_2(0) \leq C\left(s^2\lambda^2 \int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi\, \partial_\nu\beta\, |\partial_\nu q|^2\, d\sigma\, dt + s\lambda \int\int_Q e^{-2s\eta}|Hq|^2\right) \tag{2.5}$$

2.3. Lemma

We also recall a useful Lemma (see [8, 4]) for the following first order differential operator

Lemma 9. *Let P the operator defined by*

$$Pg := q_0\partial_{x_1} g + \partial_{x_2} q_0 \partial_{x_2} g = Q_0 \cdot \nabla g \text{ with } Q_0 = (q_0, \partial_{x_2} q_0).$$

Assume that Assumption 4 is satisfied and denote by $\eta_0(x) := \eta(x, 0)$ and $\varphi_0(x) := \varphi(x, 0)$. Then there exist positive constants $\lambda_1 > 0$, $s_1 > 0$ and $C = C(\Omega, \Gamma, T)$ such that for all $\lambda \geq \lambda_1$, $s \geq s_1$,

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}|g|^2 \leq C \int_\Omega \varphi_0^{-1} e^{-2s\eta_0}|Pg|^2 \tag{2.6}$$

for any g such that $g = 0$ on $\{x \in \Gamma, (Q_0 \cdot \nabla\beta)(x)(Q_0(x) \cdot \nu) > 0\}$ and in particular for any $g \in H_0^1(\Omega)$.

§3. Inverse Problem

Let q and $\widetilde{q}$ be solutions of

$$\begin{cases} i\partial_t q + \nabla\cdot(a\nabla q) + bq = 0 & \text{in} \quad \Omega\times(0,T), \\ q(x,t) = F(x,t) & \text{on} \quad \partial\Omega\times(0,T), \\ q(x,0) = q_0(x) & \text{in} \quad \Omega, \end{cases} \quad \begin{cases} i\partial_t \widetilde{q} + \nabla\cdot(\widetilde{a}\nabla \widetilde{q}) + \widetilde{b}\widetilde{q} = 0 & \text{in} \quad \Omega\times(0,T), \\ \widetilde{q}(x,t) = F(x,t) & \text{on} \quad \partial\Omega\times(0,T), \\ \widetilde{q}(x,0) = q_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where a, b, $\widetilde{a}$ and $\widetilde{b}$ satisfy Assumption 2. We suppose that Assumption 1 is checked, then we extend the functions q (resp. b) on $\widetilde{Q} = \Omega\times(-T,T)$ by the formula $q(x,t) = \overline{q}(x,-t)$ (resp. $b(x,t) = b(x,-t)$) for every $(x,t)\in\Omega\times(-T,0)$. Denote by $\Sigma_T := \partial\Omega\times(-T,T)$. If we set $u = q-\widetilde{q}$, $v = \partial_t u$, $w = \partial_t v$, $\alpha = \widetilde{a}-a$, $\gamma = \widetilde{b}-b$, $b_0(x) = b(x,0)$, $\widetilde{b}_0(x) = \widetilde{b}(x,0)$, $b_1(x) = \partial_t b(x,0)$, $\widetilde{b}_1(x) = \partial_t\widetilde{b}(x,0)$, $\gamma_0(x) = \gamma(x,0)$, $\gamma_1(x) = \partial_t\gamma(x,0)$, $q_1(x) = \partial_t\widetilde{q}(x,0)$ then u, v and w satisfy

$$\begin{cases} i\partial_t u + \nabla\cdot(a\nabla u) + bu = \nabla\cdot(\alpha\nabla\widetilde{q}) + \gamma\widetilde{q} & \text{in} \quad \widetilde{Q}, \\ u(x,t) = 0 & \text{on} \quad \Sigma_T, \\ u(x,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.2}$$

$$\begin{cases} i\partial_t v + \nabla\cdot(a\nabla v) + bv = -\partial_t b\, u + \nabla\cdot(\alpha\nabla(\partial_t\widetilde{q})) + \gamma\partial_t\widetilde{q} + \partial_t\gamma\widetilde{q} & \text{in} \quad \widetilde{Q}, \\ v(x,t) = 0 & \text{on} \quad \Sigma_T, \\ v(x,0) = \frac{1}{i}\left(\nabla\cdot(\alpha\nabla q_0) + \gamma_0 q_0\right) & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

