SOLUCIÓN — 6-II-2008

1.-) $A=\{a,b,c,d\}$

$*$	a	b	c	d
a		a	b	
b	a	b	c	b
c	b	c	b	
d		d		

a) No, no hay ninguna manera puesto que parece que (2ª col.) b es el el. neutro, pero mirando la 2ª fila. $b*d=b\neq d$.
Por los elementos de la tabla ni a, ni c ni d son el. neutro.

b) No, el elem. 2.4 y el. 4.2 no coinciden.

2.-) $W_1=\left\{\begin{pmatrix} x & y \\ z & t\end{pmatrix}\in \mathcal{M}_{2\times 2}(\mathbb{R}) \;/\; x=z,\ y=t\right\}=$

$=\left\langle \begin{pmatrix} x & y \\ x & y\end{pmatrix}\right\rangle$

$W_1 \leq \mathcal{M}_{2\times 2}(\mathbb{R})$

$\forall \lambda,\mu\in\mathbb{R},\quad \forall\, \omega_1=\begin{pmatrix} x & y \\ x & y\end{pmatrix},\ \omega_2=\begin{pmatrix} x' & y' \\ x' & y'\end{pmatrix}$

$\lambda\begin{pmatrix} x & y \\ x & y\end{pmatrix}+\mu\begin{pmatrix} x' & y' \\ x' & y'\end{pmatrix}\overset{?}{\in} W_1$

$\begin{pmatrix} \lambda x+\mu x' & \lambda y+\mu y' \\ \lambda x+\mu x' & \lambda y+\mu y'\end{pmatrix}\in W_1$

$$W_2 = \left\{ \begin{pmatrix} -\lambda & \lambda \\ \beta & -\beta \end{pmatrix} \in M_{2\times2}(\mathbb{R}),\ \lambda, \beta \in \mathbb{R} \right\}$$

$$\forall \alpha, \mu \in \mathbb{R}, \quad \forall\, \omega_1 = \begin{pmatrix} -\lambda & \lambda \\ \beta & -\beta \end{pmatrix},\ \omega_2 = \begin{pmatrix} -\lambda' & \lambda' \\ \beta' & -\beta' \end{pmatrix}$$

$$\alpha\omega_1 + \mu\omega_2 \overset{?}{\in} W_2$$

$$\alpha \begin{pmatrix} -\lambda & \lambda \\ \beta & -\beta \end{pmatrix} + \mu \begin{pmatrix} -\lambda' & \lambda' \\ \beta' & -\beta' \end{pmatrix} = \begin{pmatrix} -\lambda\alpha - \lambda'\mu & \alpha\lambda + \mu\lambda' \\ \alpha\beta + \mu\beta' & -\alpha\beta - \mu\beta' \end{pmatrix}$$

$$\in W_2$$

b) Base W_1

$$W_1 = \left\langle \begin{pmatrix} x & y \\ x & y \end{pmatrix} \right\rangle = x \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\operatorname{rg} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \Rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \text{ base de } W_1.$$

Base W_2

$$W_2 = \left\langle \begin{pmatrix} -\lambda & \lambda \\ \beta & -\beta \end{pmatrix} \right\rangle = \lambda \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\operatorname{rg} \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = 2 \Rightarrow \text{Base } W_2 = \left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

Base $\mathbb{W}_1 \cap \mathbb{W}_2$

$$A=\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \underset{\substack{P_{31}(-1)\\P_{42}(-1)}}{\sim} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \underset{P_{43}(+1)}{\sim} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rg}\, A = 3$

$\dim (\mathbb{W}_1 \cap \mathbb{W}_2) = 1$

Si $B \in \mathbb{W}_1 \cap \mathbb{W}_2 \Rightarrow$ $B \in \mathbb{W}_1 \quad B \Rightarrow x = z \quad y = t$

$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ y $B \in \mathbb{W}_2 \Rightarrow x = -y,\ z = -t$

$$\Rightarrow x = z = -y = -t$$

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} -t & t \\ -t & t \end{pmatrix} = t \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

Base de $\mathbb{W}_1 \cap \mathbb{W}_2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$\dim \mathbb{W}_1 = 2, \quad \dim \mathbb{W}_2 = 2$

$\dim(\mathbb{W}_1 + \mathbb{W}_2) = \dim \mathbb{W}_1 + \dim \mathbb{W}_2 - \dim(\mathbb{W}_1 \cap \mathbb{W}_2)$

Como $\mathbb{W}_1 \cap \mathbb{W}_2 \neq \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ <u>NO</u> es suma directa.

3-) $f: \mathbb{R}^4 \longrightarrow \mathbb{R}_2[x]$

$(a,b,c,d) \longrightarrow (a+b)x^2 - cx + d$

a) A b. canónicas

$f(1,0,0,0) = x^2$

$f(0,1,0,0) = x^2$

$f(0,0,1,0) = -1x$

$f(0,0,0,1) = 1$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

b) Núcleo. ¿Biyectiva?

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} t=0 \\ z=0 \\ x=-y \end{matrix}$$

$(-y, y, 0, 0) = y(-1, 1, 0, 0)$

$\dim N(f) = 1 \Longrightarrow$ No es inyectiva $\Rightarrow$ No biyectiva

$4 = \dim N(f) + \dim \operatorname{Im} f$

$\operatorname{rg} A = 3 = \dim \operatorname{Im} f = \dim \mathbb{R}_2[x]$

Si es suprayectiva.

c) B en $B_1 = \{(1,2,0,0), (3,0,1,0), (1,0,0,0), (0,0,0,1)\}$

$B_2 = \{1, x, x^2\}$.

