Feb_07

En el conj.

$A = \{x \,/\, x$ son dígitos impares$\}$ se han definido las siguientes operaciones:

$*$: es una operación tal que,

$a * b = c$, siendo c la cifra de las unidades del producto ab.

$\cdot$: multiplicación estandar por los elementos de $\mathbb{R}$.

Se pide hacer la tabla de $(A, *)$ y comprobar si $(A, *, \cdot\mathbb{R})$ es, o no es, espacio vectorial.

• $(A, *)$ <u>NO</u> es un grupo

$*$	1	3	5	7	9
1	1	3	5	7	9
3	3	9	5	1	7
5	5	5	5	5	5
7	7	1	5	9	3
9	9	7	5	3	1

. Es <u>ley</u> de composición interna.

. Es asociativa por serlo el prod. de nºs enteros

. Es conmutativa $a*b = b*a$ $\forall a, b \in A$

. Existe el neutro $1 \in A$ $1*a = a$

. <u>NO</u> todos los elementos de A tienen simétrico, p. ej el 5.

$\Rightarrow (A, *, \cdot\mathbb{R})$ <u>NO</u> es espacio Vectorial.

1

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big/ \; 2a+b+c-d=0, \; a-b-c=0 \right\}$$

$-2\,1^a - 2^a \qquad 1^a + 2^a$

$$T = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$-2\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \; 1 \; 1$

a) $\forall \lambda, \mu \in \mathbb{R}, \; \forall u, v \in S \Rightarrow \lambda u + \mu v \in S$

$$\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} + \begin{pmatrix} \mu a' & \mu b' \\ \mu c' & \mu d' \end{pmatrix} = \begin{pmatrix} \lambda a + \mu a' & \lambda b + \mu b' \\ \lambda c + \mu c' & \lambda d + \mu d' \end{pmatrix}$$

$$\begin{cases} 2(\lambda a + \mu a') + \lambda b + \mu b' + \lambda c + \mu c' - \lambda d - \mu d' \overset{?}{=} 0 \\ \lambda a + \mu a' - \lambda b - \mu b' - \lambda c - \mu c' \overset{?}{=} 0 \end{cases}$$

$$\lambda(2a+b+c-d) + \mu(2a'+b'+c'-d') = 0$$

$$\lambda(a-b-c) + \mu(a'-b'-c') = 0$$

b) $\begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \Rightarrow$ $b = a - c$

$2a + a - c + c - d = 0$

$3a = d$

$$\begin{pmatrix} a & a-c \\ c & 3a \end{pmatrix} = a\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Base $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rg\, T = 2$

Base de $T = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$

$$\alpha \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$\left.\begin{array}{r} -\alpha + \beta = x \\ -\beta = y \\ -\beta = z \\ -\alpha + \beta = t \end{array}\right\} \quad \begin{array}{l} y = z \\ \\ x = -\alpha + z \\ t = -\alpha - z \end{array}$$

$$T = \left\{ \begin{pmatrix} -\alpha - y & y \\ y & -\alpha - y \end{pmatrix} \;/\; \alpha, y \in \mathbb{R} \right\}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & 2 & -2 \end{pmatrix} \quad rg = 4$$