$$\begin{cases} i\partial_t w + \nabla\cdot(a\nabla w) + bw = -2\partial_t b\, v - \partial_t^2 b\, u + \nabla\cdot(\alpha\nabla(\partial_t^2\widetilde{q})) + 2\partial_t\gamma\partial_t\widetilde{q} + \partial_t^2\gamma\widetilde{q} + \gamma\partial_t^2\widetilde{q} & \text{in} \quad \widetilde{Q}, \\ w(x,t) = 0 & \text{on} \quad \Sigma_T, \\ w(x,0) = \frac{1}{i}\left(\nabla\cdot(\alpha\nabla q_1) + \gamma_0 q_1 + \gamma_1 q_0 - b_1 u(x,0) - b_0 v(x,0) - \nabla\cdot(a\nabla v(x,0))\right) & \text{in} \quad \Omega. \end{cases} \tag{3.4}$$

First we consider the case where q and $\widetilde{q}$ are solutions of (3.1) respectively associated with (a, f, g, F, q_0) and $(\widetilde{a}, \widetilde{f}, g, F, q_0)$. We recall that $b = fg$ and $\widetilde{b} = \widetilde{f}g$. Our main stability result expresses a perturbation result around the known solution $\widetilde{q}$. It is the following one

Theorem 10. *Let q and $\widetilde{q}$ be solutions of (3.1), respectively associated with (a, f, g, F, q_0) and $(\widetilde{a}, \widetilde{f}, g, F, q_0)$, such that $a-\widetilde{a}\in H_0^1(\Omega)$, $\partial_{x_2}(a-\widetilde{a})\in H_0^1(\Omega)$, $b_0-\widetilde{b}_0\in H_0^1(\Omega)$ and $b_1-\widetilde{b}_1\in H_0^1(\Omega)$. We assume that Assumptions 1-4 are satisfied.*
If $(f-\widetilde{f})(0)\neq 0$, then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough,

$$\int_\Omega \varphi_0\, e^{-2s\eta_0}(|a-\widetilde{a}|^2 + |\nabla(a-\widetilde{a})|^2) + \int_{-T}^{T}\int_\Omega e^{-2s\eta}(|b-\widetilde{b}|^2 + |\partial_t(b-\widetilde{b})|^2)$$
$$\leq C\int_{-T}^{T}\int_{\Gamma^+}\varphi\, e^{-2s\eta}\partial_\nu\beta\,[|\partial_\nu(q-\widetilde{q})|^2 + |\partial_\nu(\partial_t q - \partial_t\widetilde{q})|^2]\, d\sigma\, dt. \tag{3.5}$$

If $(f-\widetilde{f})(0) = 0$ and $(f-\widetilde{f})'(0)\neq 0$, then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough,

$$\int_\Omega \varphi_0\, e^{-2s\eta_0}(|a-\widetilde{a}|^2 + |\nabla(a-\widetilde{a})|^2) + \int_{-T}^{T}\int_\Omega e^{-2s\eta}(|b-\widetilde{b}|^2 + |\partial_t(b-\widetilde{b})|^2 + |\partial_t^2(b-\widetilde{b})|^2)$$

$$\leq C\int_{-T}^{T}\int_{\Gamma^+}\varphi\, e^{-2s\eta}\partial_\nu\beta\,[|\partial_\nu(q-\widetilde{q})|^2+|\partial_\nu(\partial_t q-\partial_t\widetilde{q})|^2+|\partial_\nu(\partial_t^2 q-\partial_t^2\widetilde{q})|^2]\,d\sigma\,dt \quad (3.6)$$

$$+C\int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0}[|\nabla\partial_{x_1}\partial_t(q-\widetilde{q})(.,0)|^2+|\Delta\partial_{x_1}\partial_t(q-\widetilde{q})(.,0)|^2].$$

Proof. We apply (2.6) given in the Lemma 2.1, to the first order partial differential equations satisfied by α and γ_0 given by the initial condition and the derivatives of this initial condition in (3.3) and we deduce the following result for s and λ large enough:

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2+|\gamma_0|^2)\leq C\int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0}\left(|v(x,0)|^2+|\nabla v(x,0)|^2\right).$$

