Interpolation lattices in several variables

J. M. Carnicer * , M. Gasca * and T. Sauer **
*University of Zaragoza, Spain, **University of Giessen, Germany

Abstract: Principal lattices are classical simplicial configurations of nodes suitable for multivariate polynomial interpolation in n dimensions. A principal lattice can be described as the set of intersection points of $n+1$ pencils of parallel hyperplanes. Using a projective point of view, Lee and Phillips extended this situation to $n+1$ linear pencils of hyperplanes. In two recent papers, two of us have introduced generalized principal lattices in the plane using cubic pencils. In this paper we analyze the problem in n dimensions, considering polynomial, exponential and trigonometric pencils, which can be combined in different ways to obtain generalized principal lattices. We also consider the case of coincident pencils. An error formula for generalized principal lattices is discussed.

1. Introduction

In contrast to the univariate case, the solvability (and uniqueness of the solution) of polynomial interpolation problems in several variables depends not only on the number but also on the *geometry* of the points where the function has to be interpolated. To be more precise, let $\mathbf{x}^1, \ldots, \mathbf{x}^N \in \mathbf{R}^n$ be a finite set of distinct points. The classical *Lagrange interpolation problem* consists of finding, for given data $y_1, \ldots, y_N \in \mathbf{R}$, a polynomial p such that $p\left(\mathbf{x}^j\right) = y_j$, $j = 1, \ldots, N$. Without further constraints, this solution is not unique, but becomes unique for $n = 1$ if the degree of the polynomial is restricted to be $\leq N-1$. This fact is often stated as that the polynomial interpolation problem with respect to arbitrary N points is *poised* or *correct* for the space Π^1_{N-1} of all (univariate) polynomials of degree $\leq N-1$. In other words, the univariate polynomials of limited degree form a *Haar space.* The natural extension would be to consider the space Π^n_m of all polynomials of *total degree* $\leq m$ in $n \geq 1$ variables. The dimension of this space is $N = \binom{n+m}{n}$, but even if the number of interpolation points coincides with the dimension, the Lagrange interpolation problem with respect to $\mathbf{x}^1, \ldots, \mathbf{x}^N$ need not be poised for Π^n_m (just consider 3 points on a straight line in $\mathbf{R}^2$ and Π^2_1).

To overcome the problem of non–poised point configurations, there have been various attempts to construct sets on $\binom{m+n}{n}$ points with respect to which the Lagrange interpolation problem is poised for Π^n_m. Since the classical paper by Chung and Yao [5], these constructions mainly consist of choosing the points as appropriate intersections of hyperplanes. The paper [6] provides constructions exploiting this idea. For more references see [7]. Recall that a *hyperplane* $H \subset \mathbf{R}^n$ is the zero set of an affine function, that is,

$$H = \{\mathbf{x} \in \mathbf{R}^n : h(\mathbf{x}) = 0\}, \qquad h(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x} + c, \quad \mathbf{v} \in \mathbf{R}^n \setminus \{0\},\, c \in \mathbf{R},$$

where the normal vector $\mathbf{v}$ and the constant c are unique up to normalization by a nonzero constant. In this paper, we will give a method to choose these hyperplanes in such a way

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that their intersections yield a proper set of interpolation points, forming a structure that generalizes what has been named *principal lattices* in the literature. The bivariate case was analyzed in [3] and [4]. Our goal is to study the case of more than two variables, which has turned out to be much more complex. We offer a variety of constructions here without claiming to cover all possibilities.

In Section 2 we briefly review the concept of principal lattices and a generalization of them. In Section 3 we present our method based on families of hyperplanes which we relate to univariate Chebyshev systems in Section 4, while Section 5 describes how to obtain lattices by combining different families. Finally, Section 6 deals with the *error of interpolation* on such a lattice, i.e., with the deviation of the interpolant from a sufficiently smooth function, showing that principal lattices admit a rather simple and geometrically appealing error formula as well as error estimates with a rather "univariate" flair.

2. A generalization of principal lattices

We shall denote $\mathbf{N}_m = \{0, 1, \dots, m\} \subset \mathbf{Z}$ and

$$\mathcal{S}_m^n := \{\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbf{N}_m^{n+1} : |\alpha| = m\}. \tag{1}$$

Principal lattices in $\mathbf{R}^n$ are distributions of points of the form

$$\mathbf{x} = \frac{\alpha_0}{m}\mathbf{a}^0 + \cdots + \frac{\alpha_n}{m}\mathbf{a}^n, \quad \alpha \in \mathcal{S}_m^n,$$

where $\mathbf{a}^i \in \mathbf{R}^n$ are the vertices of a simplex in $\mathbf{R}^n$ (see [8] for details on the history of these sets). In the case of the standard simplex, this set of points is $X = \{\alpha/m : \alpha \in \mathcal{S}_m^n\}$. Let us define

$$h_{\alpha_0}^0(\mathbf{x}) := \frac{m - \alpha_0}{m} - (x_1 + \cdots + x_n), \quad \alpha_0 \in \mathbf{N}_m,$$
$$h_{\alpha_r}^r(\mathbf{x}) := x_r - \frac{\alpha_r}{m}, \quad \alpha_r \in \mathbf{N}_m, \quad r \in \{1, \dots, n\},$$

and use the symbol $H_{\alpha_r}^r$ for the hyperplane defined by the equation $h_{\alpha_r}^r(\mathbf{x}) = 0$. The point $\mathbf{x}^\alpha = \alpha/m$, $\alpha \in \mathcal{S}_m^n$, of X is the intersection of the hyperplanes $H_{\alpha_r}^r$, $r = 0, \dots, n$.

Lee and Phillips [9] generalized this idea introducing lattices generated by $n+1$ pencils of hyperplanes

$$\{H_{\alpha_r}^r : \alpha_r \in \mathbf{N}_m\}, \quad r = 0, 1, \dots, n, \tag{2}$$

in the projective space $P^n(\mathbf{R})$. The lattice X is the set of points $\{\mathbf{x}^\alpha : \alpha \in \mathcal{S}_m^n\}$, where $\mathbf{x}^\alpha$ is the intersection of the hyperplanes $H_{\alpha_r}^r$, $r = 0, \dots, n$. In the general case, a projective coordinate system can be chosen so that the $n+1$ families of $m+1$ hyperplanes (2) have equations $h_{\alpha_r}^r(\mathbf{x}) = 0$ with

$$h_{\alpha_0}^n(\mathbf{x}) := \mu^{m-\alpha_0} x_n - x_0, \quad \alpha_0 \in \mathbf{N}_m,$$
$$h_{\alpha_r}^r(\mathbf{x}) := \mu^{\alpha_r} x_r - x_{r-1}, \quad \alpha_r \in \mathbf{N}_m, \quad r \in \{1, \dots, n\},$$

where $\mu \in \mathbf{R} \setminus \{-1, 0, 1\}$. It can be shown that X is the set of points $\mathbf{x}^\alpha$ with homogeneous coordinates

$$(\mu^{\alpha_1+\alpha_2+\cdots+\alpha_n}, \mu^{\alpha_2+\cdots+\alpha_n}, \dots, \mu^{\alpha_n}, 1),$$

where $\alpha \in \mathcal{S}^n_m$. Principal lattices can be described as the limit case when $\mu \to 1$ (cf. Section 4 of [9] and Section 2 of [3]). For more information on this kind of lattices see [11].

In [3, 4] a generalization of principal lattices in the plane was introduced in order to describe principal lattices as well as the lattices generated by $n+1$ pencils considered in [9] and further examples generated by cubic pencils. While in [9] all the $n+1$ pencils must be different, the constructions given in [3, 4] may use repeated pencils.

Let us state this definition in the n-dimensional case.

Definition 1. Let

$$H^r_i, \quad i \in \mathbf{N}_m, \quad r = 0, \dots, n,$$

be $n+1$ families of hyperplanes containing $(n+1)(m+1)$ distinct hyperplanes such that

(i) The intersection of any set of n hyperplanes $H^{r_1}_{i_1}, \dots, H^{r_n}_{i_n}$, corresponding to distinct indices $r_1, \dots, r_n \in \{0, 1, \dots, n\}$, consists of exactly one point.

(ii) we have that

$$\alpha \in \mathcal{S}^n_m \Longrightarrow \bigcap_{r=0}^{n} H^r_{\alpha_r} \neq \emptyset. \tag{3}$$

Under these assumptions the set of points

$$X := \{\mathbf{x}^\alpha : \mathbf{x}^\alpha := \bigcap_{r=0}^{n} H^r_{\alpha_r},\ \alpha = (\alpha_0, \dots, \alpha_n) \in \mathcal{S}^n_m\}, \tag{4}$$

is a *generalized principal lattice* of degree m (GPL_m) in $\mathbf{R}^n$ if it satisfies the additional condition

(iii) for any $\alpha_0, \dots, \alpha_n \in \mathbf{N}_m$ we have that

$$\bigcap_{r=0}^{n} H^r_{\alpha_r} \cap X \neq \emptyset \implies \alpha \in \mathcal{S}^n_m. \tag{5}$$

Remark 2. Let us see that a node lying on one hyperplane of each family $\mathbf{x}^\alpha = \bigcap_{r=0}^n H^r_{\alpha_r} \cap X$ cannot lie on any other hyperplane $H^r_{\beta_r}$ for some $r \in \{0, 1, \dots, n\}$, $\beta_r \in \mathbf{N}_m$. By (5), we have that $\alpha \in \mathcal{S}^n_m$. If

$$\mathbf{x}^\alpha \in H^0_{\alpha_0} \cap \cdots \cap H^{r-1}_{\alpha_{r-1}} \cap H^r_{\beta_r} \cap H^{r+1}_{\alpha_{r+1}} \cap \cdots \cap H^n_{\alpha_n} \cap X,$$

we may apply (5) to

$$(\alpha_0, \dots, \alpha_{r-1}, \beta_r, \alpha_{r+1}, \dots, \alpha_n)$$

and derive that

$$\alpha_0 + \cdots + \alpha_{r-1} + \beta_r + \alpha_{r+1} + \cdots + \alpha_n = m = \alpha_0 + \cdots + \alpha_{r-1} + \alpha_r + \alpha_{r+1} + \cdots + \alpha_n$$

and so, $\beta_r = \alpha_r$.