$f(1,0,0,0) = x^2$

$f(1,2,0,0) = 3x^2$

$f(3,0,1,0) = 3x^2 - x$

$f(0,0,0,1) = 1$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 3 & 3 & 0 \end{pmatrix} = B$$

d) P, Q t.q. $B = QAP$

$$\begin{tikzcd} f: B_c \arrow[r, "A"] & B_c \\ B_1 \arrow[u, "P"] \arrow[r, "B"'] & B_c \arrow[u, "Q"'] \end{tikzcd}$$

$B_1 = \{ (1,0,0,0), (1,2,0,0), (3,0,1,0), (0,0,0,1) \}$

$(1,0,0,0) = 1.(1,0,0,0) + 0(0,1,0,0) + 0(0,0,1,0) + 0(0,0,0,1)$

$(1,2,0,0) = (1,0,0,0) + 2(0,1,0,0) + 0(0,0,1,0) + 0(0,0,0,1)$

$(3,0,1,0) = 3(1,0,0,0) + 0(0,1,0,0) + 1(0,0,1,0) + 0(0,0,0,1)$

$(0,0,0,1) = 0(\quad) + 0(\quad) + 0(\quad) + 1(\quad)$

$$P = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$Q = I_{3\times 3}$ $\qquad$ $B = I_3 A P$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 3 & 3 & 0 \end{pmatrix} = B$$

4 - Val. prop. de A son $\{1, -1, 6\}$.

$\dim E_1(1) = \dim E_2(1) = 1$

$\dim E_1(-1) = 1$

$\dim E_2(-1) = 2$

$\dim E_3(-1) = \dim E_4(-1) = 3$

$\dim E_1(6) = \dim E_2(6) = 1$

¿ $\dim A$? ¿ J ?

$\left.\begin{array}{l} 1 \text{ val. prop. simple} \\ -1 \text{ val. prop. triple} \\ 6 \text{ val. prop. simple} \end{array}\right\}$ $A \in M_{5\times 5}$

$\underline{\lambda = 1}$ $\rightarrow$ v_1 vect. prop. asociado

$\underline{\lambda = 6}$ $\rightarrow$ v_2 vect. prop. asociado

$\underline{\lambda = -1}$

$u_3 \in E_3(-1) \setminus E_2(-1)$

$u_2 = (A + I) u_3$

$u_1 = (A + I) u_2$

$Au_1 = -u_1$

$Au_2 = u_1 - u_2$

$Au_3 = u_2 - u_3$

$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = J$

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = J$, $P = \begin{pmatrix} v_1 & v_2 & u_1 & u_2 & u_3 \end{pmatrix}$

$$p_1(x) * p_2(x) = \int_0^1 p_1(x)\, p_2(x)\, dx$$

a) $B = \{1, x\}$

$$1 * 1 = \int_0^1 1.1.dx = 1$$

$$1 * x = \int_0^1 1.x\, dx = \frac{1}{2} \qquad A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}$$

$$x * x = \int_0^1 x^2 dx = 1/3$$

b) ¿Qué angulo forman $x+3$ y $2x+4$?

$$\cos\alpha = \frac{u * v}{|u| \cdot |v|} \qquad \begin{matrix} u = x+3 \\ v = 2x+4 \end{matrix}$$

$$u * v = \int_0^1 (x+3)(2x+4)\, dx = \frac{53}{3}$$

$$|u| = \sqrt{(x+3)*(x+3)} = \sqrt{\int_0^1 (x+3)^2 dx} = \sqrt{\frac{37}{3}}$$

$$|v| = \sqrt{(2x+4)*(2x+4)} = \sqrt{\int_0^1 (2x+4)^2 dx} = \sqrt{\frac{76}{3}}$$

$$\cos\alpha = \frac{\frac{53}{3}}{\sqrt{\frac{37}{3}}\sqrt{\frac{76}{3}}} = \frac{53}{2\sqrt{703}}$$

$$\alpha = \arccos \frac{53}{2\sqrt{703}}$$

c) Base ortonormal $\{1, x\}$

$$\left.\begin{matrix} a_1 = 1 \\ a_2 = x \end{matrix}\right\} \qquad \begin{matrix} b_1 = 1 \\ b_2 = \end{matrix}$$

$$c_2 = a_2 + t b_1 = x + t$$

$$c_2 \perp b_1 \Rightarrow \int_0^1 (x+t)\, dx = 0 \Rightarrow \frac{1}{2} + t = 0 \Rightarrow t = -1/2$$

$$c_2 = x - \frac{1}{2}$$

$$\|c_2\| = \sqrt{\int_0^1 (x - 1/2)^2 dx} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}} \Rightarrow b_2 = \frac{x - 1/2}{1/2\sqrt{3}}$$

$\Rightarrow \{1, 2\sqrt{3}x - \sqrt{3}\}$ base ortonormal.

$$A=\begin{pmatrix}1 & 2 & 0\\ 2 & 3 & 1\\ 0 & 1 & 1\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc}1 & 2 & 0 & 1 & 0 & 0\\ 2 & 3 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right)\sim\cdots\sim\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & -2 & 1 & 0\\ 0 & 0 & 2 & -2 & 1 & 1\end{array}\right)$$

a) Indefinida, valores positivos y negativos en la diagonal. $sg(A)=(2,1)$

b) $P=\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -2 & 1 & 1\end{pmatrix}$

c) $\{(1,0,0),(-2,1,0),(-2,1,1)\}$

d) No, debería ser definida positiva.