$S \cap T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \Rightarrow$ Suma directa

```
  To get started, select "MATLAB Help" from the Help menu.

>> A=[1 -2 0;-2 6 1;0 1 2]

A =

     1    -2     0
    -2     6     1
     0     1     2

>> P=eye(3)

P =

     1     0     0
     0     1     0
     0     0     1

>> A=[A(1,:);A(2,:)+2*A(1,:);A(3,:)]

A =

     1    -2     0
     0     2     1
     0     1     2

>> T=[-1 1 1 0;0 -1 1 -1;0 -1 1 -1;-1 1 1 0]

T =

    -1     1     1     0
     0    -1     1    -1
     0    -1     1    -1
    -1     1     1     0

>> rref(T)

ans =

     1     0    -2     1
     0     1    -1     1
     0     0     0     0
     0     0     0     0

>> TS=[-1 1 1 0;0 -1 1 -1;0 -1 0 1;-1 1 3 0]

TS =

    -1     1     1     0
     0    -1     1    -1
     0    -1     0     1
    -1     1     3     0

>> rref(TS)

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

3-) $f(a_1+2a_2) = b_1+b_3$ $\quad$ $f(a_2) = 2b_1+b_2$

$$\left.\begin{array}{l} f(a_1+2a_2) = b_1+b_3 \\ f(a_2) = 2b_1+b_2 \end{array}\right\} \quad \begin{aligned} f(a_1) &= b_1+b_3-2f(a_2) = \\ &= b_1+b_3-2(2b_1+b_2) \\ &= -3b_1-2b_2+b_3 \end{aligned}$$

a) $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \\ 1 & 0 \end{pmatrix}$

$$f: V \longrightarrow W$$

$$\begin{tikzcd} \{a_i\} \arrow[r, "A"] & \{b_i\} \\ \{\alpha_i\} \arrow[u, "P"] \arrow[r, "B"'] & \{\beta_i\} \arrow[u, "Q"'] \end{tikzcd}$$

$$\left.\begin{array}{l} \alpha_1 = a_1+a_2 \\ \alpha_2 = a_1-a_2 \end{array}\right\}$$

$$\left.\begin{array}{l} \beta_1 = b_1+b_3 \\ \beta_2 = b_1-b_2 \\ \beta_3 = b_2 \end{array}\right\}$$

$$B = Q^{-1} A P$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad Q^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -5 \\ -1 & -3 \\ 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ -2 & -6 \\ -3 & -9 \end{pmatrix}$$

$rg(A) = 2$

$N(f) = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x = y = 0$

$\{\alpha_i\}$ $\{\beta_i\}$

c) $f(2,3) = \begin{pmatrix} 1 & 1 \\ -2 & -6 \\ -3 & -9 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -22 \\ -33 \end{pmatrix}$

Otra forma:

Si no tuviéramos la matriz en esas bases:

1º/ Expresar $\bar{x}' = (2,3)$ respecto a $B = \{a_i\}$

$\bar{x} = \underset{P}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

2º/ Calcular la imagen por f de $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ respecto a $\{b_i\}$

$f(5,-1) = \underset{A}{\begin{pmatrix} -3 & 2 \\ -2 & 1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -17 \\ -11 \\ 5 \end{pmatrix}$

3º/ Expresar en $\{\beta_i\}$

$\underset{Q^{-1}}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}} \begin{pmatrix} -17 \\ -11 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -22 \\ -33 \end{pmatrix}$

4-/

$h: V \to V$

$\{a_1, a_2, a_3, a_4\}$ base de V

$a_1, a_2 \in \text{Ker}\,(h - I) \iff$

$(A - I)\,a_1 = 0 \iff A a_1 = a_1$

$(A - I)\,a_2 = 0 \iff A a_2 = a_2$

$\boxed{1 \text{ val. prop. doble}}$

$h(a_3) = -2a_1 + 3a_3$

$A a_4 = 0 \iff \boxed{0 \text{ val. prop.}}$

a/ $v = a_1 - a_3$ vect. propio

¿$Av = \lambda v$?

$Av = Aa_1 - Aa_3 = a_1 - (-2a_1 + 3a_3) =$

$= 3a_1 - 3a_3 = 3\,(a_1 - a_3)$

$= 3v$

$\boxed{\lambda = 3 \text{ val. prop.}}$

b/ 1 val. prop. doble con 2 vect. propios
0
3
$\Big\} \Rightarrow$ Diag.

c/ $\{a_1, a_2, v, a_4\}$

$f(a_1) = a_1$
$f(a_2) = a_2$
$f(a_4) = 0$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$\dim E_1(\lambda) = 2$$

$$\dim E_2(\lambda) = 4$$

$$u_4 \in E_2 \setminus E_1$$

$$u_3 = (A - \lambda I)\, u_4 \qquad \Longrightarrow u_3 \in E_1(\lambda)$$

$$u_2 \in E_2 \setminus E_1$$

$$u_1 = (A - \lambda I)\, u_2 \qquad \Longrightarrow u_1 \in E_1(\lambda)$$

$$A u_1 = \lambda u_1$$

$$A u_2 = u_1 + \lambda u_2$$

$$A u_3 = \lambda u_3$$

$$A u_4 = u_3 + \lambda u_4$$

$$J = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$P = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}$$

Se considera el esp. vectorial de los polinomios y el producto escalar

$$p_1(x) \cdot p_2(x) = \int_0^1 p_1(x)\, p_2(x)\, dx$$

Determinar la norma de $p_1(x) = mx + 1$ sabiendo que es ortogonal al polinomio $p_2(x) = x^2$.

$$\int_0^1 (mx+1)\, x^2\, dx = \frac{m}{4} + \frac{1}{3} = 0 \implies m = -\frac{4}{3}$$

$$\int_0^1 \left(-\frac{4}{3}x+1\right)\left(-\frac{4}{3}x+1\right) dx = \frac{7}{27}$$

$$\| p_1(x) \| = \sqrt{\frac{7}{27}}$$