Now, let us show that for any node $\mathbf{x}^\alpha$, $\alpha \in \mathcal{S}^n_m$, there cannot exist any other index $\beta \in \mathcal{S}^n_m$ such that $\mathbf{x}^\beta = \mathbf{x}^\alpha$. Take any $r \in \{0, \dots, n\}$ and then $\mathbf{x}^\alpha \in H^r_{\beta_r}$. By Remark 2, we have $\beta_r = \alpha_r$. Since r is arbitrary, we have $\beta = \alpha$. So the cardinality of the set of nodes X defined in (4) is $\#\mathcal{S}_m^n = \binom{m+n}{n}$.

Let us note that, in contrast to [9], we do not impose that any family of hyperplanes is contained in a linear pencil. We shall describe constructions of GPL_m sets where each family is not contained in a linear pencil.

Generalized principal lattices satisfy the geometric characterization of Chung and Yao [5]. This property characterizes sets of $\binom{m+n}{n}$ nodes in the plane which are unisolvent for the Lagrange interpolation problem in Π_m^n and whose Lagrange polynomials are products of linear factors.

Definition 3. A set of $\binom{m+n}{n}$ nodes $X \subseteq \mathbf{R}^n$ satisfies the *geometric characterization* GC_m if for each node $\mathbf{x} \in X$, there exist m hyperplanes containing all nodes in $X \setminus \{\mathbf{x}\}$ but not $\mathbf{x}$.

Proposition 4. *Let X be a GPL_m set. Then X satisfies GC_m and hence it is a unisolvent set for the Lagrange interpolation problem in Π_m^n. A Lagrange formula for the interpolant $L_m f$ of a function $f \in C(\mathbf{R}^n)$ is given by*

$$L_m f = \sum_{\alpha \in \mathcal{S}_m^n} f(\mathbf{x}^\alpha) \prod_{r=0}^{n} \prod_{i=0}^{\alpha_r - 1} \frac{h_i^r}{h_i^r(\mathbf{x}^\alpha)}, \tag{6}$$

where $h_i^r(\mathbf{x}) = 0$ is the equation of the hyperplane H_i^r, $i \in \mathbf{N}_m$, $r = 0, \ldots, n$.

Proof: Given $\alpha \in \mathcal{S}_m^n$, the m hyperplanes

$$H_i^r, \quad 0 \le i \le \alpha_r - 1, \quad r = 0, \ldots, n, \tag{7}$$

contain all nodes of $X \setminus \{\mathbf{x}^\alpha\}$ but not $\mathbf{x}^\alpha$. ■

3. Construction of generalized principal lattices

In the bivariate case [3, 4], constructions of generalized principal lattices were obtained parameterizing the families of lines by elements of a group G. The groups arising in these constructions were the additive group of real numbers $\mathbf{R}$, $\mathbf{R} \times \mathbf{Z}_k$, where $\mathbf{Z}_k = \{0, 1, \dots, k-1\}$ is the additive group of integers modulo k, and quotient groups like $\mathbf{R}/2\pi\mathbf{Z}$.

In order to extend the constructions to more than two variables, let us assume that we have $n+1$ families of hyperplanes parameterized depending on a parameter $\omega \in G$, where G is an abelian group, that is,

$$\{H^r(\omega) : \omega \in G\}, \quad r = 0, \dots, n,$$

and assume that the following properties hold:

(P1) The intersection of any n distinct hyperplanes, $H^{r_1}(\omega_{r_1}), \dots, H^{r_n}(\omega_{r_n})$, corresponding to distinct indices $r_1, \dots, r_n \in \{0, 1, \dots, n\}$, consists of exactly one point.

(P2) Let $\omega_0, \dots, \omega_n \in G$ be such that the hyperplanes $H^r(\omega_r)$, $r = 0, \dots, n$, are distinct. Then $\bigcap_{r=0}^n H^r(\omega_r) \neq \emptyset$ if and only if $\omega_0 + \cdots + \omega_n = 0$.

It is important to mention that we need not require the families $H^r(\omega)$ to be distinct. In fact, the examples in Section 4 will even use *coincident* families

$$H^0(\omega) = H^1(\omega) = \cdots = H^n(\omega), \quad \omega \in G.$$

The following proposition shows how to construct generalized principal lattices.

Proposition 5. *Let $H^r(\omega)$, $\omega \in G$, $r = 0, \dots, n$ be $n+1$ families of hyperplanes such that (P1) and (P2) hold. Let $\delta \in G$, $\omega_{0,r} \in G$, $r = 0, \dots, n$, be such that*

$$\omega_{0,0} + \omega_{0,1} + \cdots + \omega_{0,n} + m\delta = 0. \tag{8}$$

If the $(n+1)(m+1)$ hyperplanes

$$H_i^r := H^r(\omega_{i,r}), \quad \omega_{i,r} = \omega_{0,r} + i\delta, \qquad i \in \mathbf{N}_m, \quad r = 0, \dots, n,$$

are distinct, then they define a GPL_m set in $\mathbf{R}^n$.

Proof: Since the hyperplanes are distinct, property (i) of Definition 1 clearly follows from (P1). Given any $\alpha \in \mathcal{S}_m^n$, the hyperplanes $H_{\alpha_r}^r = H(\omega_{0,r} + \alpha_r\delta)$ correspond to parameter values $\omega_{0,r} + \alpha_r\delta$. By (8), we have

$$\sum_{r=0}^n (\omega_{0,r} + \alpha_r\delta) = \omega_{0,0} + \omega_{0,1} + \cdots + \omega_{0,n} + (\alpha_0 + \cdots + \alpha_n)\delta = \omega_{0,0} + \omega_{0,1} + \cdots + \omega_{0,n} + m\delta = 0$$

and deduce from (P2), that $\bigcap_{r=0}^n H_{\alpha_r}^r \neq \emptyset$ and so (ii) of Definition 1 follows. Now we can define X as in (4) and it only remains to check (iii) of Definition 1. Let us assume that for $\alpha_r \in \mathbf{N}_m$, $r = 0, \dots, n$, $\bigcap_{r=0}^n H_{\alpha_r}^r \cap X \neq \emptyset$. By (4), the intersection point is a node

$\mathbf{x}^\beta$ with $\beta \in \mathcal{S}_m^n$. So $\mathbf{x}^\beta = \bigcap_{r=0}^n H_{\beta_r}^r = \bigcap_{r=0}^n H_{\alpha_r}^r \cap X$. Let us show that $\alpha_r = \beta_r$ for all $r \in \{0, 1, \ldots, n\}$. Taking into account that

$$\mathbf{x}^\beta \in H_{\beta_0}^0 \cap \cdots \cap H_{\beta_{r-1}}^{r-1} \cap H_{\alpha_r}^r \cap H_{\beta_{r+1}}^{r+1} \cap \cdots \cap H_{\beta_n}^n,$$

we have by (P2)

$$\omega_{0,0} + \omega_{0,1} + \cdots + \omega_{0,n} + (\beta_0 + \cdots + \beta_{r-1} + \alpha_r + \beta_{r+1} + \cdots + \beta_n)\delta = 0,$$
$$\omega_{0,0} + \omega_{0,1} + \cdots + \omega_{0,n} + (\beta_0 + \cdots + \beta_{r-1} + \beta_r + \beta_{r+1} + \cdots + \beta_n)\delta = 0,$$

and we obtain

$$(\alpha_r - \beta_r)\delta = 0.$$

Since the hyperplanes H_i^r, $i \in \mathbf{N}_m$, are distinct, we must have that $\omega_{r,0} + i\delta$ are distinct group elements and therefore $i\delta \neq 0$ for all $i \in \mathbf{Z}$, $0 < |i| \leq m$. So, we have that $\alpha_r = \beta_r$ for any $r \in \{0, 1, \ldots, n\}$. Therefore $\alpha = \beta \in \mathcal{S}_m^n$. ∎

In order to analyze the families of hyperplanes $H^r(\omega)$, we need to consider their equations

$$f_{1,r}(\omega)x_1 + \cdots + f_{n,r}(\omega)x_n + f_{0,r}(\omega) = 0.$$

The incidence properties are invariant under projective transformations and therefore we shall rather use homogeneous coordinates to express the equations in the form

$$h^r(x_0, \ldots, x_n; \omega) = 0,$$

where

$$h^r(x_0, \ldots, x_n; \omega) = \sum_{j=0}^n f_{j,r}(\omega)x_j.$$

We shall omit the dependence on the variables $x_0, \ldots, x_n$ and also write $H^r(\omega)$ instead of $h^r(x_0, \ldots, x_n; \omega)$. This notational convention means that we do not make a distinction between the hyperplane $H^r(\omega)$ and the linear function $h^r(x_0, \ldots, x_n; \omega)$ such that $h^r(1, x_1, \ldots, x_n; \omega) = 0$ is the equation defining $H^r(\omega)$.