Then from (2.5) applied for v we get

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2+|\gamma_0|^2)\leq Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\varphi e^{-2s\eta}\partial\nu\beta|\partial_\nu v|^2$$

$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}[|\alpha|^2+|\nabla\alpha|^2+|u|^2+|\gamma|^2+|\partial_t\gamma|^2].$$

Since $e^{-2s\eta}\leq e^{-2s\eta_0}$ we obtain for s and λ large enough

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2+|\gamma_0|^2)\leq Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\phi e^{-2s\eta}\partial\nu\beta|\partial_\nu v|^2$$

$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}[|u|^2+|\gamma|^2+|\partial_t\gamma|^2].$$

Using now the Carleman inequality for u we get

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2+|\gamma_0|^2)\leq Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\varphi e^{-2s\eta}\partial\nu\beta|\partial_\nu v|^2$$

$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}[|\gamma|^2+|\partial_t\gamma|^2]+Cs\lambda\int_T^T\int_{\Gamma^+}\varphi e^{-2s\eta}\partial\nu\beta|\partial_\nu u|^2+C\int_T^T\int_\Omega e^{-2s\eta}[|\alpha|^2+|\nabla\alpha|^2+|\gamma|^2].$$

Therefore

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2+|\gamma_0|^2)\leq Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\varphi e^{-2s\eta}\partial\nu\beta(|\partial_\nu v|^2+|\partial_\nu u|^2)$$

$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}[|\gamma|^2+|\partial_t\gamma|^2]. \quad (3.7)$$

We now consider the first case $(f-\widetilde{f})(0)\neq 0$.
Since $t\to|\frac{(f-\widetilde{f})(t)}{(f-\widetilde{f})(0)}|$ is bounded on $[-T,T]$ we deduce that there exists a positive constant $C=C(T,R_1)$ such that for all x,t,

$$|\gamma(x,t)|=|(\widetilde{f}(t)-f(t))g(x_1)|\leq C|\gamma(x,0)|=C|(\widetilde{f}(0)-f(0))g(x_1)|.$$

Similarly we have $|\partial_t\gamma(x,t)| = |(\widetilde{f}'(t) - f'(t))g(x_1)| \le C|\gamma(x,0)|$. So (3.7) becomes

$$s^2\lambda^2 \int_\Omega \varphi_0\, e^{-2s\eta_0}(|\alpha|^2 + |\nabla\alpha|^2) + s^2\lambda^2 \int_{-T}^{T}\int_\Omega e^{-2s\eta}(|\gamma|^2 + |\partial_t\gamma|^2)$$

$$\le Cs^2\lambda^2 \int_{-T}^{T}\int_{\Gamma^+} \varphi\, e^{-2s\eta}\partial_\nu\beta\, [|\partial_\nu v|^2 + |\partial_\nu u|^2]\, d\sigma\, dt + Cs\lambda \int_T^T \int_\Omega e^{-2s\eta}[|\gamma|^2 + |\partial_t\gamma|^2]$$

and we get (3.5).

We then consider the case $(f - \widetilde{f})(0) = 0$ and $(f - \widetilde{f})'(0) \neq 0$. Note that $\gamma_0 = 0$ and $\gamma_1(x) = (\widetilde{f}'(0) - f'(0))g(x_1)$. Applying (2.6) given in the Lemma 2.1 to the first order partial differential equations satisfied by γ_1 given by the derivative of the initial condition in (3.4), we have for s and λ large enough:

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}|\gamma_1|^2 \le C\int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0}\Big(|\partial_{x_1} w(x,0)|^2 + |\alpha|^2 + |\nabla\alpha|^2$$

$$+|u(x,0)|^2 + |\nabla u(x,0)|^2 + |v(x,0)|^2 + |\nabla v(x,0)|^2 + |\nabla\partial_{x_1} v(.,0)|^2 + |\Delta\partial_{x_1} v(.,0)|^2\Big). \qquad (3.8)$$