We next show that principal lattices are generated by families of hyperplanes satisfying (P1) and (P2). To that end, we choose $G = \mathbf{R}$ and

$$H^0(t) = tx_0 - (x_1 + \cdots + x_n), \qquad H^r(t) = tx_0 + x_r, \quad r = 1, \ldots, n.$$

The intersection of $H^r(t_r)$, $r = 0, \ldots, n$, gives rise to a system of equations

$$A(x_0, \ldots, x_n)^T = 0,$$

whose coefficient matrix A is

$$\begin{pmatrix} t_0 & -1 & -1 & \cdots & -1 \\ t_1 & 1 & 0 & \cdots & 0 \\ t_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ t_n & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{9}$$

Since any n rows of A are independent, (P1) holds. The $n+1$ hyperplanes are concurrent if and only if $\det A = 0$. Since $\det A = t_0 + \cdots + t_n$, (P2) holds.

Let us analyze the Lee and Phillips construction of lattices generated by $n+1$ pencils. We choose $G = \mathbf{R} \times \mathbf{Z}_2$ and

$$H^r(t,s) = x_r - (-1)^s \exp(t) x_{r+1},\ t \in \mathbf{R},\ s \in \mathbf{Z}_2, \quad r \in \mathbf{Z}_n.$$

Here by $r \in \mathbf{Z}_n$ we mean that the indexing is cyclic and that x_{n+1} denotes x_0. The intersection of $H^r(t_r, s_r)$, $r = 0, \ldots, n$, corresponds to nontrivial solutions of the system $A(x_0, \ldots, x_n)^T = 0$, where A is

$$\begin{pmatrix} 1 & -\mu_0 & 0 & \cdots & 0 \\ 0 & 1 & -\mu_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\mu_{n-1} \\ -\mu_n & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{10}$$

with $\mu_i = (-1)^{s_i} \exp(t_i)$. Since any n rows of A are independent, (P1) holds. The hyperplanes $H^r(t_r)$ are concurrent if and only if $0 = \det A = 1 - \mu_0 \cdots \mu_n$, that is,

$$(-1)^{s_0 + \cdots + s_n} \exp(t_0 + \cdots + t_n) = 1,$$

which is equivalent to $s_0 + \cdots + s_n = 0$ and $t_0 + \cdots + t_n = 0$. So (P2) is also satisfied.

4. Generalized principal lattices obtained from a single family

In this section we will provide a general construction process for an arbitrary number of variables with respect to the three additive groups $\mathbf{R}$, $\mathbf{R} \times \mathbf{Z}_2$ and $\mathbf{R}/2\pi\mathbf{Z}$. In the sequel, G will always stand for one of these three groups which are also locally compact Hausdorff spaces, so that the tools from Approximation Theory we are going to use in this section are well defined and accessible, cf. [10].

The goal is to find functions $f_j : G \to \mathbf{R}$, $j = 0, \ldots, n$, such that, for $\omega_0, \ldots, \omega_n \in G$ distinct,

$$\det \begin{pmatrix} f_0(\omega_0) & \ldots & f_n(\omega_0) \\ \vdots & \ddots & \vdots \\ f_0(\omega_n) & \ldots & f_n(\omega_n) \end{pmatrix} = 0 \qquad \Longleftrightarrow \qquad \sum_{j=0}^{n} \omega_j = 0, \tag{11}$$

according to the concurrency condition (P2). Clearly, such conditions are very closely related to the concept of *Chebyshev spaces*. Recall that a k-dimensional space F of functions with basis say $f_1, \ldots, f_k$ is called a *Chebyshev space* (and the basis is called a *Haar system*) on a set X if for any choice of distinct points $x_i \in X$, $i = 1, \ldots, k$, one has that

$$\Psi(x_1, \ldots, x_k) := \det(f_j(x_i) : i, j = 1, \ldots, k) \neq 0.$$

The following simple lemma is crucial to our construction.

Lemma 6. *Let $f_0, \dots, f_n, f_{n+1}$ be a Haar system on G and $f = \sum_{j=0}^{n+1} \alpha_j f_j$ a nonzero function, vanishing on distinct $\omega_0, \dots, \omega_n \in G$:*

$$f(\omega_0) = \dots = f(\omega_n) = 0.$$

Then

$$\det \begin{pmatrix} f_0(\omega_0) & \dots & f_n(\omega_0) \\ \vdots & \ddots & \vdots \\ f_0(\omega_n) & \dots & f_n(\omega_n) \end{pmatrix} = 0, \tag{12}$$

if and only if $\alpha_{n+1} = 0$, that is, f is a linear combination of $f_0, \dots, f_n$.

Proof: Let $\omega \in G \backslash \{\omega_0, \dots, \omega_n\}$. The coefficients of f with respect to the basis $f_0, \dots, f_{n+1}$ satisfy

$$\begin{pmatrix} f_0(\omega_0) & \dots & f_n(\omega_0) & f_{n+1}(\omega_0) \\ \vdots & \ddots & \vdots & \vdots \\ f_0(\omega_n) & \dots & f_n(\omega_n) & f_{n+1}(\omega_n) \\ f_0(\omega) & \dots & f_n(\omega) & f_{n+1}(\omega) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(\omega) \end{pmatrix}.$$

Observe that $f(\omega) \neq 0$ because f is a nonzero function vanishing on $\omega_0, \dots, \omega_n$, and $(f_0, \dots, f_{n+1})$ is a Haar system. By Cramer's rule, we have

$$\alpha_{n+1} = \frac{f(\omega)}{\Psi(\omega_0, \dots, \omega_n, \omega)} \det \begin{pmatrix} f_0(\omega_0) & \dots & f_n(\omega_0) \\ \vdots & \ddots & \vdots \\ f_0(\omega_n) & \dots & f_n(\omega_n) \end{pmatrix},$$

where

$$\Psi(\omega_0, \dots, \omega_n, \omega) = \det \begin{pmatrix} f_0(\omega_0) & \dots & f_n(\omega_0) & f_{n+1}(\omega_0) \\ \vdots & \ddots & \vdots & \vdots \\ f_0(\omega_n) & \dots & f_n(\omega_n) & f_{n+1}(\omega_n) \\ f_0(\omega) & \dots & f_n(\omega) & f_{n+1}(\omega) \end{pmatrix}.$$

Since $f(\omega)$ and $\Psi(\omega_0, \dots, \omega_n, \omega)$ are nonzero we have that $\alpha_{n+1} = 0$ if and only if (12) holds. ■

Lemma 6 is the key to finding functions $f_0, \dots, f_n$ such that (11) holds in order to construct pencils of hyperplanes satisfying properties (P1) and (P2).

Proposition 7. *Let $f_0, \dots, f_n, f_{n+1}$ be a Haar system on G. Assume that for any distinct $\omega_0, \dots, \omega_n \in G$, we can find a nonzero function vanishing on $\omega_0, \dots, \omega_n$,*

$$f(\omega; \omega_0, \dots, \omega_n) = \sum_{j=0}^{n+1} \alpha_j(\omega_0, \dots, \omega_n) f_j(\omega),$$

with

$$\alpha_{n+1}(\omega_0, \dots, \omega_n) = \varphi(\omega_0 + \dots + \omega_n),$$

where $\varphi : G \to \mathbf{R}$ is a function such that $\varphi(\omega) = 0$ if and only if $\omega = 0$. Let us define the pencil of hyperplanes

$$H(\omega) = \sum_{j=0}^{n} f_j(\omega)\, x_j, \quad \omega \in G.$$

Then we have

(a) Hyperplanes corresponding to distinct parameter values are distinct, that is, $H(\omega_1) \neq H(\omega_2)$, for any $\omega_1 \neq \omega_2$ in G.

(b) The coincident pencils $H^r(\omega) := H(\omega)$, $r = 0, \ldots, n$, satisfy (P1) and (P2).

(c) For any $\omega_{0,0}, \omega_{0,1}, \ldots, \omega_{0,n}, \delta \in G$ with $\omega_{0,0} + \omega_{0,1} + \cdots + \omega_{0,n} + m\delta = 0$ and such that the $(n+1)(m+1)$ values

$$\omega_{i,r} = \omega_{0,r} + i\delta, \quad i \in \mathbf{N}_m, \quad r = 0, \ldots n,$$

are all distinct, the families of hyperplanes $H_i^r = H(\omega_{i,r})$ define a GPL$_m$ *set.*

Proof: (a) and (b) If $\omega_0, \ldots, \omega_n$ are distinct, then, by Lemma 6, the matrix

$$M(\omega_0, \ldots, \omega_n) := \begin{pmatrix} f_0(\omega_0) & \ldots & f_n(\omega_0) \\ \vdots & \ddots & \vdots \\ f_0(\omega_n) & \ldots & f_n(\omega_n) \end{pmatrix}$$

has zero determinant if and only if $\varphi(\omega_0 + \cdots + \omega_n) = 0$, that is, $\omega_0 + \cdots + \omega_n = 0$.

For any distinct $\omega_0, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n \in G$, let us define

$$\omega_i = -(\omega_0 + \cdots + \omega_{i-1}) - (\omega_{i+1} + \cdots + \omega_n)$$

and consider any $\omega \in G \setminus \{\omega_0, \ldots, \omega_n\}$. Then the matrix

$$M(\omega_0, \ldots, \omega_{i-1}, \omega, \omega_{i+1}, \ldots, \omega_n) = \begin{pmatrix} f_0(\omega_0) & \ldots & f_n(\omega_0) \\ \vdots & \ddots & \vdots \\ f_0(\omega_{i-1}) & \ldots & f_n(\omega_{i-1}) \\ f_0(\omega) & \ldots & f_n(\omega) \\ f_0(\omega_{i+1}) & \ldots & f_n(\omega_{i+1}) \\ \vdots & \ddots & \vdots \\ f_0(\omega_n) & \ldots & f_n(\omega_n) \end{pmatrix}$$

has nonzero determinant because

$$\varphi(\omega_0 + \ldots + \omega_{i-1} + \omega + \omega_{i+1} + \cdots + \omega_n) = \varphi(\omega - \omega_i) \neq 0.$$

Therefore any n rows of $M(\omega_0, \ldots, \omega_{i-1}, \omega, \omega_{i+1}, \ldots, \omega_n)$ are linearly independent and the intersection of the n hyperplanes $\bigcap_{r \neq i} H(\omega_r)$ is exactly one point. Since this can be done for any $i \in \{0, \ldots, n\}$, the hyperplanes $H(\omega_0), \ldots, H(\omega_n)$, corresponding to distinct parameters $\omega_0, \ldots, \omega_n$, are distinct and moreover (P1) holds. Finally, (P2) follows from the fact that, for any choice of distinct $\omega_0, \ldots, \omega_n$, $\det M(\omega_0, \ldots, \omega_n) = 0$ if and only if $\omega_0 + \cdots + \omega_n = 0$.

Part (c) of the proposition follows from (a), (b) and Proposition 5. ∎

Now we are ready to show some relevant examples of the above construction of GPL$_m$ sets.

Example 1. For the group $G = \mathbf{R}$, we choose the Haar system defined by $f_j(t) = t^j$, $j = 0, \dots, n-1$, $f_n(t) = t^{n+1}$ and $f_{n+1}(t) = t^n$. For any selection of distinct points $t_0, \dots, t_n$, the coefficient of the polynomial

$$f(t) = (t - t_0) \cdots (t - t_n) = \sum_{j=0}^{n+1} \alpha_j f_j(t),$$

in t^n is $\alpha_{n+1} = -(t_0 + \cdots + t_n)$ and therefore we find that the pencil

$$H(t) = \sum_{j=0}^{n-1} t^j \, x_j + t^{n+1} \, x_n, \quad t \in \mathbf{R}, \tag{13}$$

satisfies the hypotheses of Proposition 7.

Example 2. For the group $G = \mathbf{R} \times \mathbf{Z}_2$ we write $\omega = (t, s) \in \mathbf{R} \times \mathbf{Z}_2$ and apply the transformation $\mu := (-1)^s \exp t \in \mathbf{R}^* := \mathbf{R} \setminus \{0\}$. Then we use the Haar system on the multiplicative group $\mathbf{R}^*$

$$f_j(\mu) = \mu^j, \; j = 0, \dots, n-1, \quad f_n(\mu) = \mu^n - (-1)^n \mu^{-1}, \quad f_{n+1}(\mu) = \mu^{-1}$$

and define for distinct $\mu_0, \dots, \mu_n$ in $\mathbf{R}^*$

$$f(\mu) = \mu^{-1} \prod_{j=0}^{n} \left(1 - \frac{\mu}{\mu_j} \right) = \sum_{j=0}^{n+1} \alpha_j f_j(\mu).$$

The coefficient with respect to f_{n+1} is then computed to be $\alpha_{n+1} = 1 - \mu_0^{-1} \cdots \mu_n^{-1}$. Then $\alpha_{n+1} = 0$ if and only if $\mu_0 \cdots \mu_n = 1$, that is, $s_0 + \cdots + s_n = 0$ and $t_0 + \cdots + t_n = 0$. Therefore the pencil

$$H(t,s) = \sum_{j=0}^{n-1} (-1)^{js} \exp(jt) x_j + [(-1)^{ns} \exp(nt) - (-1)^{n-s} \exp(-t)] x_n, \quad t \in \mathbf{R}, s \in \mathbf{Z}_2 \tag{14}$$

satisfies the hypotheses of Proposition 7.

Example 3. In the last case $G = \mathbf{R}/2\pi\mathbf{Z}$, the function f to be considered will be

$$f(t) = \prod_{j=0}^{n} \sin \frac{t - t_j}{2} = \sum_{j=0}^{n+1} \alpha_j f_j(t), \tag{15}$$

but the Haar system $(f_0, \dots, f_{n+1})$ will depend on n. If n is odd, $n = 2k+1$, we set $f_0(t) = 1$, $f_{2j-1}(t) = \cos(jt)$ and $f_{2j}(t) = \sin(jt)$, $j = 1, \dots, k+1$, while for the even case $n = 2k$ we choose the functions $f_{2j}(t) = \sin \frac{(2j+1)t}{2}$ and $f_{2j+1}(t) = \cos \frac{(2j+1)t}{2}$, $j = 0, \dots, k$.

In order to identify the coefficient α_{n+1} in both cases, we rewrite (15) in terms of complex exponentials and expand it, using the abbreviation $T = t_0 + \cdots + t_n$, into

$$
\begin{aligned}
(2i)^{n+1} f(t) &= \prod_{j=0}^{n} \left(e^{i(t-t_j)/2} - e^{-i(t-t_j)/2} \right) \\
&= e^{i((n+1)t-T)/2} + (-1)^{n+1} e^{-i((n+1)t-T)/2} \\
&\quad - \sum_{j=0}^{n} \left(e^{i((n-1)t-T+2t_j)/2} + (-1)^{n+1} e^{-i((n-1)t-T+2t_j)/2} \right) + \cdots
\end{aligned}
\tag{16}
$$

For $n = 2k+1$, (16) yields

$$
\begin{aligned}
f(t) &= (-1)^{(n+1)/2} 2^{-n} \left[\cos\left(\frac{(n+1)t}{2} - \frac{T}{2} \right) - \sum_{j=0}^{n} \cos\left(\frac{(n-1)t}{2} - \frac{T-2t_j}{2} \right) + \cdots \right] \\
&= (-1)^{k+1} 2^{-(2k+1)} \left[\cos((k+1)t)\, \cos\frac{T}{2} + \sin((k+1)t)\, \sin\frac{T}{2} + \cdots \right].
\end{aligned}
$$

The coefficient α_{n+1} is

$$
\alpha_{2k+2} = (-1)^{k+1} 2^{-(2k+1)} \sin\frac{T}{2}
$$

and vanishes if and only if $T \in 2\pi\mathbf{Z}$. Therefore the pencil

$$
H(t) = x_0 + \sum_{j=1}^{k+1} x_{2j-1} \cos(jt) + \sum_{j=1}^{k} x_{2j} \sin(jt)
$$

satisfies the hypotheses of Proposition 7.

If $n = 2k$, on the other hand, the expansion (16) becomes

$$
\begin{aligned}
f(t) &= (-1)^{n/2} 2^{-n} \left[\sin\left(\frac{(n+1)t}{2} - \frac{T}{2} \right) - \sum_{j=0}^{n} \sin\left(\frac{(n-1)t}{2} - \frac{T-2t_j}{2} \right) + \cdots \right] \\
&= (-1)^{k} 2^{-2k} \left[\sin\frac{(2k+1)t}{2} \cos\frac{T}{2} - \cos\frac{(2k+1)t}{2} \sin\frac{T}{2} + \cdots \right].
\end{aligned}
$$

The coefficient α_{n+1} is

$$
\alpha_{2k+1} = (-1)^{k+1} 2^{-2k} \sin\frac{T}{2}
$$

and therefore

$$
H(t) = \sum_{j=0}^{k} x_{2j} \sin\frac{(2j+1)t}{2} + \sum_{j=0}^{k-1} x_{2j+1} \cos\frac{(2j+1)t}{2}
$$

satisfies the hypotheses of Proposition 7.

Remark 8. Changes of variables in the projective space lead to other families of hyperplanes generating GPL_m sets, which are projective images of the sets defined above. In the first example these families are of the form

$$H(t) = f_0(t)x_0 + \cdots + f_n(t)x_n,$$

where $f_0, \ldots, f_n$ form a basis of the polynomial space generated by $1, t, \ldots, t^{n-1}, t^{n+1}$. In the second example they can be written as

$$H(t, s) = f_0((-1)^s \exp(t))x_0 + \cdots + f_n((-1)^s \exp(t))x_n,$$

where $f_0, \ldots, f_n$ form a basis of the space of Laurent polynomials generated by 1, μ, ..., μ^{n-1}, $\mu^n - (-1)^n\mu^{-1}$. Finally, in the third example a general form of the pencil is

$$H(t) = f_0(t)x_0 + \cdots + f_n(t)x_n,$$

where $f_0, \ldots, f_n$ form a basis of the subspace of trigonometric polynomials generated by $1, \cos t, \sin t, \ldots, \cos((n/2 - 1)t), \sin((n/2 - 1)t), \cos(nt/2)$, if n is even and of the space generated by $\sin(t/2), \cos(t/2), \ldots, \sin((n - 2)t/2), \cos((n - 2)t/2t), \sin(nt/2)$, if n is odd.

5. Generalized principal lattices combining different families

In the preceding section we have shown how to check conditions (P1) and (P2) for generalized principal lattices, finding systems of functions satisfying (11). However, we have used Lemma 6 to avoid the explicit computation of the determinant.

In this section we want to combine the examples of Section 4 in order to generate new examples. In this case, it is not so easy to apply Lemma 6. A general method will be provided for examples 1 and 2 and a particular instance will be shown for Example 3. We shall apply Proposition 5 to check the validity of our constructions.