From (3.7) and (3.8),

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2 + |\nabla\alpha|^2 + |\gamma_1|^2) \le Cs^2\lambda^2 \int_T^T \int_{\Gamma^+} \phi e^{-2s\eta}\partial\nu\beta(|\partial_\nu v|^2 + |\partial_\nu u|^2)$$

$$+Cs\lambda \int_T^T \int_\Omega e^{-2s\eta}[|\gamma|^2 + |\partial_t\gamma|^2] + C\int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0}\Big(|\nabla w(x,0)|^2 + +|u(x,0)|^2 + |\nabla u(x,0)|^2\Big)$$

$$+C\int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0}(|v(x,0)|^2 + |\nabla v(x,0)|^2) + C\int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0}(|\nabla\partial_{x_1} v(.,0)|^2 + |\Delta\partial_{x_1} v(.,0)|^2).$$

Then using (2.5) for u, v, w defined by (3.2), (3.3), (3.4) we have

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2 + |\nabla\alpha|^2 + |\gamma_1|^2) \le Cs\lambda \int_T^T \int_\Omega e^{-2s\eta}[|\gamma|^2 + |\partial_t\gamma|^2]$$

$$+Cs^2\lambda^2 \int_T^T \int_{\Gamma^+} \phi e^{-2s\eta}\partial\nu\beta(|\partial_\nu v|^2 + |\partial_\nu u|^2 + |\partial_\nu w|^2)$$

$$+Cs\lambda \int_T^T \int_\Omega e^{-2s\eta}[|Hu|^2 + |Hv|^2 + |Hw|^2] + C\int_\Omega \varphi_0^{-1}\, e^{-2s\eta_0}(|\nabla\partial_{x_1} v(.,0)|^2 + |\Delta\partial_{x_1} v(.,0)|^2).$$

Since

$$|Hu|^2 \le C(|\alpha|^2 + |\nabla\alpha|^2 + |\gamma|^2), \quad |Hv|^2 \le C(|u|^2 + |\alpha|^2 + |\nabla\alpha|^2 + |\gamma|^2 + |\partial_t\gamma|^2) \qquad (3.9)$$

and

$$|Hw|^2 \le C(|u|^2 + |v|^2 + |\alpha|^2 + |\nabla\alpha|^2 + |\gamma|^2 + |\partial_t\gamma|^2 + |\partial_t^2\gamma|^2).$$

Therefore

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2+|\gamma_1|^2)\le Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}(|u|^2+|v|^2)$$

$$+Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\phi e^{-2s\eta}\partial\nu\beta(|\partial_\nu v|^2+|\partial_\nu u|^2+|\partial_\nu w|^2)$$

$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}[|\gamma|^2+|\partial_t\gamma|^2+|\partial_t^2\gamma|^2]+C\int_\Omega \varphi_0^{-1}\ e^{-2s\eta_0}(|\nabla\partial_{x_1}v(.,0)|^2+|\Delta\partial_{x_1}v(.,0)|^2).$$

We apply again (3.9) and the Carleman estimate for u and v and we have

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2+|\gamma_1|^2)\le Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\phi e^{-2s\eta}\partial\nu\beta(|\partial_\nu v|^2+|\partial_\nu u|^2+|\partial_\nu w|^2)$$

$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}[|\gamma|^2+|\partial_t\gamma|^2+|\partial_t^2\gamma|^2]+C\int_\Omega \varphi_0^{-1}\ e^{-2s\eta_0}(|\nabla\partial_{x_1}v(.,0)|^2+|\Delta\partial_{x_1}v(.,0)|^2).$$

Using the same argument as in the first case we deduce that there exists a positive constant $C = C(T, R_1)$ such that for all $|\gamma|\le C|\gamma_1|$, $|\partial_t\gamma|\le C|\gamma_1|$ and $|\partial_t^2\gamma|\le C|\gamma_1|$. We conclude as in the first case

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}(|\alpha|^2+|\nabla\alpha|^2)+s^2\lambda^2\int_T^T\int_\Omega e^{-2s\eta}[|\gamma|^2+|\partial_t\gamma|^2+|\partial_t^2\gamma|^2]$$

$$\le Cs^2\lambda^2\int_T^T\int_{\Gamma^+}\phi e^{-2s\eta}\partial\nu\beta(|\partial_\nu v|^2+|\partial_\nu u|^2+|\partial_\nu w|^2)$$

$$+Cs\lambda\int_T^T\int_\Omega e^{-2s\eta}[|\gamma|^2+|\partial_t\gamma|^2+|\partial_t^2\gamma|^2]+C\int_\Omega \varphi_0^{-1}\ e^{-2s\eta_0}(|\nabla\partial_{x_1}v(.,0)|^2+|\Delta\partial_{x_1}v(.,0)|^2).$$

and we get (3.6). □

By the same way, if we now consider the case where q and $\widetilde{q}$ are solutions of (3.1) respectively associated with $(a,\ g,\ f,\ F,\ q_0)$ and $(\widetilde{a},\ \widetilde{g},\ f,\ F,\ q_0)$, the stability result is (with $b = fg$ and $\widetilde{b} = f\widetilde{g}$)