We introduce the notation

$$V(g; t_0, \ldots, t_n) := \begin{pmatrix} 1 & t_0 & \cdots & t_0^{n-1} & g(t_0) \\ 1 & t_1 & \cdots & t_1^{n-1} & g(t_1) \\ 1 & t_2 & \cdots & t_2^{n-1} & g(t_2) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & t_n & \cdots & t_n^{n-1} & g(t_n) \end{pmatrix}$$

and, for the usual Vandermonde matrix, we write

$$V(t_0, \ldots, t_n) := V((\cdot)^n; t_0, \ldots, t_n).$$

The *divided difference* $[t_0, \ldots, t_n]g$ is the coefficient c_n of t^n in the interpolating polynomial $p(t) = c_0 + c_1 t + \cdots + c_n t^n$ of g at $t_0, \ldots, t_n$. Observe that

$$(c_0, \ldots, c_n)^T = V(t_0, \ldots, t_n)^{-1}(g(t_0), \ldots, g(t_n))^T$$

for distinct $t_0, \ldots, t_n$. So, it follows that

$$V(t_0, \ldots, t_n)^{-1} V(g; t_0, \ldots, t_n) = \begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & [t_0, \ldots, t_n]g \end{pmatrix}. \tag{17}$$

In Example 1, we know that the polynomial of degree at most n interpolating $g(t) = t^{n+1}$ at $t_0, \ldots, t_n$ is

$$t^{n+1} - (t - t_0) \cdots (t - t_n) = (t_0 + \cdots + t_n)t^n + \text{ lower degree terms}$$

and so

$$[t_0, \ldots, t_n]g = t_0 + \cdots + t_n, \text{ for } g(t) = t^{n+1}. \tag{18}$$

In Example 2, the polynomial of degree not greater that n interpolating $g(\mu) = \mu^n - (-1)^n \mu^{-1}$ at $\mu_0, \ldots, \mu_n$ is

$$\mu^n - (-1)^n \mu^{-1} - (\mu_0^{-1}\mu - 1) \cdots (\mu_n^{-1}\mu - 1)\mu^{-1} = (1 - \mu_0^{-1} \cdots \mu_n^{-1})\mu^n + \text{ lower degree terms}$$

and so

$$[\mu_0, \ldots, \mu_n]g = 1 - (\mu_0 \cdots \mu_n)^{-1}, \text{ for } g(\mu) = \mu^n - (-1)^n \mu^{-1}. \tag{19}$$

Next, we point out how to combine several families of Example 1. We choose p different families (13) of degrees $k_1, \ldots, k_p$, with $k_1 + \cdots + k_p = n + 1$. Let us partition the set of indices $I := \{0, 1, \ldots, n\}$, into p subsets $I = \bigcup_{l=1}^p I_l$, $\# I_l = k_l$, $l = 1, \ldots, p$,

$$I_l := \{r \in I : k_1 + \cdots + k_{l-1} \leq r < k_1 + \cdots + k_l\}. \tag{20}$$

We take

$$H^r(t) = \sum_{j=0}^{k_1 - 2} t^j x_j + t^{k_1} x_{n-p+1} - t^{k_1 - 1} \sum_{j=n-p+2}^{n} x_j \tag{21}$$

for $r \in I_1$ and

$$H^r(t) = \sum_{j=0}^{k_l - 2} t^j x_{k_1 + \cdots + k_{l-1} - (l-1) + j} + t^{k_l} x_{n-p+1} + t^{k_l - 1} x_{n-p+l} \tag{22}$$

for $r \in I_l$, $l = 2, \ldots, p$.

Let us apply Proposition 5 to construct generalized principal lattices. In order to check (P1) and (P2) we need to deal with the coefficient matrix of the linear system $H^r(t_r) = 0$, $r = 0, \ldots, n$,

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & B_1 & C_1 \\ 0 & A_{22} & \ddots & \vdots & B_2 & C_2 \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & A_{pp} & B_p & C_p \end{pmatrix}.$$

In the above formula

$$A_{ll} := (t_i^{j-1})_{i\in I_l; j\in\{1,\dots,k_l-1\}}$$

is the $k_l \times (k_l-1)$ matrix formed by the k_l-1 first columns of the Vandermonde matrix $V(t_i; i\in I_l)$. The matrix

$$B_l := (t_i^{k_l})_{i\in I_l},$$

is a $k_l \times 1$ matrix. The $k_1\times(p-1)$ matrix C_1 is formed by $p-1$ repeated columns $(-t_i^{k_1-1})_{i\in I_1}$ and the $k_l\times(p-1)$ matrix C_l, $l=2,\dots,p$, is formed by zero columns, except the $(l-1)$-th column which is of the form $(t_i^{k_l-1})_{i\in I_l}$. Observe that any nonzero column of any C_l is the last column of the Vandermonde matrix $V(t_i; i\in I_l)$.

Let us show that

$$\det A = (-1)^\sigma (t_0+\cdots+t_n)\prod_{l=1}^p \prod_{i>j\in I_l}(t_i-t_j), \quad \sigma\in\{0,1\} \tag{23}$$

where σ depends only on $k_1,\dots,k_p$. If, for some $l\in\{1,\dots,p\}$, we have $t_i=t_j$, $i,j\in I_l$, then two rows of A are equal and $\det A=0$ and formula (23) holds. Otherwise the matrix $W:=\operatorname{diag}(V_1,\dots,V_p)$, where $V_l := V(t_i; i\in I_l)$, is nonsingular and

$$E := W^{-1}A = \begin{pmatrix} V_1^{-1}A_{11} & 0 & \cdots & 0 & V_1^{-1}B_1 & V_1^{-1}C_1 \\ 0 & V_2^{-1}A_{22} & \ddots & \vdots & V_2^{-1}B_2 & V_2^{-1}C_2 \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & V_p^{-1}A_{pp} & V_p^{-1}B_p & V_p^{-1}C_p \end{pmatrix}.$$

Then

$$\det A = \det W \det E = \det E \prod_{l=1}^p \det V_l = \det E \prod_{l=1}^p \prod_{i>j\in I_l}(t_i-t_j). \tag{24}$$

Let us compute $\det E$. Observe that $V_l^{-1}A_{ll}$ is the $k_l\times(k_l-1)$ matrix formed by the first k_l-1 columns of the identity. Expanding the determinant of E by the first $n+1-p$ columns we get that $\det E$ coincides up to a sign (which only depends on the size of the blocks) with the determinant of the $p\times p$ submatrix $\hat{E}$ formed by the last p columns and the last row of each block

$$\hat{E} := (e_{k_1+\cdots+k_l,n-p+j})_{l,j\in\{1,\dots,p\}}. \tag{25}$$

Taking into account (17) and (18) we have

$$\hat{e}_{l,1} = e_{k_1+\cdots+k_l,n-p+1} = [t_i; i\in I_l](\cdot)^{k_l} = \sum_{i\in I_l} t_i, \quad l=1,\dots,p.$$

Similarly, we can deduce that

$$\hat{e}_{1,j} = e_{k_1,n-p+j} = -[t_i; i\in I_1](\cdot)^{k_1-1} = -1, \quad j=2,\dots,p.$$

We also have that

$$\hat{e}_{l,j} = e_{k_1+\cdots+k_l,j} = 0, \quad j,l = 2,\dots,n, \quad j \neq l,$$

because it corresponds to a zero column of $V_l^{-1}C_l$. Finally

$$\hat{e}_{l,l} = e_{k_1+\cdots+k_l,l} = [t_i; i \in I_l](\cdot)^{k_l-1} = 1, \quad l = 2,\dots,n.$$

So, we have that

$$\hat{E} = \begin{pmatrix} \sum_{i\in I_1} t_i & -1 & -1 & \cdots & -1 \\ \sum_{i\in I_2} t_i & 1 & 0 & \cdots & 0 \\ \sum_{i\in I_3} t_i & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \sum_{i\in I_p} t_i & 0 & \cdots & 0 & 1 \end{pmatrix}$$

is a matrix analogous to (9) and has determinant $t_0+t_1+\cdots+t_n$. This observation allows us to obtain $\det E = (-1)^\sigma(t_0+\cdots+t_n)$, $\sigma \in \{0,1\}$, and, by (24), formula (23) holds.

Now, let us verify (P1) and (P2). Take $t_0,\dots,t_n \in \mathbf{R}$ such that

$$t_i, \quad i \in I_l \text{ are distinct,} \tag{26}$$

for all $l \in \{1,\dots,p\}$. For any $r \in I_l$ and $l \in \{1,\dots,p\}$, we take $t \in \mathbf{R}$ such that

$$t \neq t_i, \quad \forall i \in I_l \setminus \{r\}, \quad t \neq -\sum_{i\in I_l, i\neq r} t_i.$$

The coefficient matrix of the system

$$\begin{aligned} &H^i(t_i) = 0, \quad i \in \{0,1,\dots,n\}, \quad i \neq r \\ &H^r(t) = 0, \end{aligned}$$

has nonzero determinant. Therefore, the set of solutions of the homogeneous system of equations $H^i(t_i) = 0$, $i \in \{0,1,\dots,n\}$, $i \neq r$, is a one-dimensional vector subspace of $\mathbf{R}^{n+1}$, that is, the hyperplanes $H^i(t_i)$, $i \in \{0,1,\dots,n\}$, $i \neq r$, are distinct and intersect exactly at one point. So (P1) holds. Finally, (P2) follows from (23), which implies that, the determinant of the matrix of the system $H^r(t_i) = 0$, $r = 0,\dots,n$, is zero if and only if $t_0+\cdots+t_n = 0$. According to Proposition 5, we can define a generalized principal lattice taking, $t_{0,0},\dots,t_{0,n}$ and t in $\mathbf{R}$ such that $\sum_{r=0}^n t_{0,r} + mt = 0$ and the hyperplanes $H_i^r = H^r(t_{0,r}+it)$, $i \in \mathbf{N}_m$, $r \in \{0,\dots,n\}$. In order to have distinct hyperplanes we only need that, for each $l \in \{1,\dots,p\}$, the values

$$t_{0,r} + it, \quad i \in \mathbf{N}_m, \quad r \in I_l,$$

are distinct.

Now we want to combine several families of Example 2. We choose p different families (14) corresponding to spaces of Laurent polynomials $\langle \mu^{-1}, 1, \ldots, \mu^{k_l-1}\rangle$, $l = 1, \ldots, p$, where $k_1 + \cdots + k_p = n + 1$. Let us partition the set of indices $I = \{0, 1, \ldots, n\}$, in p subsets $I = \bigcup_{l=1}^{p} I_l$, $\# I_l = k_l$, $l = 1, \ldots, p$, as in (20). For $t \in \mathbf{R}$, $s \in \mathbf{Z}_2$, we define the families

$$
\begin{aligned}
H^r(t,s) &:= \sum_{j=0}^{k_l-2} (-1)^{js} \exp(jt) x_{k_1+\cdots+k_{l-1}-(l-1)+j} + (-1)^{(k_l-1)s} \exp((k_l-1)t) x_{n-p+l} + \\
&\quad (-1)^{k_l-s} \exp(-t) x_{n-p+l+1}, \quad r \in I_l, \quad l = 1, \ldots, p-1, \\
H^r(t,s) &:= \sum_{j=0}^{k_p-2} (-1)^{js} \exp(jt) x_{k_1+\cdots+k_{p-1}-(p-1)+j} + (-1)^{(k_p-1)s} \exp((k_p-1)t) x_n + \\
&\quad (-1)^{k_p-s} \exp(-t) x_{n-p+1}, \quad r \in I_p.
\end{aligned}
\tag{27}
$$

The coefficient matrix of the linear system $H^r(t_r, s_r) = 0$, $r = 0, \ldots, n$, can be written in the form

$$
A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & F_1 \\ 0 & A_{22} & \ddots & \vdots & F_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & A_{pp} & F_p \end{pmatrix}.
$$

Let us describe the nonzero blocks of this matrix in terms of $\mu_r = (-1)^{s_r} \exp(t_r)$, $r = 0, \ldots, n$. The block

$$
A_{ll} := (\mu_i^{j-1})_{i \in I_l; j \in \{1, \ldots, k_l-2\}}
$$

is the $k_l \times (k_l - 1)$ matrix formed by the $k_l - 1$ first columns of the Vandermonde matrix $V(\mu_i; i \in I_l)$. The block F_l is the $k_l \times p$ matrix formed by zero columns, except for the l-th column, which is of the form $(\mu_i^{k_l-1})_{i \in I_l}$, and the $(l+1)$-th column, which is of the form $((-1)^{k_l} \mu_i^{-1})_{i \in I_l}$ for $l = 1, \ldots, p-1$. In the case $l = p$ the role of the $(l+1)$-th column of F_l is played by the first column, that is, all columns of F_p are zero except for the first column which is of the form $((-1)^{k_p} \mu_i^{-1})_{i \in I_p}$ and the last column which is of the form $(\mu_i^{k_p-1})_{i \in I_p}$.

Let us show that

$$
\det A = (-1)^{\sigma} (1 - \mu_0^{-1} \cdots \mu_n^{-1}) \prod_{l=1}^{p} \prod_{i > j \in I_l} (\mu_i - \mu_j), \quad \sigma \in \{0, 1\}, \tag{28}
$$

where σ depends only on $k_1, \ldots, k_p$. As in the preceding example, we define

$$
E := W^{-1} A = \begin{pmatrix} V_1^{-1} A_{11} & 0 & \cdots & 0 & V_1^{-1} F_1 \\ 0 & V_2^{-1} A_{22} & \ddots & \vdots & V_2^{-1} F_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & V_p^{-1} A_{pp} & V_p^{-1} F_p \end{pmatrix},
$$

where $W := \mathrm{diag}(V_1, \dots, V_p)$, $V_l := V(\mu_i; i \in I_l)$. Then

$$\det A = \det E \prod_{l=1}^{p} \prod_{i>j \in I_l} (\mu_i - \mu_j).$$

The determinant of E coincides up to a sign with the determinant of the $p \times p$ submatrix (25) formed by the last p columns and the last row of each block. All elements of $\hat{E}$ are zero (because they correspond to zero columns of A) except for

$$\hat{e}_{ll} = [\mu_i; i \in I_l](\cdot)^{k_l - 1} = 1, \quad l = 1, \dots, p,$$

$$\hat{e}_{l,l+1} = (-1)^{k_l} [\mu_i; i \in I_l](\cdot)^{-1} = -\prod_{i \in I_l} \mu_i^{-1}, \quad l = 1, \dots, p-1,$$

and

$$\hat{e}_{p,1} = (-1)^{k_p} [\mu_i; i \in I_p](\cdot)^{-1} = -\prod_{i \in I_p} \mu_i^{-1},$$

that is

$$\hat{E} = \begin{pmatrix} 1 & -\prod_{i \in I_1} \mu_i^{-1} & 0 & \cdots & 0 \\ 0 & 1 & -\prod_{i \in I_2} \mu_i^{-1} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & 1 & -\prod_{i \in I_{p-1}} \mu_i^{-1} \\ -\prod_{i \in I_p} \mu_i^{-1} & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The matrix $\hat{E}$ is analogous to the matrix (10) and so we obtain

$$\det E = (-1)^{\sigma} (1 - (\mu_0 \cdots \mu_n)^{-1}),$$

where $\sigma \in \{0, 1\}$. As in the previous example, it is straightforward to check (P1) and (P2) for the families of hyperplanes defined by (27). The construction of generalized principal lattices using Proposition 5 is completely analogous. We take $t_{0,0}, \dots, t_{0,n}$ and t in $\mathbf{R}$, $s_{0,0}, \dots, s_{0,n}$ and s in $\mathbf{Z}_2$ such that $\sum_{r=0}^{n} t_{0,r} + mt = 0$, $\sum_{r=0}^{n} s_{0,r} + ms = 0$ and choose the hyperplanes $H_i^r = H^r(t_{0,r} + it, s_{0,r} + is)$, $i \in \mathbf{N}_m$, $r \in \{0, \dots, n\}$. In order to have distinct hyperplanes we only need that for each $l \in \{1, \dots, p\}$,

$$(t_{0,r} + it, s_{0,r} + is) \quad i \in \mathbf{N}_m, \quad r \in I_l,$$

are distinct group elements.

Let us finish this section providing a three dimensional example, combining two families with $G = \mathbf{R}/2\pi\mathbf{Z}$

$$H_1(t) = H_2(t) = \cos(t)x_0 + \sin(t)x_1 + x_2,$$
$$H_3(t) = H_4(t) = \cos(t)x_0 - \sin(t)x_1 + x_3.$$

The intersection of $H_r(t_r) = 0$, $r = 0, 1, 2, 3$, leads to a system with coefficient matrix

$$A = \begin{pmatrix} \cos(t_0) & \sin(t_0) & 1 & 0 \\ \cos(t_1) & \sin(t_1) & 1 & 0 \\ \cos(t_2) & -\sin(t_2) & 0 & 1 \\ \cos(t_3) & -\sin(t_3) & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} \det A &= -\det \begin{pmatrix} \cos(t_1) - \cos(t_0) & \sin(t_1) - \sin(t_0) \\ \cos(t_3) - \cos(t_2) & -\sin(t_3) + \sin(t_2) \end{pmatrix} \\ &= -4\sin((t_1 - t_0)/2)\sin((t_3 - t_2)/2) \det \begin{pmatrix} -\sin((t_0 + t_1)/2) & \cos((t_0 + t_1)/2) \\ -\sin((t_2 + t_3)/2) & -\cos((t_2 + t_3)/2) \end{pmatrix} \\ &= -4\sin((t_1 - t_0)/2)\sin((t_3 - t_2)/2)\sin((t_0 + t_1 + t_2 + t_3)/2). \end{aligned}$$

So, if $t_1 - t_0, t_3 - t_2 \notin 2\pi\mathbf{Z}$, the determinant of A vanishes if and only if $t_0 + t_1 + t_2 + t_3 \in 2\pi\mathbf{Z}$. Hence (P1) and (P2) hold.

In contrast with the previous examples of this section, we have not been able to derive a general construction with parameterizations using trigonometric functions of the families of hyperplanes. Our first attempts indicate that there may be a restriction on the sizes of the blocks $k_1, \ldots, k_p$.

6. Remainder formulas for principal lattices

In this section, we will derive some general facts on the error of interpolation, i.e., on the function $f - L_m f$, provided that f is a sufficiently smooth function. It will turn out that these formulas are in accordance with the geometric nature of the interpolation nodes in GPL_m sets. To that end, we first note that condition (i) of Definition 1 ensures that the points

$$\mathbf{x}^\alpha := \bigcap_{j=1}^{n} H^j_{\alpha_j} \in \mathbf{R}^n, \qquad \alpha \in \mathbf{N}^n_m,\ |\alpha| \le m \tag{29}$$

indexed by the *dehomogeneized* multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ are well–defined. Observe that there is no loss of information with this change of notation because we can recover α_0 by $\alpha_0 = m - |\alpha|$. For $|\alpha| \le m$ we also define the polynomials

$$p_\alpha := \prod_{j=1}^{n} \prod_{k=0}^{\alpha_j - 1} \frac{h^j_k}{h^j_k(\mathbf{x}^\alpha)} =: \frac{\phi_\alpha}{\phi_\alpha(\mathbf{x}^\alpha)} \tag{30}$$

of total degree $|\alpha| \le m$. We observe that $\phi_\alpha(\mathbf{x}^\alpha) \ne 0$ because the hyperplanes (7) do not contain $\mathbf{x}^\alpha$. We also introduce the partial ordering "$\le$" on $\mathbf{N}^n_m$, writing $\alpha \le \beta$ iff $\alpha_j \le \beta_j$ for $j = 1, \ldots, n$, and $\alpha < \beta$ iff $\alpha \le \beta$ and $\alpha \ne \beta$.