Theorem 11. *Let q and $\widetilde{q}$ be solutions of (3.1), respectively associated with $(a,\ g,\ f,\ F,\ q_0)$ and $(\widetilde{a},\ \widetilde{g},\ f,\ F,\ q_0)$, such that $a-\widetilde{a}\in H_0^1(\Omega)$, $\partial_{x_2}(a-\widetilde{a})\in H_0^1(\Omega)$, $b_0-\widetilde{b}_0\in H_0^1(\Omega)$ and $b_1-\widetilde{b}_1\in H_0^1(\Omega)$. We assume that Assumptions 1-4 are satisfied.*
If $f(0)\neq 0$, then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough (3.5) *is satisfied.*
If $f(0) = 0$ and $f'(0)\neq 0$, then there exists a positive constant $C = C(\Omega,\Gamma,T,R_1)$ such that for s and λ large enough (3.6) *is satisfied.*

Remark 1. Note that Theorems 10 and 11 are available with weaker hypotheses on $a-\widetilde{a}$ and $b_0-\widetilde{b}_0$. Indeed if moreover we suppose that $(Q_0\cdot\nabla\beta)(x)(Q_0(x)\cdot\nu)\le 0$ for all $x\in\Gamma$, then Lemma 9 holds and therefore we still have (3.5) and (3.6) without assuming that $a-\widetilde{a}$, $b_0-\widetilde{b}_0$ and their derivatives are null functions on Γ.

References

[1] Baudouin, L., and Puel, J.-P. Uniqueness and stability in an inverse problem for the Schrödinger equation. *Inverse Problems 18* (2002), 1537–1554.

[2] Bukhgeim, A. L. *Volterra Equations and Inverse Problems*. Inverse and Ill-Posed Problems Series. VSP, Utrecht, 1999.

[3] Bukhgeim, A. L., and Klibanov, M. V. Uniqueness in the large of a class of multidimensional inverse problems. *Soviet. Math. Dokl. 17* (2006), 244–247.

[4] Cardoulis, L., Cristofol, M., and Gaitan, P. Inverse problem for the Schrödinger operator in an unbounded strip. *Journal of Inverse and Ill-posed Problem 16*, 2 (2008), 127–146.

[5] Cardoulis, L., and Gaitan, P. Identification of two independent coefficients with one observation for the Schrödinger operator in an unbounded strip. *CRAS Paris Ser I 348* (2010), 149–153.

[6] Cardoulis, L., and Gaitan, P. Simultaneous identification of the diffusion coefficient and the potential for the Schrödinger operator with one observation. *Inverse Problems 26* (2010), 035012.

[7] Cristofol, M., and Soccorsi, E. Stability estimate in an inverse problem for non autonomous magnetic Schrödinger equations. *Applicable Analysis 90*, 10 (2011), 1499–1520.

[8] Immanuvilov, O. Y., and Yamamoto, M. Carleman estimates for the non-stationary Lamé system and the application to an inverse problem. *ESAIM Control Optim. Calc. Var. 11*, 1 (2005), 1–56.

[9] Klibanov, M. V. Inverse problems in the large and Carleman bounds. *Differential Equations 20* (1984), 755–760.

[10] Klibanov, M. V. Inverse problems and carleman estimates. *Inverse Problems 8* (1992), 575–596.

[11] Klibanov, M. V., and Timonov, A. *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*. VSP, Utrecht, 2004.

Laure Cardoulis
Université de Toulouse, UT1 Ceremath , CNRS, Intitut de Mathématiques de Toulouse, UMR 5219
21 Allées de Brienne, 31042 Toulouse, France
laure.cardoulis@univ-tlse1.fr