Lemma 9. *The polynomials p_α and the points $\mathbf{x}^\alpha$, $|\alpha| \le m$, satisfy the following properties:*

(i) If $p_\alpha\left(\mathbf{x}^\beta\right) \ne 0$, then $\alpha \le \beta$.

(ii) $p_\alpha\left(\mathbf{x}^\beta\right) = \delta_{\alpha,\beta}$, for any α, β with $|\beta| \le |\alpha| \le m$.

Proof: If there exists $1 \le j \le n$ such that $\beta_j < \alpha_j$, then $\mathbf{x}^\beta \in H^j_{\beta_j} \subseteq H^j_0 \cup \cdots \cup H^j_{\alpha_j - 1}$ and therefore $h^j_k\left(\mathbf{x}^\beta\right) = 0$ for some $0 \le k \le \alpha_j - 1$, which proves (i). Condition (ii), on the other hand, follows from (i) and taking into account that $\beta \ge \alpha$ implies that either $|\alpha| < |\beta|$ or $\alpha = \beta$. ∎

In the terminology of [12], Lemma 9 says that the polynomials p_α, $|\alpha| \le m$, form a *Newton basis* for interpolation in Π^n_m. The coefficients $\lambda_\alpha f$ in the *Newton form* of the interpolation polynomial

$$L_m f = \sum_{|\alpha| \le m} \lambda_\alpha f \, p_\alpha,$$

the so-called *finite differences*, measure the error of lower degree interpolation at the interpolation nodes,

$$\lambda_\alpha f = \left(f - L_{|\alpha|-1} f\right)(\mathbf{x}^\alpha), \tag{31}$$

and admit an integral representation that has been given in [12]. To state this formula, we need some more notation. A *path* μ of length $k+1$ is a vector $(\mu_0, \ldots, \mu_k)$ of multiindices $\mu_j \in \mathbf{N}^n_m$ such that $|\mu_j| = j$, $j = 0, \ldots, m$. Denoting the totality of all such paths by Λ_m, we associate to any path $\mu \in \Lambda_m$

(i) a set $X_\mu := \{\mathbf{x}^{\mu_0}, \ldots, \mathbf{x}^{\mu_m}\}$ of interpolation nodes,

(ii) an m–th order partial differential operator

$$D^m_\mu := D_{\mathbf{x}^{\mu_m} - \mathbf{x}^{\mu_{m-1}}} \cdots D_{\mathbf{x}^{\mu_1} - \mathbf{x}^{\mu_0}} \tag{32}$$

following the directional derivatives along the path,

(iii) a number

$$\pi_\mu := \prod_{j=0}^{m-1} p_{\mu_j}\left(\mathbf{x}^{\mu_{j+1}}\right). \tag{33}$$

By means of the *simplex spline* integral

$$\int_{[X]} f = \int_{\Delta_N} f(Xu)\, du, \qquad X \subset \mathbf{R}^n,\ \#X = N,$$

where

$$\Delta_N := \{(u_1, \ldots, u_N) : u_1, \ldots, u_N \ge 0, u_1 + \cdots + u_N = 1\}, \quad Xu := u_1 \mathbf{x}^1 + \cdots + u_N \mathbf{x}^N,$$

we then obtain the following error formula that is valid for any $f \in C^{m+1}(\mathbf{R}^n)$, see Theorem 3.4 and Theorem 3.6 of [12]:

$$f(\mathbf{x}) - L_m f(\mathbf{x}) = \sum_{\mu \in \Lambda_m} p_{\mu_m}(\mathbf{x})\, \pi_\mu \int_{[X_\mu, \mathbf{x}]} D_{\mathbf{x} - \mathbf{x}^{\mu_m}}\, D^m_\mu f. \tag{34}$$

Note that, despite of the innocent appearance of this formula, the number of terms in the sum is tremendous: even in two variables it is already $(m+1)!$ while in general it will be of the form

$$\prod_{j=0}^{m-1} \binom{n+j}{j+1} = \binom{n}{1}\binom{n+1}{2}\cdots\binom{n+m-1}{m},$$

which makes it very difficult to even derive useful error estimates for the interpolation polynomial. In the case of our configuration of interpolation points formula (34) can be significantly simplified due to the very special form of the Newton polynomials. The first observation is that, due to Lemma 9, we have that

$$\pi_\mu \neq 0 \quad \Rightarrow \quad \mu \in \widehat{\Lambda}_m := \{\mu \in \Lambda_m : \mu_j \leq \mu_{j+1},\, j = 0,\dots,m-1\},$$

reducing the number of terms in the sum (34) to a mere n^m. A closer look at π_μ reveals even more structure: by definition we have for $\mu \in \Lambda_m$ that

$$p_{\mu_m}(\mathbf{x})\,\pi_\mu = \frac{\phi_{\mu_m}(\mathbf{x})}{\phi_{\mu_m}(\mathbf{x}^{\mu_m})} \prod_{j=0}^{m-1} \frac{\phi_{\mu_j}(\mathbf{x}^{\mu_{j+1}})}{\phi_{\mu_j}(\mathbf{x}^{\mu_j})} = \phi_{\mu_m}(\mathbf{x}) \prod_{j=0}^{m-1} \frac{\phi_{\mu_j}(\mathbf{x}^{\mu_{j+1}})}{\phi_{\mu_{j+1}}(\mathbf{x}^{\mu_{j+1}})}. \tag{35}$$

Since $\mu \in \widehat{\Lambda}_m$, we have that $\mu_j \leq \mu_{j+1}$ and $|\mu_{j+1}| = j+1 = |\mu_j|+1$, hence $\mu_{j+1} = \mu_j + \epsilon_{\ell(\mu,j)}$, where ϵ_k, $k=1,\dots,n$, denotes the unit multiindices of order 1. Fixing j for the moment and writing $\alpha = \mu_j$, $\ell = \ell(\mu,j)$, it then follows that

$$\frac{\phi_{\mu_j}}{\phi_{\mu_{j+1}}} = \frac{\phi_\alpha}{\phi_{\alpha+\epsilon_\ell}} = \frac{1}{h^\ell_{\alpha_\ell}}.$$

We write the affine functions associated to the hyperplanes as $h^\ell_{\alpha_\ell}(\mathbf{x}) = \mathbf{v}^{\ell,\alpha_\ell}\cdot\mathbf{x} + c_{\ell,\alpha_\ell}$ for $\mathbf{v}^{\ell,\alpha_\ell} \in \mathbf{R}^n$, $c_{\ell,\alpha_\ell} \in \mathbf{R}$, and assume that $\|\mathbf{v}^{\ell,\alpha_\ell}\|_2 = 1$ for all of these vectors; if we want to make them unique, we could, for example, require that the first nonzero entry of the vector is positive. Using this explicit form, we now obtain that

$$h^\ell_{\alpha_\ell}\left(\mathbf{x}^{\alpha+\epsilon_\ell}\right) = h^\ell_{\alpha_\ell}(\mathbf{x}^\alpha) + \mathbf{v}^{\ell,\alpha_\ell}\cdot\left(\mathbf{x}^{\alpha+\epsilon_\ell} - \mathbf{x}^\alpha\right) = \mathbf{v}^{\ell,\alpha_\ell}\cdot\left(\mathbf{x}^{\alpha+\epsilon_\ell} - \mathbf{x}^\alpha\right),$$

since $\mathbf{x}^\alpha \in H^\ell_{\alpha_\ell}$. Let us introduce the notation $\mathbf{v}^{\mu,j} := \mathbf{v}^{\ell,(\mu_j)_\ell}$ and recall that $\alpha = \mu_j$, $\alpha + \epsilon_\ell = \mu_j + \epsilon_{\ell(\mu,j)} = \mu_{j+1}$. The above formula can be written

$$\frac{\phi_{\mu_j}(\mathbf{x}^{\mu_{j+1}})}{\phi_{\mu_{j+1}}(\mathbf{x}^{\mu_{j+1}})} = \frac{1}{\mathbf{v}^{\mu,j}\cdot\left(\mathbf{x}^{\mu_{j+1}} - \mathbf{x}^{\mu_j}\right)}.$$

Substituting back into (35), we thus find that

$$p_{\mu_m}(\mathbf{x})\,\pi_\mu = \phi_{\mu_m}(\mathbf{x}) \prod_{j=0}^{m-1} \frac{1}{\mathbf{v}^{\mu,j}\cdot\left(\mathbf{x}^{\mu_{j+1}} - \mathbf{x}^{\mu_j}\right)}. \tag{36}$$

Recall from (29) that $\mathbf{x}^{\mu_j}$ is the intersection of $H^1_{\alpha_1}, \dots, H^n_{\alpha_n}$ while $\mathbf{x}^{\mu_{j+1}}$ is the common point of $H^1_{\alpha_1}, \dots, H^{\ell-1}_{\alpha_{\ell-1}}, H^{\ell}_{\alpha_\ell+1}, H^{\ell+1}_{\alpha_{\ell+1}}, \dots, H^n_{\alpha_n}$. Thus, $\mathbf{x}^{\mu_{j+1}} - \mathbf{x}^{\mu_j}$ is a multiple of the normalized directional vector $\mathbf{y}^{\mu,j}$ of the straight line

$$\bigcap_{k\in\{1,\dots,n\}\setminus\{\ell\}} H^k_{\alpha_k}$$

and since $H^\ell_{\alpha_\ell}$ intersects this straight line in precisely one point, the normal vector of $H^\ell_{\alpha_\ell}$ cannot be perpendicular to the directional vector $\mathbf{y}^{\mu,j}$ and therefore we have that

$$0 \neq \mathbf{v}^{\mu,j} \cdot \left(\mathbf{x}^{\mu_{j+1}} - \mathbf{x}^{\mu_j}\right) =: \rho_{\mu,j} \|\mathbf{x}^{\mu_{j+1}} - \mathbf{x}^{\mu_j}\|_2,$$

where $\rho_{\mu,j}$ denotes the nonzero cosine of the angle of intersection. In particular, (36) is indeed well–defined. Defining the *normalized* directional derivatives

$$\begin{aligned}\widehat{D}^m_\mu &:= D_{\mathbf{y}^{\mu,m-1}} \dots D_{\mathbf{y}^{\mu,0}} = \frac{D_{\mathbf{x}^{\mu_m}-\mathbf{x}^{\mu_{m-1}}}}{\|\mathbf{x}^{\mu_m} - \mathbf{x}^{\mu_{m-1}}\|_2} \cdots \frac{D_{\mathbf{x}^{\mu_1}-\mathbf{x}^{\mu_0}}}{\|\mathbf{x}^{\mu_1} - \mathbf{x}^{\mu_0}\|_2} \\ &= \frac{\rho_{\mu,m-1}}{\mathbf{v}^{\mu,m-1} \cdot \left(\mathbf{x}^{\mu_m} - \mathbf{x}^{\mu_{m-1}}\right)} D_{\mathbf{x}^{\mu_m}-\mathbf{x}^{\mu_{m-1}}} \cdots \frac{\rho_{\mu,0}}{\mathbf{v}^{\mu,0} \cdot \left(\mathbf{x}^{\mu_1} - \mathbf{x}^{\mu_0}\right)} D_{\mathbf{x}^{\mu_1}-\mathbf{x}^{\mu_0}}\end{aligned}$$

and the product $\rho_\mu := \prod_{j=0}^{m-1} \rho_{\mu,j} \neq 0$, we thus get from (36) the error formula

$$f(\mathbf{x}) - L_m f(\mathbf{x}) = \sum_{\mu\in\widehat{\Lambda}_m} \phi_{\mu_m}(\mathbf{x})\, \rho_\mu^{-1} \int_{[X_\mu,\mathbf{x}]} D_{\mathbf{x}-\mathbf{x}^{\mu_m}} \widehat{D}^m_\mu f. \tag{37}$$

This formula has a remarkable property: the differential operator $\hat{D}^m_\mu$ under the integral is invariant under scaling, a property very similar to the univariate case. From (37) we will finally derive an error estimate for the interpolant. To that end, let $\Omega \subset \mathbf{R}^n$ be a compact set which contains all the interpolation points $\mathbf{x}^\alpha$, $|\alpha| \le n$, and define the *diameter* of Ω as usual as

$$\operatorname{diam}(\Omega) := \max_{\mathbf{x},\mathbf{y}\in\Omega} \|\mathbf{x} - \mathbf{y}\|_2.$$

For a function f defined on Ω we consider the following family of (semi)norms

$$\|f\|_\Omega := \max_{\mathbf{x}\in\Omega} |f(\mathbf{x})|, \qquad \left\|f^{(k)}\right\|_\Omega = \max_{\mathbf{x}\in\Omega} \max_{|\alpha|=k} \left|\frac{\partial^k f}{\partial \mathbf{x}^\alpha}(\mathbf{x})\right|, \quad k = 1, 2, \dots$$

Moreover, let

$$\rho := \min\left\{|\rho_{\mu,j}| : \mu \in \widehat{\Lambda}_m,\ j = 0, \dots, m-1\right\}$$

denote the cosine of the smallest angle of intersection.

Theorem 10. *If $f \in C^{m+1}(\mathbf{R}^n)$ and $\Omega \subset \mathbf{R}^n$ is a convex compact set containing the interpolation points, then the interpolant $L_m f$ satisfies*

$$\|f - L_m f\|_\Omega \le n^{2m+1} \frac{\operatorname{diam}(\Omega)^{m+1}}{\rho^m} \frac{\left\|f^{(m+1)}\right\|_\Omega}{(m+1)!}. \tag{38}$$

Proof: We set $\alpha := \mu_m$ and consider

$$\phi_\alpha(\mathbf{x}) = \prod_{j=1}^{n} \prod_{k=0}^{\alpha_j - 1} h_k^j(\mathbf{x}).$$

For each j,k choose $\mathbf{x}^{j,k}$ such that $h_k^j\left(\mathbf{x}^{j,k}\right) = 0$ and note that again

$$h_k^j(\mathbf{x}) = h_k^j\left(\mathbf{x}^{j,k}\right) + \mathbf{v}^{j,k} \cdot \left(\mathbf{x} - \mathbf{x}^{j,k}\right) = \mathbf{v}^{j,k} \cdot \left(\mathbf{x} - \mathbf{x}^{j,k}\right),$$

from which we conclude for $\mathbf{x} \in \Omega$ that

$$|\phi_\alpha(\mathbf{x})| = \prod_{j=1}^{n} \prod_{k=0}^{\alpha_j - 1} \left|\mathbf{v}^{j,k} \cdot \left(\mathbf{x} - \mathbf{x}^{j,k}\right)\right| \le \prod_{j=1}^{n} \prod_{k=0}^{\alpha_j - 1} \left\|\mathbf{x} - \mathbf{x}^{j,k}\right\|_2 \le \operatorname{diam}(\Omega)^m. \tag{39}$$

Next, we define for $\mu \in \widehat{\Lambda}_m$ the directions

$$\mathbf{y}^m = \mathbf{x} - \mathbf{x}^{\mu_m}, \qquad \mathbf{y}^j = \frac{\mathbf{x}^{\mu_{j+1}} - \mathbf{x}^{\mu_j}}{\left\|\mathbf{x}^{\mu_{j+1}} - \mathbf{x}^{\mu_j}\right\|_2}, \quad j = 0, \dots, m-1, \tag{40}$$

and consider the differential operator $D_{\mathbf{y}^0} \cdots D_{\mathbf{y}^m} f$ on Ω, where we have that

$$\left|D_{\mathbf{y}^0} \cdots D_{\mathbf{y}^m} f\right| \le \left\|\mathbf{y}^0\right\|_2 \left\|D_{\mathbf{y}^1} \cdots D_{\mathbf{y}^m} \nabla f\right\|_2 \le \left\|\mathbf{y}^0\right\|_2 \left\|D_{\mathbf{y}^1} \cdots D_{\mathbf{y}^m} \nabla f\right\|_1,$$

and, by induction and taking into account (40),

$$\begin{aligned}
\left|D_{\mathbf{y}^0} \cdots D_{\mathbf{y}^m} f\right| &\le \prod_{j=0}^{m} \|\mathbf{y}^j\|_2 \left\|f^{(m+1)}\right\|_1 = \prod_{j=0}^{m} \|\mathbf{y}^j\|_2 \sum_{|\alpha|=m+1} \binom{m+1}{\alpha} \left|\frac{\partial^{m+1} f}{\partial \mathbf{x}^\alpha}\right| \\
&\le \operatorname{diam}(\Omega)\, n^{m+1} \left\|f^{(m+1)}\right\|_\Omega .
\end{aligned}$$

Substituting this estimate and (39) into (37), the claim (38) follows immediately from recalling that $\int_{[X_\mu, \mathbf{x}]} 1 = 1/(m+1)!$. ■

The error estimate (38) is indeed remarkable as it is practically the univariate one, except two "natural" multivariate ingredients: the *dimension curse* n^{2m+1} (which could be reduced to n^m by a modification of the seminorm $\left\|f^{(m+1)}\right\|$) and the *geometry term* ρ^{-m} depending on the angles of intersection between the hyperplanes. Note that $\rho \le 1$ and that $\rho = 1$ if and only if all hyperplanes intersect perpendicularly which essentially corresponds to the situation that $\mathbf{x}^\alpha = \alpha/m$, the *triangular grid* which has already been investigated in [1], cf. [2].

References

1. Biermann, O., Über näherungsweise Kubaturen. Monatsh. Math. Phys. **14** (1903), 211–225.
2. Biermann, O., *Vorlesungen über mathematische Näherungsmethoden.* F. Vieweg, Braunschweig 1905.
3. J. M. Carnicer and M. Gasca, Interpolation on lattices generated by cubic pencils, to appear in Adv. in Comp. Math.
4. J. M. Carnicer and M. Gasca, Generation of lattices of points for bivariate interpolation, Numer. Algor. **39** (2005), 69–79
5. K. C. Chung and T. H. Yao, On lattices admitting unique Lagrange interpolations, SIAM J. Numer. Anal. **14** (1977) 735–743.
6. M. Gasca and J. I. Maeztu, On Lagrange and Hermite interpolation in $\mathbf{R}^n$, Numer. Math. 39 (1982) 1–14.
7. M. Gasca and T. Sauer, Polynomial interpolation in several variables, Adv. Comp. Math. **12** (2000) 377–410.
8. M. Gasca and T. Sauer, On the history of multivariate polynomial interpolation, JCAM, **122** (2000) 23–35.
9. S. L. Lee and G. M. Phillips, Construction of Lattices for Lagrange Interpolation in Projective Space, Constr. Approx. **7** (1991), 283–297.
10. G. G. Lorentz, *Approximation of functions*, Chelsea Publishing Company, New York, 1966.
11. G. M. Phillips, *Interpolation and Approximation by Polynomials*, Springer, New York, 2003.
12. T. Sauer and Yuan Xu, On multivariate Lagrange interpolation, Math. Comp. **64** (1995) 1147–1170.